ANALYSIS ON GROUPS:
AMENABLE ACTIONS

Definition 1 (amenable action)**.** Let Γ be a group, let X be a set, and let $\Gamma \curvearrowright^{\sigma} X$ be an action of Γ on X. We say that the action σ is *amenable* if there is a Γ-invariant mean on X.

Recall that a mean is the same as an element $m \in \ell^{\infty}(X)^*$ such that $m(\xi) \geq 0$ if $\xi \geq 0$, $m(\mathbb{1}_X) = 1$, and that m is Γ-invariant means that $m(g.\xi) = m(\xi)$ for all $g \in \Gamma, \xi \in \ell^{\infty}(X)$.

Exercise 2. Observe that a group Γ is amenable if and only if the (left-) action $\Gamma \curvearrowright \Gamma$ by multiplication, is amenable. More generally, show that if Γ is amenable then *any* action of Γ is amenable and that, conversely, if $\Gamma \curvearrowright X$ is an amenable *free* action of Γ (i.e. if $g.x = x$ for any $x \in X$, then $g = \mathbb{1}$), then Γ is amenable.

For the following exercise, look carefully at the proof we did, in the lectures, of the fixed point property for amenable groups, and try to suss out precisely which point in the proof is suitable for "abstractification".

Exercise 3 (fixed point property)**.** Recall that a group Γ is amenable if and only if for any weak-$*$ compact convex set $C \subseteq \mathcal{E}^*$, with $\Gamma \curvearrowright \mathcal{E}$ and isometric action on a Banach space, such that $\Gamma.C \subseteq C$, there is an invariant point $\eta \in C$. Formulate and prove an analogous statement, characterising amenable actions.

Exercise 4 (Bonus!)**.** If you're feeling really zealous, try to show that if $\Gamma \curvearrowright X$ amenably on, say a countable X, then there is a Følner sequence in X. (Of course, first you have to define what this means.[1]) Of course, the converse is clear!

Definition 5 (relative amenability)**.** Let Γ be a group and H a subgroup. We say that H is co-amenable (relative to Γ), if the action $\Gamma \curvearrowright \Gamma/H$ is amenable.

Exercise 6. Can you say anything neat about co-amenability and amenable actions? E.g. try to consider things like orbit spaces, stabiliser groups, normal subgroups, &c.

The following example shows that the concept of amenable action(s) is, perhaps contrarywise to immediate impressions (?), quite non-trivial. Recall that an action $\Gamma \curvearrowright^{\sigma} X$ is called *faithful* if the homomorphism $\sigma\colon \Gamma \to \mathrm{Perm}(X)$ is injective, i.e. if the following holds: $\forall g \in \Gamma : (\forall x \in X : g.x = x) \Rightarrow g = \mathbb{1}$.

Let $\Gamma := \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ be the free product of three copies of $\mathbb{Z}/2\mathbb{Z}$, say with generators labeled a, b, c. That is, in generators and relations, $\Gamma = \langle a, b, c \mid a^2 = b^2 = c^2 = \mathbb{1} \rangle$.

Let $X := \mathbb{Z}$ as a set; for each pair $(n, n+1)$ choose a label in $\{a, b, c\}$, in such a way that $(n, n+1)$ and $(n+1, n+2)$ have different labels for each n, and such that each letter appears at least once. We let $\Gamma \curvearrowright X$ by specifying that a letter $v \in \{a, b, c\}$ interchanges n and $n+1$ if and only if the label at $(n, n+1)$ is precisely v.

Exercise 7. Prove that this really does specify and action of Γ, and that it is faithful.

Exercise 8. Let $S_n := \{-n, -n+1, \ldots, n-1, n\}, n \in \mathbb{N}$ be the usual Følner sequence in $\mathbb{Z}$. Prove that for any $g \in \Gamma$ we have

$$\frac{\sharp(g.S_n \Delta S_n)}{\sharp S_n} \to 0, \quad \text{as } n \to \infty.$$

[1] get it?

(In fact there is nothing special about S_n: the analogous statement is true for *any* $\mathbb{Z}$-Følner sequence.) Conclude that the action $\Gamma \curvearrowright X$ is amenable.

Exercise 9. Show that Γ contains a copy of F_2, the free group on two generators. By picking labels carefully, can you arrange that the action of your copy of F_2 on X is *transitive*? (recall this means that for any two $x, y \in X$ there is a $g \in F_2$ such that $g.x = y$). In any case, observe that it must have an infinite orbit. The conclusion then is that there is a subgroup H of F_2 such that H is co-amenable in F_2, yet $\cap_{g \in F_2} g^{-1} H g = \{\mathbb{1}\}$. Wow!

Exercise 10. Can you construct an example with $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ instead of Γ ?

Note that without transitivity, it is not hard to construct examples of faithful, amenable actions: Recall that a group Γ is called *residually finite* if for any $g \in \Gamma \setminus \{\mathbb{1}\}$, there is a finite group G and a homomorphism $\varphi \colon \Gamma \to G$ such that $\varphi(g) \neq \mathbb{1}$. If Γ is countable, this is equivalent to the existence of a sequence $H_1 \supseteq H_2 \supseteq \cdots$ of normal subgroups $H_n \trianglelefteq \Gamma, n \in \mathbb{N}$, such that Γ / H_n is finite for every n, and such that $\cap H_n = \{\mathbb{1}\}$.

Exercise 11. Show that if Γ is residually finite, then the natural action of Γ on $\dot{\cup}_n H_n$ (disjoint union!) is amenable and faithful.

Exercise 12. Observe that $SL_n(\mathbb{Z})$ is residually finite.

It is non-trivial, but true, that the free group F_2 is residually finite.

– HDP, April 15, 2015