LIE GROUPS IN THE SYMMETRIC GROUP: REDUCING ULAM'S PROBLEM TO THE SIMPLE CASE

ANNALISA CONVERSANO AND NICOLAS MONOD

Abstract. Ulam asked whether all Lie groups can be represented faithfully on a countable set. We establish a reduction of Ulam's problem to the case of simple Lie groups.

In particular, we solve the problem for all solvable Lie groups and more generally Lie groups with a linear Levi component. It follows that every amenable locally compact second countable group acts faithfully on a countable set.

1. Introduction

A group is called **countably representable** if it can be realised as a permutation group of a countable set. For obvious cardinality reasons, this only makes sense for groups that are not larger than the continuum $\mathfrak{c} = |\mathbf{R}| = 2^{\aleph_0}$, and it is trivial for countable groups.

Therefore the rich world of *Polish groups* is a prime location to study countable representability, even though (or perhaps because) this notion is non-topological. Some examples and counter-examples are discussed in [Mon22].

Schreier and Ulam observed in 1935 that the group $\mathbf{R}$ is countably representable [Ula58]. This prompted the following problem [Ula60], still open to this day (Problem 15.8.b in [KM22]).

Problem (Ulam)**.** *Is every Lie group countably representable?*

In 1999, Thomas [Tho99, §2], then Kallman [Kal00] and later Ershov–Churkin [EC04] proved that this is the case for every *linear* Lie group. There are of course many non-linear Lie groups, and although the non-linearity is in a way caused only by the center, it is well-known that the center is a fundamental obstruction to such questions, see e.g. [McK71], [Chu05] and the discussion in [Mon22].

Nonetheless, it was recently established that every *nilpotent* Lie group is countably representable [Mon22]. Our first result applies to many non-linear groups beyond the nilpotent case:

Theorem 1.1. *Every solvable Lie group is countably representable.*

Date: November 8, 2023.

Turning now to the most general case of Ulam's problem, recall that any Lie group G admits a maximal connected solvable normal subgroup $R = \mathrm{Rad}(G)$, called the **radical** of G, which is a closed Lie subgroup. Moreover, Levi's decomposition theorem provides an *immersed* connected semi-simple Lie group S in G such that the neutral component G^0 of G can be written $G^0 = RS$ with $R \cap S$ central in S, hence countable. By Malcev's theorem [Mal42] (see [HN12, 5.6]), S is unique up to conjugacy and we hence abusively refer to it as "the" **semi-simple Levi component** of G. Unlike the radical, S is not always closed, as witnessed by the following classical example.

Example 1.2. Let $a \in \mathbf{SO}_2(\mathbf{R})$ be a rotation through an angle incommensurable with π, and let $b \in \widetilde{\mathbf{SL}}_2(\mathbf{R})$ be a generator of the center of the universal cover of $\mathbf{SL}_2(\mathbf{R})$. Set $H = \mathbf{SO}_2(\mathbf{R}) \times \widetilde{\mathbf{SL}}_2(\mathbf{R})$ and Γ to be the central subgroup generated by (a, b) in H. Define $G = H/\Gamma$.

The radical R of G and the semisimple Levi component S are respectively the images of $\mathbf{SO}_2(\mathbf{R}) \times \{e\}$ and $\{e\} \times \widetilde{\mathbf{SL}}_2(\mathbf{R})$ under the natural homomorphism $H \to H/\Gamma$. Since $\langle a \rangle$ is dense in $\mathbf{SO}_2(\mathbf{R})$, it follows that S is not closed in G.

The main result of this article is the following strengthening of Theorem 1.1:

Theorem 1.3. *A Lie group is countably representable if and only if its semi-simple Levi factor is countably representable.*

A preliminary step in the proof is to consider the easier situation where the semi-simple Levi factor is linear, in which case it is possible to combine the proof of Theorem 1.1 and Thomas's theorem (see Theorem 4.1 below). This includes notably the case of compact Levi components and therefore we can apply the solution to Hilbert's fifth problem and deduce a statement far beyond Lie groups:

Corollary 1.4. *Every locally compact second countable group which is amenable is countably representable.*

(The corollary holds for all groups that are amenable *as locally compact groups*, not only for those amenable as abstract groups.)

