II_1 factors with at most one Cartan subalgebra

Narutaka OZAWA
Joint work with Sorin POPA

University of Tokyo

Analytic Properties of Infinite Groups,
Genève, August 2008

Introduction

geared for rigidity phenomena

Travel supported by JSPS

What do we classify?

Γ	countable discrete group
(X, μ)	standard probability measure space
$\Gamma \curvearrowright (X, \mu)$	(ergodic) measure preserving action

$\Gamma \curvearrowright X$ is said to be *ergodic* if

$$A \subset X \text{ and } \Gamma A = A \Rightarrow \mu(A) = 0, 1.$$

We only consider either

- $(X, \mu) \cong ([0,1], \text{Lebesgue})$ and
  $\Gamma \curvearrowright X$ is *essentially-free* i.e. $\mu(\{x : gx = x\}) = 0 \ \forall g \in \Gamma \setminus \{1\}$;

or

- $X = \{\text{pt}\}$.

How do we classify?

GA: $\Gamma \curvearrowright (X, \mu)$ → **OE**: $L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$ → **vN**: $\mathrm{vN}(X \rtimes \Gamma)$

GP: Γ → **vN**: $\mathrm{vN}(\Gamma)$

To what extent do vN/OE remember OE/GA/GP?

Group measure space constructions

$$\Gamma \curvearrowright (X,\mu) \quad \text{p.m.p.} \quad \longleftrightarrow \quad \begin{array}{l} \sigma\colon \Gamma \curvearrowright L^\infty(X,\mu) \\ \sigma_g(f)(x) = f(g^{-1}x) \\ \int \sigma_g(f)\,d\mu = \int f\,d\mu \end{array}$$

The unitary element $u_g = \sigma_g \otimes \lambda_g \in \mathbb{B}(L^2(X) \otimes \ell_2(\Gamma))$ satisfies

$$u_g\, f\, u_g^* = \sigma_g(f)$$

for all $f \in L^\infty(X,\mu)$, identified with $f \otimes 1 \in \mathbb{B}(L^2(X) \otimes \ell_2(\Gamma))$.
We encode the information of $\Gamma \curvearrowright X$ into a single vN algebra

$$\mathrm{vN}(X \rtimes \Gamma) := \{\sum_{g\in\Gamma}^{\text{finite}} f_g\, u_g : f_g \in L^\infty(X)\}'' \subset \mathbb{B}(L^2(X) \otimes \ell_2(\Gamma)).$$

$\mathrm{vN}(X \rtimes \Gamma)$ is same as the crossed product vN algebra $L^\infty(X) \rtimes \Gamma$.

Group measure space constructions

$\mathrm{vN}(X \rtimes \Gamma)$ is a vN algebra of type II_1, with the trace τ given by

$$\tau(\sum_g f_g\, u_g) = \langle \sum_g f_g\, u_g\, (\mathbf{1} \otimes \delta_1), (\mathbf{1} \otimes \delta_1) \rangle = \int f_1\, d\mu.$$

(It follows $\tau(xy) = \tau(yx)$.)

The subalgebra $L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$ has a special property.

Definition

A von Neumann subalgebra $A \subset M$ is called a *Cartan subalgebra* if it is a maximal abelian subalgebra such that the normalizer

$$\mathcal{N}(A) = \{u \in M : \text{unitary} \quad uAu^* = A\}$$

generates M as a von Neumann algebra.

Orbit Equivalence Relation

Theorem (Singer, Dye, Krieger, Feldman-Moore 1977)

Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be ess-free p.m.p. actions, and

$$\theta\colon (X,\mu) \to (Y,\nu)$$

be an isomorphism. Then, the isomorphism

$$\theta^*\colon L^\infty(Y,\nu) \ni f \mapsto f\circ\theta \in L^\infty(X,\mu)$$

extends to a $*$-isomorphism

$$\pi\colon \mathrm{vN}(Y\rtimes\Lambda) \to \mathrm{vN}(X\rtimes\Gamma)$$

if and only if θ preserves the orbit equivalence relation:

$$\theta(\Gamma x) = \Lambda\theta(x) \quad \text{for } \mu\text{-a.e. } x.$$

Lack of rigidity

GA		OE		vN
$\Gamma \curvearrowright (X, \mu)$	→	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	→	$\mathrm{vN}(X \rtimes \Gamma)$

Theorem (Hakeda-Tomiyama, Sakai 1967)

$\mathrm{vN}(X \rtimes \Gamma)$ *is injective (amenable)* $\Leftrightarrow \Gamma$ *is amenable.*

E.g. Solvable groups and subexponential groups are amenable.
Non-abelian free groups $\mathbb{F}_r$ are not.

