Ultralimits and fixed-point properties, after Gromov

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Context (1)

In this talk:

- G, Γ denote finitely generated groups;
- X, Y, Z denote complete metric spaces.

We are concerned with isometric actions $G \curvearrowright X$.

Let S be a finite generating set of G. For all $x \in X$, we set

$$\delta_S(x) := \max\{d(x, s \cdot x) : s \in S\}$$

and call δ_S the displacement function.

Remark

$x \in X$ is a fixed point iff $\delta_S(x) = 0$.

Context (2)

Remark

If T is another finite generating set, then δ_S and δ_T are bilipschitz equivalent: $\delta_T(x) \leq (\max_{t \in T} |t|_S) \cdot \delta_S(x)$.

Definition

A sequence of almost fixed points (for an action $G \curvearrowright X$) is a sequence (x_n) in X such that

$$\lim_{n \to \infty} \delta_S(x_n) = 0 \ .$$

If there is no such sequence, the action is said to be uniform.

Remark

1. This Definition does not depend on S;
2. The action has almost fixed points iff $\inf_{x \in X} \delta_S(x) = 0$.

Fixed points vs. almost fixed points

Let $G = \mathbb{Z} = \langle t \rangle$ and $X = \mathbb{H}^2$ (Poincaré upper half-plane).
Let G act on X by $t \cdot z = z + 1$ (horizontal translation).

This action has almost fixed points (take $x_n = in$), but satisfies

$$\forall z \in X \quad \lim_{g \to \infty} d(z, g \cdot z) = +\infty \ .$$

This is far from having fixed points . . .

Fixed points vs. almost fixed points (2)

The same phenomenon occurs with Hilbert spaces.

Let $G = \mathbb{Z} = \langle t \rangle$ act (affinely) on $X = \ell^2(\mathbb{Z})$ by $t \cdot \xi = S\xi + \delta_0$, where

- S is the shift operator: $(S\xi)(k) = \xi(k-1)$;
- δ_0 is the Dirac mass at 0.

This action has almost fixed points (take $x_n = \sum_{j=0}^{n} \frac{n-j}{n}\delta_j$), but, again, satisfies

$$\forall \xi \in X \qquad \lim_{g\to\infty} d(\xi, g\cdot\xi) = +\infty \ .$$

Serre's Property (FH)

Definition

We say that G has Property (FH) if every G-action on a (real, affine) Hilbert space has a fixed point.

Example

1. $SL_n(\mathbb{Z})$ has property (FH) for $n \geq 3$;
2. A free group $\mathbb{F}_n$ does not have Property (FH).

Results about Property (FH)

Theorem (Korevaar-Schoen, 1997; Shalom, 2000)

A finitely generated group G has property (FH) if and only if every G-action on a Hilbert space has almost fixed points.

Theorem (Shalom, 2000)

Every finitely generated group with property (FH) is a quotient of a finitely presented group with property (FH).

A result by Gromov

We shall use:

Theorem (Gromov, 2003)

Consider actions $G \curvearrowright (X_n, d_n)$ for $n \in \mathbb{N}$ and suppose there are no fixed points. Let $x_n \in X_n$ for all n.

Then, there exist points $y_n \in X_n$ such that $\delta_S(y_n) \leq \delta_S(x_n)$ for all n and G acts uniformly on the ultralimit

$$(X_\omega, d_\omega, y_\omega) := \lim_{n \to \omega} \left(X_n, \frac{d_n}{\delta_S(y_n)}, y_n \right)$$

for any non-principal ultrafilter ω on $\mathbb{N}$.

How to deduce the Korevaar-Schoen Theorem

Theorem (Korevaar-Schoen, 1997; Shalom, 2000)

A finitely generated group G has property (FH) if and only if every G-action on a Hilbert space has almost fixed points.

Suppose G acts without fixed point on some Hilbert space (X, d) and set $(X_n, d_n) = (X, d)$ for all n.

