ANALYSIS ON GROUPS: SOME PROPERTIES OF AMENABLE GROUPS

The following three exercises can be completed in many different ways. We will even do some in the lectures. It is a good idea to come back to these every time you get a new definition of 'amenable' and re-do them with that. Just to try it out.

Exercise 1. Let G, H be amenable groups. Show that $G \times H$ is amenable. For any $d \in \mathbb{N}$, give an explicit example of a Følner sequence in $\mathbb{Z}^d$.

For the following exercise, recall that a subgroup N of G is called *normal* if $g^{-1}Ng = N$ for all $g \in G$. When this is so, the quotient space G/N is a group. As a *set with a right-N-action*, we have $G = (G/N) \times N$.

Exercise 2. Let $\mathbb{1} \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to \mathbb{1}$ be a short exact sequence of groups. That is, ι is injective, and π is surjective with $\ker \pi = \iota(N)$. Show that G is amenable if and only if N and Q are both amenable.

Exercise 3. Let G be a group and let $(H_i)_{i \in I}$ be an increasing net (or, if you prefer, just a sequence) of subgroups: that is, $H_i \subseteq H_j$ whenever $i \preceq j$ in I. Show that if $G = \cup_i H_i$ and each H_i is amenable, then G is amenable.

Exercise 4. Let $S_{(\mathbb{N})}$ be the group consisting of all bijections $\sigma \colon \mathbb{N} \xrightarrow{\sim} \mathbb{N}$ such that $\sigma(n) = n$ for all but finitely many n. Show that $S_{(\mathbb{N})}$ is amenable, and that so is $A_{(\mathbb{N})}$, the subgroup of even permutations. Show that $A_{(\mathbb{N})}$ is simple (i.e. that is has no non-trivial normal subgroups).

Remark 5. You can show that $A_{(\mathbb{N})}$ is not finitely generated. It was, until recently, an open problem to find examples of *finitely generated* (infinite) simple amenable groups.

Finally, here is a bonus exercise if you know the Riesz Representation Theorem (if not, don't worry about it!):[1]

Exercise 6. Let Γ be an amenable group and suppose that Γ acts by homeomorphisms on a compact set X.

(i) Show that there is a (Borel) probability measure μ on X such that $g.\mu = \mu$ for all $g \in \Gamma$. (Recall: this means that $\mu(g^{-1}.A) = \mu(A)$ for every Borel measurable set A.)

(ii) The Riesz Representation Theorem holds also in general for any locally compact X, identifying the measures with the dual of $C_0(X)$. Explain *why* the argument in (i) does not extend to locally compact X in general. What is the obvious counterexample?

– HDP, March 3, 2015

[1] See Theorem 2.14 in W. Rudin, *Real and Complex Analysis*.