Theorem (Connes 1974, Ornstein-Weiss, C-Feldman-W 1981)

Amenable **vN** *and* **OE** *are unique modulo center.*

Lack of rigidity

GA		OE		vN
$\Gamma \curvearrowright (X, \mu)$	$\longrightarrow$	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	$\longrightarrow$	$\mathrm{vN}(X \rtimes \Gamma)$

Theorem (Connes-Jones 1982)

OE $\longrightarrow$ **vN** *is not one-to-one,*
i.e. $\exists$ *a* II_1*-factor with non-conjugate Cartan subalgebras.*

Example (Oz-Popa 2008)

$$\mathrm{vN}\Big((\varprojlim(\mathbb{Z}/k_n\mathbb{Z})^2) \rtimes (\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))\Big)$$

has at least two Cartan subalg $L^\infty(\varprojlim(\mathbb{Z}/k_n\mathbb{Z})^2)$ and $\mathrm{vN}(\mathbb{Z}^2)$.

Lack of rigidity

GA: $\Gamma \curvearrowright (X, \mu)$ → **OE**: $L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$ → **vN**: $\mathrm{vN}(X \rtimes \Gamma)$

GP: Γ → **vN**: $\mathrm{vN}(\Gamma)$

Theorem (Connes 1975)

$\exists$ *a* II_1*-factor which is not* $*$*-isomorphic to its complex conjugate.*

Theorem (Voiculescu 1994)

$\mathrm{vN}(\mathbb{F}_r)$ *does not have a Cartan subalgebra.*

Rigidity

GA		OE		vN
$\Gamma \curvearrowright (X,\mu)$	→	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	→	$\mathrm{vN}(X \rtimes \Gamma)$

Theorem (Furman 1999, (Monod-Shalom,) Popa, Kida, Ioana)

Some **OE** *fully remembers* **GA**. *E.g.,* $\mathrm{SL}(3,\mathbb{Z}) \curvearrowright \mathbb{T}^3$.

Theorem (Oz-Popa 2007, 2008)

Some **vN** *fully remembers* **OE**, *i.e.,* $\exists$ *a (non-amenable)* II_1*-factor with a unique Cartan subalgebra up to unitary conjugacy.*

Note: Popa (2000) proved $\mathrm{vN}(\mathbb{Z}^2) \subset \mathrm{vN}(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))$ is a unique "Cartan subalgebra with the relative property (T)."

GA $\Gamma \curvearrowright (X,\mu)$ → **OE** $L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$ → **vN** $\mathrm{vN}(X \rtimes \Gamma)$

GP Γ → **vN** $\mathrm{vN}(\Gamma)$

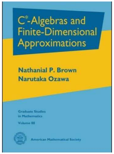
Problem

- Is there **vN** which fully remembers **GA**?
- Is there **vN** which fully remembers **GP**?
- $\mathrm{vN}(\mathbb{F}_r) \ncong \mathrm{vN}(\mathbb{F}_s)$?

Note: Popa (2004) proved $\mathrm{vN}([0,1]^\Gamma \rtimes \Gamma) \cong \mathrm{vN}(Y \rtimes \Lambda)$ implies $(\Gamma \curvearrowright [0,1]^\Gamma) \cong (\Lambda \curvearrowright Y)$ provided that Λ has the property (T). Further results by Popa and Vaes.

From **vN** to **OE**

NEW!



Groups with the Haagerup property

Definition

A 1-cocycle of a loc. cpt group G consists of a conti. unitary rep $(\pi, \mathcal{H})$ and a conti. map $b\colon G \to \mathcal{H}$ such that

$$\forall g, h \in G, \quad b(gh) = b(g) + \pi_g b(h).$$

(i.e., $\theta_g \xi = \pi_g \xi + b(g)$ defines an affine isometric action θ on $\mathcal{H}$. Schönberg: $\exp(-t\|b(g)\|^2)$ is a 1-pr semigr of pos type functions.)
The 1-cocycle b is *proper* if $\|b(g)\| \to \infty$ as $g \to \infty$.
A group G has the *Haagerup property* if it admits a proper 1-cocycle $(\pi, \mathcal{H}, b)$. The group G has the *property* (HH) if in addition π can be taken non-amenable (i.e., no $\mathrm{Ad}\pi$-invariant state on $\mathbb{B}(\mathcal{H})$).

Groups with the property (HH)

Observation

A group G with the property (HH) *is not inner-amenable.*
In particular, (infinite amenable) $\times \Gamma$ *does not have* (HH).

Proof.