In conclusion, let us consider the status of Ulam's general problem in the light of Theorem 1.3. The semi-simple Levi factor can be further decomposed as a commuting product of simple Lie groups. Ulam's problem remains open because we do not know whether non-linear simple Lie groups are countably representable. Indeed we show that this is the *only* obstruction:

Theorem 1.5. *The following statements are equivalent:*

(i) every Lie group is countably representable;
(ii) every connected simple Lie group with finite center is countably representable.

2. Preliminaries

A subgroup $H < G$ of a group G is said to have **countable index** (in G) if the coset space G/H is countable. The term *cocountable* is also used in the literature. We shall constantly use the basic observation that G is countably representable if and only if it admits a sequence $(H_n)_{n\in\mathbf{N}}$ of countable index subgroups $H_n < G$ whose intersection $\bigcap_n H_n$ is trivial.

We recall from Lemma 11 in [Mon22] that if G admits a countable index subgroup that is countably representable, then G itself is countably representable.

We shall routinely reduce ourselves from Lie groups to the case of *connected* Lie groups. Indeed, since Lie groups are second countable and locally connected, they have at most countably many connected components. Therefore the connected component of the identity G^0 has countable index in G, and we can apply the above principle.

For general locally compact groups (Corollary 1.4), our statement assumes second countability to ensure that the cardinality does not exceed $\mathfrak{c}$.

One could counter that this sufficient condition seems not to be necessary: there are locally compact groups of cardinality $\leq \mathfrak{c}$ which are not second countable, e.g. discrete ones. However, that generality would allow for amenable groups that are not countably representable, such as McKenzie's example [McK71].

3. The solvable case

The goal of this section is to prove that every solvable Lie group is countably representable, as announced in Theorem 1.1.

The main tool for this case is as follows. Recall that polycyclic groups include notably all finitely generated nilpotent groups, see for instance [Seg83, Chap. 1].

Proposition 3.1. *Let m be a positive integer and $\Gamma < \mathbf{GL}_m(\mathbf{R})$ a polycyclic subgroup. There exists a countable index subgroup $H < \mathbf{GL}_m(\mathbf{R})$ which intersects Γ trivially.*

This statement will easily be reduced to the case where Γ is cyclic, which is the object of the following lemma.

Lemma 3.2. *Let m be a positive integer and $\gamma \in \mathbf{SL}_m(\mathbf{R})$ any element. There exists a countable index subgroup $H < \mathbf{SL}_m(\mathbf{R})$ which intersects trivially the cyclic subgroup generated by γ.*

The proof of this lemma will use a field isomorphism between $\mathbf{C}$ and the field $\bigcup_{q\geq 1} \overline{\mathbf{Q}}((t^{1/q}))$ of Puiseux series over $\overline{\mathbf{Q}}$, as did Thomas's proof. This provides a valuation $\mathbf{C} \to \mathbf{Q} \cup \{\infty\}$ with the property that the group $\mathbf{SL}_m(V_\mathbf{C})$ associated to the corresponding valuation ring $V_\mathbf{C}$ in $\mathbf{C}$ has countable index in $\mathbf{SL}_m(\mathbf{C})$, see Theorem 2.5 in [Tho99]. It will be important for us to choose the field isomorphism suitably.

Proof of Lemma 3.2. We can argue in the larger group $\mathbf{SL}_m(\mathbf{C})$ since all conclusions are preserved when taking the intersection of a countable index subgroup $H < \mathbf{SL}_m(\mathbf{C})$ with $\mathbf{SL}_m(\mathbf{R})$.

We choose the identification of $\mathbf{C}$ with Puiseux series in the indeterminate t in such a way that *if γ has any transcendental Eigenvalue, then t is one such Eigenvalue.* This is possible since the automorphism group of $\mathbf{C}$ acts transitively on transcendentals.

