Infinite Groups with Fixed Point Properties

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Let $\mathcal{X}$ be a class of top. spaces and let G be a gp.

Def. We will say that G has property FP$(\mathcal{X})$ if $\forall\, X \in \mathcal{X}$ any action of G on X has a global fixed point.

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Ex. 1. Denote by $\mathcal{A}$ the class of all simplicial trees.
Then $\mathrm{FP}^{sim}(\mathcal{A}) \Leftrightarrow$ (Serre's property FA).

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Thm. (J.-P. Serre). *A countable gp. G has $\mathrm{FP}^{sim}(\mathcal{A})$ iff all of the following hold:*

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(ii) *$\mathbb{Z}$ is not a homomorphic image of G;*

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(i) G is f.g.;

(ii) $\mathbb{Z}$ is not a homomorphic image of G;

(iii) G is doesn't split in a non-trivial amalgamated product.

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Ex. 2. Kazhdan's property (T) $\Leftrightarrow \mathrm{FP}^{iso}(\mathcal{H})$ where $\mathcal{H}$ is the class of Hilbert spaces (Delorme-Guichardet).

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Ex. 3. Let $\mathcal{C}_0$ be the class of all complete f.d. CAT(0)-spaces. Lemma of center implies that any finite gp. G has $\mathrm{FP}^{iso}(\mathcal{C}_0)$.

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Ex. 4. Denote by $\mathcal{X}_c$ the class of all contractible top. spaces of finite covering dim.
Smith Theory: $\forall$ prime p, any finite p-group G has $\mathrm{FP}^{hom}(\mathcal{X}_c)$.

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Question. Does there exist a countable infinite group with $\mathrm{FP}^{iso}(\mathcal{C}_0)$ or $\mathrm{FP}^{hom}(\mathcal{X}_c)$?

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Helly's Theorem: *Let $K_1, \ldots, K_m$ be a collection of convex subsets of the space $X = \mathbb{R}^n$, $m > n$. If*

$$\bigcap_{j=1}^{n+1} K_{i_j} \neq \emptyset \text{ for any } \{i_1, \ldots, i_{n+1}\} \subset \{1, \ldots, m\}$$

then $\bigcap_{i=1}^{m} K_i \neq \emptyset$.

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Analogues of this thm. were proved by Serre (when X is a tree, $n = 1$) and by Bridson (when X is a complete CAT(0)-space of covering dimension n).

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Consequence: the affine reflection group $\tilde{A}_{n+1}$ has $\mathrm{FP}^{iso}(\mathcal{C}_0^{(n)})$, where $\mathcal{C}_0^{(n)}$ is the class of all complete CAT(0)-spaces of dimension n.

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Observation: suppose that $\mathcal{X} = \bigcup_{n\in\mathbb{N}} \mathcal{X}^{(n)}$ and the group G_n has $\mathrm{FP}(\mathcal{X}^{(n)})$, $n \in \mathbb{N}$. If Q is a common quotient of $\{G_n \mid n \in \mathbb{N}\}$ then Q has $\mathrm{FP}(\mathcal{X})$.

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Problem: the only non-trivial common quotient of the family $\{\tilde{A}_{n+1} \mid n \in \mathbb{N}\}$ is $\mathbb{Z}/2\mathbb{Z}$.

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Thm. A. *Any countable family of non-elem. (rel.) hyp. gps. possesses an infinite common quotient.*

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Thm. A. *Any countable family of non-elem. (rel.) hyp. gps. possesses an infinite common quotient.*

(The proof is based on Small Cancellation Theory over (rel.) hyp. gps.)

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Thm. B. $\forall$ *prime* p *and* $\forall\, n \in \mathbb{N}$ *there is a non-elem. hyp. gp.* $G_{n,p}$ *s.t.* $G_{n,p} = \langle S \rangle$, $|S| = n+2$ *and* $\forall\, P \subsetneqq S$, $\langle P \rangle$ *is a finite* p*-sbgp. of* $G_{n,p}$.

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(The gps. $G_{n,p}$ are constructed as non-positively curved simplices of finite gps.)

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$\mathcal{X}_c = \bigcup_n \mathcal{X}_c^{(n)}$, where $\mathcal{X}_c^{(n)} = \{X \in \mathcal{X}_c \mid dim(X) = n\}$.

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Thm. C. *The group* $G_{n,p}$ *, as above, has* $\mathrm{FP}^{hom}(\mathcal{X}_c^{(n)})$.

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<u>Thm. C.</u> *The group* $G_{n,p}$ *, as above, has* $\mathrm{FP}^{hom}(\mathcal{X}_c^{(n)})$.

(The proof uses Smith Theory together with a cohomological version of Helly's Thm.)

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Thm. 1. (A.-B.-J.-L-M.-Ś.) *There exists an infinite f.g. gp.* Q *with* $\mathrm{FP}^{hom}(\mathcal{X}_c)$. *Moreover, for any countable gp.* C, Q *can be chosen to satisfy the following:*

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(c) $C \hookrightarrow Q$.

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Thm. 2. (A.-B.-J.-L-M.-Ś.) *There exists an infinite f.g. gp. Q with $\mathrm{FP}^{hom}(\mathcal{X}_c)$. such that*

(a) *Q is simple;*

(b) *Q has Kazhdan's property (T);*

(c)′ *Q is periodic.*

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(a) *Q is simple;*

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(c) *$C \hookrightarrow Q$.*

Remark 1. Any infinite hyp. gp. G acts without global fixed points on some finite-dimensional contractible simplicial complex (Rips Complex).

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Remark 2. Any gp. G acts on the space $M(G)$ of finitely-additive probability measures on G. The space $M(G)$ is contractible, compact and Hausdorff. This G-action has a fixed point iff G is amenable.

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Remark 3. Any infinite amenable gp. acts isometrically without global fixed points on the Hilbert space.

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(a) *Q is simple;*

(b) *Q has Kazhdan's property (T);*

(c) *$C \hookrightarrow Q$.*

Remark 4. Let Q be a simple gp. with $\mathrm{FP}^{hom}(\mathcal{X}_c)$. Then for any proper metric space $X \in \mathcal{X}_c$ any isometric action of Q on X is trivial.

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Question 1. Does there exist a non-trivial torsion-free f.g. gp. having $\mathrm{FP}^{hom}(\mathcal{X}_c)$?

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Question 1. Does there exist a non-trivial torsion-free f.g. gp. having $\mathrm{FP}^{hom}(\mathcal{X}_c)$, or, at least, $\mathrm{FP}^{iso}(\mathcal{C}_0)$?

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Question 1. Does there exist a non-trivial torsion-free f.g. gp. having $\mathrm{FP}^{hom}(\mathcal{X}_c)$, or, at least, $\mathrm{FP}^{iso}(\mathcal{C}_0)$?

Question 2. Does there exist an infinite finitely presented gp. with $\mathrm{FP}^{iso}(\mathcal{C}_0)$ ($\mathrm{FP}^{hom}(\mathcal{X}_c)$)?