Believe that the space X_ω provided by Gromov's result is a Hilbert space; G acts uniformly on it.

The other direction is trivial.

A remark about the spaces we deal with

In the preceeding argument, Hilbert spaces may be replaced by any class $\mathcal{X}$ of metric spaces which is close under rescaling and ultralimits, e.g.

- L^p spaces;
- CAT(0) spaces;
- $\mathbb{R}$-trees, but not simplicial trees.

For such a class $\mathcal{X}$ and a group G, the following are equivalent:

1. Every action $G \curvearrowright X$ with $X \in \mathcal{X}$ has a fixed point;
2. Every action $G \curvearrowright X$ with $X \in \mathcal{X}$ has almost fixed points;

Ultralimits (1)

Definition

1. An ultrafilter on $\mathbb{N}$ is a finitely-additive $\{0,1\}$-valued measure $\omega : \mathcal{P}(\mathbb{N}) \to \{0,1\}$ such that $\omega(\mathbb{N}) = 1$;
2. It is non-principal if finite sets have measure 0;
3. A sequence (z_n) in (Z,d) converges to z w.r.t. ω if, for any $\varepsilon > 0$, the set $\{n \in \mathbb{N} : d(z,z_n) < \varepsilon\}$ has ω-measure 1.

An important property for us: bounded sequences in $\mathbb{R}$ all converge w.r.t. any ultrafilter ω (the limit may depend on ω).

Ultralimits (2)

Let $(X_n, d_n, *_n)_{n\in\mathbb{N}}$ be pointed metric spaces;
Let ω be a non-principal ultrafilter on $\mathbb{N}$.

We set

$$\mathcal{B} = \left\{ (x_n) \in \prod_{n\in\mathbb{N}} X_n : \text{ the sequence } \big(d_n(*_n, x_n)\big)_{n\in\mathbb{N}} \text{ is bounded} \right\}$$

Definition

The ultralimit of the sequence $(X_n, d_n, *_n)_{n\in\mathbb{N}}$ w.r.t. ω is the pointed metric space obtained by separating the pseudo-metric space

$$\big(\mathcal{B}, d_\omega, (*_n)_n\big) \text{ where } d_\omega((x_n),(y_n)) = \lim_{n\to\omega} d_n(x_n, y_n)\ .$$

We denote it $\lim_{n\to\omega}(X_n, d_n, *_n)$.

Ultralimits (3)

Remark

If G acts on all spaces X_n, the diagonal G-action on $\prod_{n\in\mathbb{N}} X_n$ stabilizes $\mathcal{B}$ iff the sequence $d_n(*_n, s \cdot *_n)$ is bounded for every generator s of G.

In this case, we obtain an (isometric) action on each ultralimit $\lim_{n\to\omega}(X_n, d_n, *_n)$.

Some particular cases of ultralimits:

- asymptotic cone: $\lim_{n\to\omega}(X, \alpha_n \cdot d, *_n)$, with $\alpha_n \to 0$;
- tangent cone (at $* \in X$): $\lim_{n\to\omega}(X, \alpha_n \cdot d, *)$, with $\alpha_n \to +\infty$.

How to deduce Shalom's Theorem

Theorem (Shalom, 2000)

If Γ has property (FH), then it is a quotient of a finitely presented group with property (FH).

Write $\Gamma =: \mathbb{F}_k/N$ with $N =: \{r_1, r_2, \ldots\}$.

Set $N_n := \langle\langle r_1, \ldots, r_n \rangle\rangle$ and $\Gamma_n := \mathbb{F}_k/N_n$ (which is finitely presented).

Assume by contradiction that no Γ_n has property (FH) and take actions $\Gamma_n \curvearrowright \mathcal{H}_n$ without fixed points on Hilbert spaces.

Consider induced actrions $\mathbb{F}_k \curvearrowright \mathcal{H}_n$. The ultralimit $(\mathcal{H}_\omega, d_\omega)$ arising form Gromov's Theorem is a Hilbert space on which:

- $\mathbb{F}_k$ acts uniformly;
- N acts trivially (since N_n acts trivially on $\mathcal{H}_n$ and the N_n's cover N).