Let $L_{\mathbf{C}} < V_{\mathbf{C}}$ be the maximal ideal and denote the corresponding congruence subgroup by $\mathbf{SL}_m(V_{\mathbf{C}}; L_{\mathbf{C}})$. In other words, $\mathbf{SL}_m(V_{\mathbf{C}}; L_{\mathbf{C}})$ is the kernel of the reduction morphism $\mathbf{SL}_m(V_{\mathbf{C}}) \to \mathbf{SL}_m(\overline{\mathbf{Q}})$ since $\overline{\mathbf{Q}}$ is the residue field. In particular, $\mathbf{SL}_m(V_{\mathbf{C}}; L_{\mathbf{C}})$ also has countable index in $\mathbf{SL}_m(\mathbf{C})$ since $\overline{\mathbf{Q}}$ is countable.

We now proceed to choose H as a suitable conjugate of $\mathbf{SL}_m(V_{\mathbf{C}}; L_{\mathbf{C}})$, as follows. Since $\mathbf{C}$ is algebraically closed, γ admits a Jordan normal form. Therefore, after a conjugation we may assume that γ is in Jordan normal form and we take $H = \mathbf{SL}_m(V_{\mathbf{C}}; L_{\mathbf{C}})$ in that conjugation.

We now claim that if $\gamma^p \in H$ for any $p \in \mathbf{Z}$, then γ^p is the identity.

Suppose first that some Eigenvalue of γ is transcendental. Then the same holds for γ^p whenever $p \neq 0$. Since γ is upper triangular, the eigenvalues of γ^p appear as diagonal matrix coefficients. This excludes $\gamma^p \in H$ because the diagonal coefficients of elements in H have valuation zero by the definition of congruence subgroups, whereas some Eigenvalue of γ^p is t^p, which has valuation p.

We are now in the case where all Eigenvalues of γ are algebraic. Assume $\gamma^p \in H$. It follows that all diagonal coefficients of γ^p are 1. Supposing for a contradiction that γ^p is not the identity, it follows that γ itself was not diagonal. Consider now any non-trivial Jordan block of γ with corresponding Eigenvalue $\lambda \in \overline{\mathbf{Q}}$. Recall that the first subdiagonal of the Jordan block consists of 1s, and that consequently the corresponding coefficients on the first subdiagonal of γ^p are $p\lambda^{p-1}$. This is a non-zero algebraic number, which contradicts $\gamma^p \in H$ because off-diagonal coefficients of elements of H must be in the valuation ideal.

This contradiction proves the claim and hence the lemma. $\square$

Proof of Proposition 3.1. Note that in our statement $\mathbf{GL}_m(\mathbf{R})$ can be replaced by $\mathbf{SL}_m(\mathbf{R})$ since $\mathbf{GL}_m(\mathbf{R})$ embeds into $\mathbf{SL}_{m+1}(\mathbf{R})$.

By definition, Γ admits a subnormal series

$$1 = \Gamma_0 \lhd \Gamma_1 \lhd \cdots \lhd \Gamma_d = \Gamma$$

with cyclic quotient Γ_j/Γ_{j-1} for each $j = 1, \ldots, d$. We proceed by induction on the minimal length d of such a subnormal series.

The base case $d = 0$ holds trivially. We now suppose that the statement has been established for all minimal lengths $\leq d$ and consider length $d +$

1. Let γ be a generator of the cyclic group Γ_1. By Lemma 3.2, there is a countable index subgroup $H_1 < \mathbf{SL}_m(\mathbf{R})$ intersecting Γ_1 trivially. We now consider the group $\Delta = \Gamma \cap H_1$. This is a polycyclic subgroup of $\mathbf{SL}_m(\mathbf{R})$ and in fact a subnormal series witnessing polycyclicity is given by $\Delta_j = \Gamma_j \cap H_1$ since Δ_j/Δ_{j-1} embeds into Γ_j/Γ_{j-1}.

However, the length $d+1$ of this series is not minimal; indeed $\Delta_1 = \Delta_0$ by the choice of H_1. Therefore, we can apply the inductive hypothesis to Δ and obtain a countable index subgroup $H_0 < \mathbf{SL}_m(\mathbf{R})$ intersecting Δ trivially. Now the group $H = H_0 \cap H_1$ satisfies the desired conclusion. □

Combining Proposition 3.1 with Thomas's theorem for linear groups, we obtain:

Theorem 3.3. *Let m be a positive integer, $G < \mathbf{GL}_m(\mathbf{R})$ any subgroup and $\Gamma \lhd G$ a polycyclic normal subgroup of G.*

Then the quotient group G/Γ is countably representable.