Let $(\pi, \mathcal{H}, b)$ be a proper 1-cocycle, and suppose that $\exists$ a singular $\mathrm{Ad}G$-invariant state μ on $L^\infty(G)$. For $x \in \mathbb{B}(\mathcal{H})$, we define $f_x \in L^\infty(G)$ by $f_x(g) = \|b(g)\|^{-2}\langle xb(g), b(g)\rangle$.
Let $h \in G$ be fixed. Since $\lim_g \|b(g)\| = \infty$ and

$$\|b(h^{-1}gh) - \pi_h^{-1} b(g)\| = \|b(h^{-1}) + \pi_{h^{-1}g} b(h)\| \leq 2\|b(h)\|,$$

one has $(\mathrm{Ad}h)(f_x) - f_{\pi_h x \pi_h^*} \in C_0(G)$.
It follows that $x \mapsto \mu(f_x)$ is an $\mathrm{Ad}\pi$-invariant state. □

Groups with the property (HH)

Observation

A group G with the property $\mathrm{(HH)}$ *is not inner-amenable.*
In particular, (infinite amenable) $\times\,\Gamma$ *does not have* $\mathrm{(HH)}$.

The converse...

Theorem (Haagerup 1978, De Cannière-H. 1985, Cowling 1983)

The connected simple Lie groups $\mathrm{SO}(n,1)$ *with* $n \geq 2$ *and* $\mathrm{SU}(n,1)$ *have the property* $\mathrm{(HH)}$. *In particular, lattices of products of* $\mathrm{SO}(n,1)$ *with* $n \geq 2$ *and* $\mathrm{SU}(n,1)$ *have the property* $\mathrm{(HH)}$.
Moreover, they have the complete metric approximation property.

At Most One Cartan Subalgebra

Theorem **A** (Oz-Popa 2008)

Let Γ be a countable group with the property (HH) and the CMAP. Then, $\mathrm{vN}(\Gamma)$ has no Cartan subalgebra. Moreover, if $\Gamma \curvearrowright X$ is profinite action, then $L^\infty(X)$ is the unique Cartan subalgebra in $\mathrm{vN}(X \rtimes \Gamma)$.

Definition

An ergodic action $\Gamma \curvearrowright X$ is *profinite* if $X = \varprojlim \Gamma/\Gamma_n$ for some finite index subgroups $\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$;
or equivalently $\exists A_1 \subset A_2 \subset \cdots \subset L^\infty(X)$ finite-dimensional Γ-invariant vN subalgebras with dense union. $(A_n = \ell_\infty(\Gamma/\Gamma_n).)$

$$\mathrm{vN}(X \rtimes \Gamma) = \Big(\bigcup \mathrm{vN}((\Gamma/\Gamma_n) \rtimes \Gamma)\Big)'' \cong \Big(\bigcup \mathbb{M}_{[\Gamma:\Gamma_n]}(\mathrm{vN}(\Gamma_n))\Big)''.$$

Weak Compactness

Theorem (Oz-Popa 2007)

Suppose that M has CMAP and A is an amenable vN subalgebra. Then, $A \subset M$ is weakly compact in the following sense: $\exists\ \eta_n \in L^2(A \bar{\otimes} \bar{A})_+$ such that

- $\|\eta_n - (u \otimes \bar{u})\eta_n\|_2 \to 0$ *for every* $u \in \mathcal{U}(A)$;
- $\|\eta_n - \mathrm{Ad}(u \otimes \bar{u})\eta_n\|_2 \to 0$ *for every* $u \in \mathcal{N}(A)$;
- $\langle (x \otimes 1)\eta_n, \eta_n \rangle = \tau(x) = \langle \eta_n, (1 \otimes \bar{x})\eta_n \rangle$ *for every* $x \in M$.

If $M = A \rtimes \Gamma$ and $\exists\ A_1 \subset A_2 \subset \cdots \subset A$ finite-dim. Γ-invariant vN subalgebras with dense union, then $A \subset M$ is weakly compact with $\eta_n = c\,\mathrm{Id}_{L^2(A_n)} \in L^2(A_n \bar{\otimes} \bar{A}_n)$.

Rough Proof of Theorem A

Let Γ be a group having the property (HH) with $(\pi, \mathcal{H}, b)$ and the CMAP, and suppose there is a Cartan subalgebra $A \subset \mathrm{vN}(\Gamma)$.
Let $\eta_n \in \ell^2(\Gamma) \otimes \ell^2(\Gamma)$ be as in the previous slide.
We define a state φ on $\mathbb{B}(\mathcal{H})$ by

$$\varphi(x) = \operatorname*{Lim}_{n\to\infty} \sum_{g,g'} \langle x\, \eta_n(g,g') \frac{b(g)}{\|b(g)\|}, \eta_n(g,g') \frac{b(g)}{\|b(g)\|} \rangle.$$

Since η_n is approx. conjugate invariant and

$$b(h^{-1}gh) = b(h^{-1}) + \pi_{h^{-1}} b(g) + \pi_{h^{-1}g} b(h) \approx \pi_{h^{-1}} b(g)$$

as $g \to \infty$, the state φ is $\mathrm{Ad}\pi$-invariant. Thus π is amenable. $\square$

The real proof involves spectral analysis of the quantum Markov semigroup associated with a closable derivation (Sauvageot, et al. and Peterson).