Thus, Γ acts uniformly on $(\mathcal{H}_\omega, d_\omega)$, a contradiction.

Another application of Gromov's result

Definition

G has Kazhdan's property (T) if, for any orthogonal representation $(\pi, \mathcal{H})$, the restriction on the unit sphere $\mathcal{S}(\mathcal{H})$ either has a fixed point, or is uniform.

Theorem (Guichardet, 1977)

If G has property (FH), then it has property (T).

The converse is due to Delorme (1977).

Remark

Delorme's and Guichardet's results would not require the "finitely generated" assumption. In fact, both properties (FH) and (T) imply finite genration.

A proof by Gromov and Schoen (1)

Suppose G has not property (T) and take an orthogonal representation $(\pi, \mathcal{H})$, whose restriction on the unit sphere $\mathcal{S}(\mathcal{H})$ has almost fixed points $(\eta_n)_{n\in\mathbb{N}}$, but no fixed point.

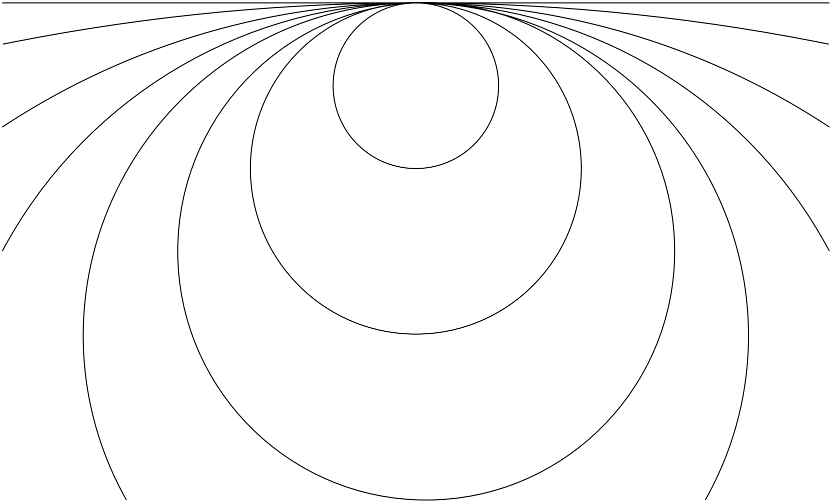
By Gromov's Theorem, we obtain almost fixed points $(\xi_n)_n$ in $\mathcal{S}(\mathcal{H})$ such that G acts uniformly on the ultralimit

$$(X_\omega, d_\omega, \xi_\omega) := \lim_{n\to\omega} \left(\mathcal{S}(\mathcal{H}), \frac{d}{\delta_S(\xi_n)}, \xi_n \right)$$

for any non-principal ultrafilter ω on $\mathbb{N}$.

A proof by Gromov and Schoen (2)

Since $\lim_{n\to\infty} \delta_S(\xi_n) = 0$, $(X_\omega, d_\omega, \xi_\omega)$ is an ultralimit of spheres in $\mathcal{H}$ whose radii tend to $+\infty$.



A proof by Gromov and Schoen (3)

We see that $(X_\omega, d_\omega, \xi_\omega)$ identifies with an ultralimit of (a codimension 1 affine subspace of) $\mathcal{H}$, that is with an affine Hilbert space.

G acts uniformly on X_ω (as said before).

Thus G does not have property (FH). Q.E.D.