Proof. Proposition 3.1 provides a countable index subgroup $H < \mathbf{GL}_m(\mathbf{R})$ which intersects Γ trivially. Since $\mathbf{GL}_m(\mathbf{R})$ is countably representable ([Tho99, §2]), so is H. Now $H \cap G$ has countable index in G and is countably representable. Since H intersects Γ trivially, the image J of $H \cap G$ in G/Γ is isomorphic to $H \cap G$ and hence J is countably representable. Since J has countable index in G/Γ, we can apply Lemma 11 in [Mon22] and conclude that G/Γ itself is countably representable. □

We can now handle all solvable Lie groups:

Proof of Theorem 1.1. Let G be a solvable Lie group; as explained in Section 2, we shall assume without loss of generality that G is connected. Let $\pi\colon \widetilde{G} \twoheadrightarrow G$ be the universal covering map. Recall that the kernel $\ker \pi = \pi_1(G)$ is a finitely generated abelian group, see Corollary 14.2.10(iv) in [HN12]. By a Theorem of Malcev, $\widetilde{G}$ is a linear group, see [OV94, 2§7]. Therefore, G is a quotient as those considered in Theorem 3.3 and it follows from that theorem that G is countably representable. □

For future reference, we also record a straightforward bootstrap of Proposition 3.1:

Proposition 3.4. *Let m be a positive integer, $G < \mathbf{GL}_m(\mathbf{R})$ any subgroup, $\Gamma \lhd G$ a polycyclic normal subgroup of G and $\Delta < G/\Gamma$ any polycyclic subgroup of the quotient.*

Then there is a countable index subgroup $H < G/\Gamma$ which meets Δ trivially.

Proof. The pre-image $\widetilde{\Delta}$ of Δ in G is also polycyclic. Therefore Proposition 3.1 ensures that G contains a countable index subgroup $K < G$ with $K \cap \widetilde{\Delta}$ trivial. It now suffices to take for H the image of K in G/Γ. □

4. Linear Levi component and amenable groups

We recall the Levi decomposition, see [OV94, 1§4] for references:

Let G be any connected Lie group, let $R = \mathrm{Rad}(G)$ be its radical, i.e. the maximal connected solvable normal subgroup of G, which is automatically a closed Lie subgroup.

By Levi's decomposition theorem, there exists an immersed (not necessarily closed) connected semi-simple Lie group S in G such that $G = RS$ with $R \cap S$ countable.

In the special case where G is *simply connected*, the following hold:

(i) both R and S are simply connected [Mos50, §5];
(ii) any connected semi-simple Lie subgroup of G is closed [Mos50, §6];
(iii) $R \cap S$ is trivial and therefore we have a semi-direct product decomposition [OV94, 1§4].

Theorem 4.1. *Let G be a connected Lie group. If the semi-simple Levi component of G is linear, then G is countably representable.*

Proof. Let $\pi\colon \widetilde{G} \twoheadrightarrow G$ be the universal covering map of G and recall that its kernel $\Gamma = \pi_1(G)$ is a finitely generated abelian group (Corollary 14.2.10(iv) in [HN12]). In particular Γ is central since $\widetilde{G}$ is connected. Denote by $\widetilde{G} = \widetilde{R}\widetilde{S}$ a Levi decomposition of $\widetilde{G}$. Since Levi decompositions of connected Lie groups are determined by Levi decompositions of the corresponding Lie algebra, we can assume that π restricts to covering maps $\widetilde{R} \twoheadrightarrow R$ and $\widetilde{S} \twoheadrightarrow S$ for a Levi decomposition $G = RS$ of G. Moreover, since both $\widetilde{R}$ and $\widetilde{S}$ are simply connected as recalled above, they are in fact the universal covers of R and S respectively, justifying the notation.