From **OE** to **GA**

Orbit Equivalence Relation

Theorem (Singer, Dye, Krieger, Feldman-Moore 1977)

Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be ess-free p.m.p. actions, and

$$\theta\colon (X,\mu) \to (Y,\nu)$$

be an isomorphism. Then, the isomorphism

$$\theta^*\colon L^\infty(Y,\nu) \ni f \mapsto f \circ \theta \in L^\infty(X,\mu)$$

extends to a $*$-isomorphism

$$\pi\colon \mathrm{vN}(Y \rtimes \Lambda) \to \mathrm{vN}(X \rtimes \Gamma)$$

if and only if θ preserves the orbit equivalence relation:

$$\theta(\Gamma x) = \Lambda\theta(x) \quad \text{for } \mu\text{-a.e. } x.$$

From OE to Cocycle (after Zimmer)

Suppose $(\Gamma \curvearrowright X) \cong_{\mathrm{OE}} (\Lambda \curvearrowright Y)$, i.e. $\exists\, \theta \colon X \stackrel{\sim}{\to} Y$ such that

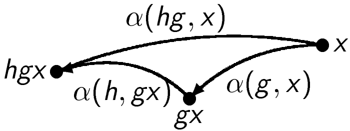
$$\theta(\Gamma x) = \Lambda\theta(x) \quad \text{for } \mu\text{-a.e. } x.$$

Define $\alpha \colon \Gamma \times X \to \Lambda$ by

$$\theta(gx) = \alpha(g, x)\theta(x).$$

Then, α satisfies the cocycle identity:

$$\alpha(h, gx)\alpha(g, x) = \alpha(hg, x).$$



A cocycle α is a *homomorphism* if ess. independent of the second variable.
Cocycles α and β are *equivalent* if $\exists\, \phi \colon X \to \Lambda$ such that

$$\beta(g, x) = \phi(gx)\alpha(g, x)\phi(x)^{-1}.$$

Theorem (Zimmer)

$(\Gamma \curvearrowright X) \cong (\Lambda \curvearrowright Y)$ *if and only if* α *is equivalent to a homomorphism.*

From Cocycle to Group Action

Theorem (Cocycle Superrigidity)

With some assumption on $\Gamma \curvearrowright X$ (and not on Λ), any cocycle

$$\alpha \colon \Gamma \times X \to \Lambda$$

is equivalent to a homomorphism β.

Applied to the Zimmer cocycle, one obtains (virtual) isomorphism $(\Gamma \curvearrowright X) \cong (\Lambda \curvearrowright Y)$ via the homomorphism $\beta \colon \Gamma \to \Lambda$.

Examples

- Γ higher rank lattice + Λ simple Lie group (Zimmer)
- Γ Kazhdan (T) / product + $\Gamma \curvearrowright X$ Bernoulli (Popa)
- Γ Kazhdan (T) + $\Gamma \curvearrowright X$ profinite (Ioana)

New von Neumann Rigidity

By adapting Ioana's arguments, we obtain a cocycle superrigidity result for some profinite actions of property (τ) groups with residually-finite targets. There are groups with the property (HH) and the property (τ).

Corollary

Let $\Gamma_i = \mathrm{PSL}(2, \mathbb{Z}[\sqrt{2}])$ *and* $p_1 < p_2 < \cdots$ *be prime numbers.*
Let $\Gamma = \Gamma_1 \times \Gamma_2$ *act on* $X = \varprojlim \mathrm{PSL}(2, (\mathbb{Z}/p_1 \cdots p_n\mathbb{Z})[\sqrt{2}])$ *by the left-and-right translation.*
Let $\Lambda \curvearrowright Y$ *be any (free ergodic prob.m.p.) action of a residually-finite group* Λ *such that* $\mathrm{vN}(X \rtimes \Gamma) \cong \mathrm{vN}(Y \rtimes \Lambda)$*. Then,* $\Gamma \curvearrowright X$ *and* $\Lambda \curvearrowright Y$ *are virtually isomorphic.*

GA	→	OE	→	vN
$\Gamma \curvearrowright (X, \mu)$		$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$		$\mathrm{vN}(X \rtimes \Gamma)$