Proof of Gromov's Theorem (1)

Lemma

Consider an action $G \curvearrowright X$ with no fixed point. Let $x \in X$ and $r > 0$. Then, there exists $y \in X$ such that $\delta_S(y) \leq \delta_S(x)$ and

$$\textit{for all } z \in \bar{B}(y, r \cdot \delta_S(y)), \quad \delta_S(z) \geq \frac{\delta_S(y)}{2} .$$

Idea of Proof: Assume the conclusion does not hold.
We get a (Cauchy) sequence (x_k) such that $x_0 = x$,

$$x_{k+1} \in \bar{B}(x_k, r \cdot \delta_S(x_k)) \text{ and } \delta_S(x_{k+1}) < \frac{\delta_S(x_k)}{2} .$$

As X is complete, we get a limit $\ell = \lim_{k \to \infty} x_k$.

Then $\delta_S(\ell) = 0$, a contradiction since the action has no fixed point.

Proof of Gromov's Theorem (2)

Theorem (Gromov, 2003)

Consider actions $G \curvearrowright (X_n, d_n)$ for $n \in \mathbb{N}$ and suppose there are no fixed points. Let $x_n \in X_n$ for all n.

Then, there exist points $y_n \in X_n$ such that $\delta_S(y_n) \leq \delta_S(x_n)$ for all n and G acts uniformly on the ultralimit

$$(X_\omega, d_\omega, y_\omega) := \lim_{n \to \omega} \left(X_n, \frac{d_n}{\delta_S(y_n)}, y_n \right)$$

for any non-principal ultrafilter ω on $\mathbb{N}$.

Proof: The Lemma gives points $y_n \in X_n$ such that $\delta_S(y_n) \leq \delta_S(x_n)$ and

$$\text{for all } w \in \bar{B}_{d_n}(y_n, n \cdot \delta_S(y_n)), \quad \delta_S(w) \geq \frac{\delta_S(y_n)}{2} \; . \tag{1}$$

Let ω be a non-principal ultrafilter.

Proof of Gromov's Theorem (3)

The displacement function on $(X_n, \tilde{d}_n := \frac{d_n}{\delta_S(y_n)})$ is $\tilde{\delta}_S(x) = \frac{\delta_S(x)}{\delta_S(y_n)}$.

G acts "diagonally" on the ultralimit $\lim_{n\to\omega}(X_n, \tilde{d}_n, y_n)$ since

$$\tilde{d}_n(y_n, s \cdot y_n) \leq \tilde{\delta}_S(y_n) = 1 \ .$$

Remains to show: G acts uniformly.

Take $z = [z_n]_\omega \in \lim_{n\to\omega}(X_n, \tilde{d}_n, y_n)$; Then $\left(\tilde{d}_n(y_n, z_n)\right)$ is bounded.

Re-write (1) with distance $\tilde{d}_n$ instead of d_n:

$$\text{for all } w \in \bar{B}_{\tilde{d}_n}(y_n, n), \quad \tilde{\delta}_S(w) \geq \frac{1}{2} \ .$$

Thus, $\tilde{\delta}_S(z_n) \geq \frac{1}{2}$ for n sufficiently large.

Proof of Gromov's Theorem (4)

Set now (for $s \in S$)

$$A_s := \left\{ n \in \mathbb{N} : \tilde{d}_n(z_n, s \cdot z_n) \geq \frac{1}{2} \right\} \ ;$$

we have $n \in \bigcup_{s \in S} A_s$ for n large enough, hence $\omega(\bigcup_{s \in S} A_s) = 1$.

As S is finite, there exists $s(z) \in S$ such that $\omega(A_{s(z)}) = 1$, hence

$$d_\omega(z, s(z) \cdot z) = \lim_{n \to \omega} d_n(z_n, s(z) \cdot z_n) \geq \frac{1}{2} \ .$$

Thus, G acts uniformly on (X_ω, d_ω). Q.E.D.

Possible generalisations

Gromov's result may be stated in more general contexts. One may replace:

1. isometries by (uniformly) Lipschitz transformations;
2. groups by semigroups;

A question

Definition

We say that G has Serre's Property (FA) if every G-action on a simplicial tree (without inversion) has a fixed point.

Question

Is every finitely generated group with Property (FA) a quotient of a finitely presented group with Property (FA) ?