Define $N = \Gamma \cap \widetilde{S}$. Note that N is normal in $\widetilde{G}$ because Γ is central. Hence we can consider the connected Lie group $L = \widetilde{G}/N$, which is a cover of G. For the reasons indicated in the first paragraph of the proof, a Levi decomposition $L = R'S'$ of L is given by the images of $\widetilde{R}$ and $\widetilde{S}$ in L.

We claim that R' is linear. Since $\widetilde{R} \cap \widetilde{S}$ is trivial, the map $\widetilde{R} \twoheadrightarrow R'$ is an isomorphism and hence R' is simply connected. Thus the claim follows from Malcev's linearity criterion for solvable Lie groups, see [OV94, 2§7].

Next we claim that S' is linear. The kernel of the covering map $\widetilde{S} \twoheadrightarrow S$ is the intersection between $\widetilde{S}$ and the kernel Γ of $\widetilde{G} \twoheadrightarrow G$. Therefore the quotient group $\widetilde{S}/N$ is isomorphic to both S' and S. Since S is linear by assumption, S' is linear as claimed.

It now follows that $L = R'S'$ is linear by a theorem of Malcev, see [OV94, 1§5.4]. Finally, since G is the quotient of L by the finitely generated abelian group Γ/N, we can apply Theorem 3.3 and deduce that G is countably representable. □

Corollary 4.2. *Every amenable Lie group is countably representable.*

Proof. Let G be an amenable Lie group; we can assume G connected. Let $G = RS$ be a Levi decomposition. In view of Theorem 4.1, it suffices to prove that S is linear.

Note that the quotient $S/Z(S)$ by the center of S is a quotient of G/R, which is amenable as a quotient of G. Thus $S/Z(S)$ is amenable center-free connected semi-simple Lie group, which implies that it is compact, see Theorem 1.6 in [Fur63]. Since S is a cover of the compact semi-simple group $S/Z(S)$, Weyl's theorem [Kna02, Theorem 4.69] implies that S itself is compact. It is therefore linear by an application of Peter–Weyl [Kna02, Corollary 4.22]. □

Proof of Corollary 1.4. It was exposed in Proposition 2 of [Mon22] how a positive solution to Ulam's problem would imply that *every* locally compact second countable group G would be countably representable. The reduction to Lie groups through the solution of Hilbert's fifth problem used in that proof (specifically: the approximation theorem as stated in §4.6, p. 175 in [MZ55]) produces a family of Lie groups that are all quotients of an open subgroup G_1 of G. In particular, when G is amenable, all those Lie groups are amenable and hence we can apply Corollary 4.2 instead of relying on a hypothetical solution to Ulam's problem. □

5. A reduction to simple groups

We finally turn to the most substantial result of this article, namely that an arbitrary Lie group is countably representable if and only if its semi-simple Levi component is countably representable (Theorem 1.3).

To prepare for the proof, recall that given any connected Lie group L, the **lineariser** of L, denoted by $\Lambda(L)$, is the intersection of the kernels of all finite-dimensional linear representations of L [Hoc60]. The fundamental property of the lineariser (apparently due to Goto, see [HM57, §7]) is that $L/\Lambda(L)$ is a linear Lie group, see Theorem 16.2.7 in [HN12] for a proof.

Lemma 5.1. *Let G be a connected Lie group and L any immersed (not necessary closed) connected Lie subgroup. Then the lineariser $\Lambda(L)$ of L is central in G.*

Proof of Lemma 5.1. Since G is connected, the adjoint representation

$$G \longrightarrow G/Z(G) \longrightarrow \mathrm{Aut}(G)$$

descends to a *faithful* representation on the quotient of G by its center $Z(G)$, see Lemma 9.2.21 in [HN12]. However, the Lie group automorphism group $\mathrm{Aut}(G)$ is linear, see Theorem 1 in [Hoc52]. Therefore, the adjoint representation is trivial on the subgroup $\Lambda(L)$, which means that $\Lambda(L)$ is contained in $Z(G)$ as claimed. □

We are now ready for the proof of Theorem 1.3, a significant part of which is devoted to circumventing the intersection $R \cap S$ in the Levi decomposition $G = RS$.

