ANALYSIS ON GROUPS: WEAK TOPOLOGIES

Recall that a topological vector space is, formally, a pair $(\mathcal{E}, \mathscr{S})$, but that we typically just refer to $\mathcal{E}$ when no confusion can arise from this. In this note, confusion can arise! (But we still refer to just $\mathcal{E}$ as the topological vector space.) For completeness, some of the following exercises are stated in slightly greater generality than what is strictly needed. If you want to simplify things a bit, try to consider in general the situation where every topological vector space is a Banach space, and in particular every dual space is the dual of a Banach space.

Definition 1 (weak topologies)**.** Let $\mathcal{E}$ be a vector space, let $\mathcal{F}$ be a topological vector space, and let V be a set of linear maps $\mathcal{E} \to \mathcal{F}$. The weak topology on $\mathcal{E}$ induced by V (the 'V-weak topology' for short) is defined as the weakest topology $\mathcal{S}$ on $\mathcal{E}$ such that φ is continuous for every $\varphi \in V$. That is, $\mathscr{S} = \cap_{\mathscr{T}} \mathscr{T}$, over all topologies such that $\mathscr{T}$ on $\mathcal{E}$ such that φ is $\mathscr{T}$-continuous for every $\varphi \in V$. The V-weak topology on $\mathcal{E}$ is denoted by $\sigma(\mathcal{E}, V)$.

Exercise 2. Observe that the V-weak topology is always well-defined.

(i) Show that for any set V as in the previous definition, $\sigma(\mathcal{E}, V) = \sigma(\mathcal{E}, \operatorname{span}(V))$. That is, we may as well always assume that V is a vector subspace (and we generally will implicitly assume this in the sequel).

(ii) Show that $\sigma(\mathcal{E}, V)$ is generated by the basis of neighbourhoods $V(\varepsilon, \xi, \varphi_1, \dots, \varphi_n), \varepsilon > 0, \xi \in \mathcal{E}, \varphi_i \in V$ given by
$$V(\xi, \varphi_1, \dots, \varphi_n) := \{\eta \in \mathcal{E} \mid \forall i : |\varphi_i(\xi - \eta)| < \varepsilon\}.$$

(iii) Equivalently, show that a *net* $(\xi_i)_{i \in I} \subseteq \mathcal{E}$ converges to $\xi \in \mathcal{E}$ in the V-weak topology, if and only if $\varphi(\xi_i) \to_i \varphi(\xi)$ in $\mathcal{F}$ for all $\varphi \in V$.

(iv) Conclude that $\mathcal{E}$ with the V-weak topology is indeed a (not necessarily Hausdorff) topological vector space. Observe that weak topologies are always locally convex. Precisely when are they Hausdorff?

(v) Let V be a subspace of the space of linear maps $\varphi \colon \mathcal{E} \to \mathbb{R}$. Show that if $\psi \colon \mathcal{E} \to \mathbb{R}$ is a linear map which is $\sigma(\mathcal{E}, V)$-continuous, then in fact $\psi \in V$.

By the previous exercise, on any locally convex topological vector space $\mathcal{E}$, the $\mathcal{E}^*$-weak topology, which for brevity is usually just called *the* weak topology on $\mathcal{E}$, is a Hausdorff vector topology on $\mathcal{E}$.

Exercise 3. Show that a convex set in $(\mathcal{E}, \mathscr{S})$ is closed if and only if it is weakly closed. (That is, if nad only if it is closed in the weak topology.)

Remark 4. Let's digress a bit. If you want to be really fancy, here is another way to construct $\sigma(\mathcal{E}, V)$ in such a way that it is, by construction, a vector space topology. Let $\mathscr{E}_{fin}$ be the system of all *finite dimensional* subspaces of $\mathcal{E}$. We can then define a vector space topology $\mathscr{S}_{max}$ on $\mathcal{E}$ as the *strongest* (locally convex) vector space topology such that the embedding $\iota_{\mathcal{F}} \colon \mathcal{F} \to \mathcal{E}$ is continuous for every $\mathcal{F} \in \mathscr{E}_{fin}$.

It can be shown that this is well-defined, and that *every* linear map $\varphi \colon \mathcal{E} \to \mathcal{E}'$ is $\mathscr{S}_{max}$-continuous for any topological vector space $\mathcal{E}'$. Hence in the definition of weak topology, we can in fact restrict to taking the intersection just over (locally convex) vector space topologies.

Let $\mathcal{E}$ be a Banach space, $\mathcal{E}^*$ its continuous dual (that is, $\mathcal{E}^* := \{\varphi \colon \mathcal{E} \to \mathbb{R} \mid \text{continuous, linear}\}$) with the norm topology, and let $\mathcal{E}^{**} := (\mathcal{E}^*)^*$ be the double dual of $\mathcal{E}$.

Exercise 5. Consider the map $\iota\colon \mathcal{E} \to \mathcal{E}^{**}$ given by $\iota(\xi)(\varphi) = \varphi(\xi), \xi \in \mathcal{E}, \varphi \in \mathcal{E}^*$. Show that ι is an isometric linear map.

Then on $\mathcal{E}^*$ we get in particular two weak topologies:

(i) The $\sigma(\mathcal{E}^*, \mathcal{E}^{**})$-topology. That is, the usual weak topology on $\mathcal{E}^*$.
(ii) The $\sigma(\mathcal{E}^*, \iota(\mathcal{E}))$-topology. This is weaker than the weak topology. It is called the *weak-$*$ topology* on $\mathcal{E}^*$.

Exercise 6. Show that $\iota(\mathcal{E})$ is weak-$*$ dense in $\mathcal{E}^{**}$.

Remark 7. Note that the definition of the weak-$*$ topology makes sane on the dual of any topological vector space.

Let $\mathcal{F}, \mathcal{F}$ be Banach spaces and $T\colon \mathcal{E} \to \mathcal{F}$ be a continuous linear map. Then we can define a continuous linear map $T^*\colon \mathcal{F}^* \to \mathcal{E}^*$ by

$$(T^*\varphi)(\xi) = \varphi(T\xi), \quad \varphi \in \mathcal{F}^*, \xi \in \mathcal{E}.$$

Exercise 8. Show that a norm-continuous linear map $S\colon \mathcal{F}^* \to \mathcal{E}^*$ of dual Banach spaces, is continuous with respect to the weak-$*$ topologies if and only if there exists a continuous $T\colon \mathcal{E} \to \mathcal{F}$ such that $S = T^*$.

– HDP, February 17, 2015