Proof of Theorem 1.3. We can assume that our arbitrary Lie group G is connected. Let $G = RS$ be a Levi decomposition of G. Assuming S is countably representable, we will deduce that G is countably representable. The other implication is obvious.

Set $\Lambda = \Lambda(S)$, the lineariser of S. By Lemma 5.1, Λ is central in G and we can thus consider the quotient group G/Λ. Its semi-simple Levi component is the linear group S/Λ. Therefore G/Λ is countably representable on account of Theorem 4.1. Hence there is a sequence of countable index subgroups $G_n < G$ whose intersection is Λ. Indeed, such a sequence can be found by taking pre-images in G of a sequence of countable index subgroups of G/Λ whose intersection is trivial.

In order to prove the theorem, it now suffices to find another sequence of countable index subgroups $H_n < G$ such that the intersection of all H_n meets Λ trivially.

Since Λ is central in the semi-simple group S, it is a finitely generated abelian group. In particular it is countable, and therefore it suffices to find for any given $\lambda_0 \neq e$ in Λ some countable index subgroup $H < G$ avoiding λ_0. The assumption made on S implies that there is a countable index subgroup $S_0 < S$ avoiding λ_0. The remainder of the proof will deal with the fact that the naive candidate $RS_0 < G$ might nonetheless contain λ_0, and therefore we need to choose a smaller subgroup $H < RS_0$.

Denote by $\Gamma < G$ the intersection $R \cap S$. Since Γ is a normal solvable subgroup of the connected semi-simple group S, it is central in S. In particular, Γ is a finitely generated abelian group.

We use the conjugation S-action on R to form the semi-direct product $\widehat{G} := R \rtimes S$, which is a connected Lie group with a natural quotient map $\widehat{G} \twoheadrightarrow G$ given by the multiplication in G. The kernel of this map is the group $\{(\gamma^{-1}, \gamma) : \gamma \in \Gamma\}$ isomorphic to Γ. Thus the pre-image $\widehat{\Lambda}$ of Λ in $\widehat{G}$ is

$$\widehat{\Lambda} = \{(\gamma^{-1}, \gamma\lambda) : \gamma \in \Gamma, \lambda \in \Lambda\}.$$

By construction this is a finitely generated metabelian group; in fact, abelian because Λ acts trivially on R.

Viewing Λ in $\widehat{G}$ (as $\{e\} \times \Lambda$), we can consider the quotient group $\widehat{G}/\Lambda = R \rtimes (S/\Lambda)$, using again Lemma 5.1. Since its Levi component S/Λ is a linear group, we are again in the situation of Theorem 4.1. As explained in the proof of that theorem, the group $\widehat{G}/\Lambda = R \rtimes (S/\Lambda)$ admits a *linear* (connected) cover of the form $L = \widetilde{R} \rtimes (S/\Lambda)$. The kernel of the quotient map $L \twoheadrightarrow \widehat{G}/\Lambda$ is finitely generated abelian (as a subgroup of the fundamental group of $\widehat{G}/\Lambda$, Corollary 14.2.10(iv) in [HN12])) and therefore we can invoke Proposition 3.4 to find a subgroup of countable index J in $\widehat{G}/\Lambda$ which meets trivially the image in $\widehat{G}/\Lambda$ of the finitely generated abelian group $\widehat{\Lambda}$.

The pre-image $\widehat{J}$ in $\widehat{G}$ of J is a countable index subgroup of $\widehat{G}$ such that $\widehat{J} \cap \widehat{\Lambda}$ is $\{e\} \times \Lambda$.

We claim that the image $H < G$ of the group $\widehat{J} \cap (R \rtimes S_0) < \widehat{G}$ under the map $\widehat{G} \twoheadrightarrow G$ has the desired property of avoiding λ_0.

If not, then there is $(r,s) \in \widehat{J} \cap (R \rtimes S_0)$ with $rs = \lambda_0$. In particular, $r = \lambda_0 s^{-1}$ implies $r \in \Gamma$ and then $s = r^{-1}\lambda_0$ implies $(r,s) \in \widehat{\Lambda}$. The choice of $\widehat{J}$ now implies $(r,s) \in \{e\} \times \Lambda$. It follows $(r,s) = (e, \lambda_0)$, which contradicts $(r,s) \in R \rtimes S_0$ by the choice of S_0. This confirms the claim and hence completes the proof. □

It remains to establish Theorem 1.5. This requires two additional observations recorded in the lemmata below.

Lemma 5.2. *Let G be a connected Lie group. If every finite-sheeted cover of G is countably representable, then every cover of G is so too.*

Proof. Let $\widetilde{G}$ be the universal cover of G and $\Gamma < Z(\widetilde{G})$ be the fundamental group of G viewed as a finitely generated abelian subgroup of $\widetilde{G}$ (Corollary 14.2.10(iv) in [HN12]).

Let $\widehat{G}$ an arbitrary cover of G, so that $\widehat{G} = \widetilde{G}/\Lambda$ for some subgroup $\Lambda < \Gamma$. Since Γ/Λ is finitely generated abelian, it admits a nested sequence of finite index subgroups with trivial intersection. Taking pre-images in Γ, we obtain a nested sequence of finite index subgroups $\Gamma_n < \Gamma$ with $\bigcap_n \Gamma_n = \Lambda$.

Recalling that Γ is central since $\widetilde{G}$ is connected, we can define the groups $G_n := \widetilde{G}/\Gamma_n$, which are finite-sheeted covers of G and quotients of $\widehat{G}$. They form an inverse system with a natural morphism from $\widehat{G}$ to the inverse limit $\varprojlim_n G_n$. This morphism is injective because $\bigcap_n \Gamma_n = \Lambda$.

Our assumption implies that each G_n is countably representable. Since we realised $\widehat{G}$ as a subgroup of an inverse limit of a *countable* system of countably representable groups, it follows from Lemma 9 in [Mon22] that $\widehat{G}$ is itself countably representable. □

Next, recall that every connected semi-simple group S is a (not necessarily direct) product $S = S_1 \cdots S_k$ of finitely many connected normal subgroups $S_1, \ldots, S_k$ having simple Lie algebra, and such that for all $i \neq j$ the subgroups S_i, S_j commute (elementwise). These subgroups S_i are referred to as **the simple factors** of S. This decomposition follows from the decomposition of any semi-simple Lie algebra into a direct sum of simple ideals, see for instance Proposition 5.5.11 in [HN12].

Lemma 5.3. *Let S be a connected semi-simple Lie group and assume that the center $Z(S)$ is finite. If all the simple factors of S are countably representable, then so is S.*

Proof. Let $S_1, \ldots, S_k$ be the simple factors of S. By assumption each S_i admits a sequence of countable index subgroups $S_{i,n} < S_i$ with trivial intersection.

The center $Z(S)$ is equal to the product of the centers $Z(S_i)$ since the factors commute elementwise. In particular, all simple factors have finite center. Therefore, there is no loss of generality in assuming that $S_{i,n} \cap Z(S_i)$ is trivial for all i, n.

The set of products $H_n = S_{1,n} \cdots S_{k,n}$ is a subgroup of S since the factors S_i commute elementwise, and this subgroup has countable index in S for that same reason.

We claim that H_n is the direct product of the groups $S_{i,n}$. Indeed, any element $z \in S_{i,n} \cap S_{j,n}$ with $i \neq j$ is centralised by $S_{j,n}$. Since the latter is dense in S_j, it follows that z is centralised by S_j. But z being in S_j, the choice of $S_{j,n}$ shows that z is trivial as claimed.

It now follows that the intersection of all H_n is trivial, completing the proof. □

End of the proof of Theorem 1.5. Under the hypothesis that every connected simple Lie group with finite center is countably representable, we need to show that an arbitrary Lie group G is countably representable. We can suppose G connected and we consider a Levi decomposition $G = RS$. Thanks to Theorem 1.3, it suffices to show that S is countably representable.

Since S is a cover of the group $S/Z(S)$, Lemma 5.2 shows that it suffices to show that every finite-sheeted cover of $S/Z(S)$ is countably representable. Such a finite-sheeted cover is a connected semi-simple Lie group with finite center, and therefore Lemma 5.3 reduces the question to the simple factors of $S/Z(S)$. Each of these factors still has finite center and therefore is accommodated by our hypothesis. □

Despite our reduction of Ulam's general problem to simple factors, there is a slight disharmony between the statements of Theorem 1.3 and Theorem 1.5. Namely, it is unclear to the authors whether for each *given* Lie group it suffices to consider the simple factors of its semi-simple Levi component.

References

[Chu05] Valerii Avdeevich Churkin, *Examples of groups with continuum cardinality that are not embeddable into the permutation group of a countable set (in Russian)*, Algebra and model theory 5, Novosibirsk State Tech. Univ., Novosibirsk, 2005, pp. 39–43.

[EC04] Yuri Leonidovich Ershov and Valerii Avdeevich Churkin, *One problem of Ulam*, Dokl. Math. **70** (2004), no. 3, 896–898.

[Fur63] Harry Furstenberg, *A Poisson formula for semi-simple Lie groups*, Ann. of Math. (2) **77** (1963), 335–386.

[HM57] Gerhard Paul Hochschild and George Daniel Mostow, *Representations and representative functions of Lie groups*, Ann. of Math. (2) **66** (1957), 495–542.

[HN12] Joachim Hilgert and Karl-Hermann Neeb, *Structure and geometry of Lie groups*, Springer Monographs in Mathematics, Springer, New York, 2012.

[Hoc52] Gerhard Paul Hochschild, *The automorphism group of Lie group*, Trans. Amer. Math. Soc. **72** (1952), 209–216.

[Hoc60] ———, *The universal representation kernel of a Lie group*, Proc. Amer. Math. Soc. **11** (1960), 625–629.

[Kal00] Robert Richard Kallman, *Every reasonably sized matrix group is a subgroup of S_∞*, Fund. Math. **164** (2000), no. 1, 35–40.

[KM22] Evgeny I. Khukhro and Viktor Danilovich Mazurov (eds.), *The Kourovka notebook*, Sobolev Institute of Mathematics. Russian Academy of Sciences. Siberian Branch, Novosibirsk, 2022, Unsolved problems in group theory, Twentieth edition, June 2022 update.

[Kna02] Anthony William Knapp, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston Inc., Boston, MA, 2002.

[Mal42] Anatoly Ivanovich Malcev, *On the representation of an algebra as a direct sum of the radical and a semi-simple subalgebra*, Dokl. Akad. Nauk SSSR, **36** (1942), no. 2, 42–45.

[McK71] Ralph Nelson Whitfield McKenzie, *A note on subgroups of infinite symmetric groups*, Nederl. Akad. Wetensch. Proc. Ser. A **74** = Indag. Math. **33** (1971), 53–58.

[Mon22] Nicolas Monod, *Lie groups as permutation groups: Ulam's problem in the nilpotent case*, J. Group Theory **25** (2022), no. 5, 851–865.

[MZ55] Deane Montgomery and Leo Zippin, *Topological Transformaton Groups*, Interscience, 1955.

[Mos50] George Daniel Mostow, *The extensibility of local Lie groups of transformations and groups on surfaces*, Ann. of Math. (2) **52** (1950), 606–636.

[OV94] A. L. Onishchik and È. B. Vinberg (eds.), *Lie groups and Lie algebras, III*, Encyclopaedia of Mathematical Sciences, vol. 41, Springer-Verlag, Berlin, 1994, Structure of Lie groups and Lie algebras.

[Seg83] Dan Segal, *Polycyclic groups*, Cambridge Univ. Press, 1983.

[Tho99] Simon Thomas, *Infinite products of finite simple groups. II*, J. Group Theory **2** (1999), no. 4, 401–434.

[Ula58] Stanisław Marcin Ulam, *Untitled*, Typed manuscript sent to Edward Thomas Copson, based on the Scottish Book, 1958.

[Ula60] ———, *A collection of mathematical problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York-London, 1960.

Massey University Auckland, New Zealand

EPFL, Switzerland