ANALYSIS ON GROUPS: COMPLEMENTS TO THE REITER PROPERTY

Recall that for a (discrete) group Γ we denote by $\mathscr{M}_1^+(\Gamma)$ the set of probability measures on Γ, that is $\mathscr{M}_1^+(\Gamma) := \{\xi \in \ell^1\Gamma \mid \xi \geq 0, \|\xi\|_1 = 1\}$.

Exercise 1. Let Γ be an amenable group. Following the lectures, show that for any finite set $\{g_k\}_k \subseteq \Gamma$ there is a *sequence* $\{\xi_i\}_{i\in\mathbb{N}} \subseteq \mathscr{M}_1^+(\Gamma)$ such that for all k we have $\|g_k.\xi_i - \xi_i\|_1 \to 0$ as $i \to \infty$. Then show that this in fact holds for any *countable* subset of Γ.

Recall that a sequence $\{\xi_i\}_i$ in a Banach space $\mathcal{E}$ with an isometric action of Γ, such that $\|g.\xi_i - \xi_i\| \to 0$ for all $g \in \Gamma$ is called a sequence of almost invariant vectors. If such a sequence exists, we say that $\Gamma \curvearrowright \mathcal{E}$ almost has invariant vectors.

Exercise 2. Show that if $\Gamma \curvearrowright \ell^p\Gamma$ almost has invariant vectors for some $p \in [1, \infty)$ (and the usual Γ-action), then $\Gamma \curvearrowright \ell^1\Gamma$ almost has invariant vectors.

Exercise 3. As in the lectures, consider a $\xi \in \mathscr{M}_1^+(\Gamma)$ written as

$$\xi = \sum_{k=1}^{n} t_k \mathbb{1}_{A_k}, \quad t_k \geq 0, A_k \supseteq A_{k+1}.$$

Show that for $h \notin A_k$, we have (say, letting $A_0 := \Gamma$ with the usual convention regarding summing over empty/ill-defined intervals; if this confuses you, sorting it out and writing it explicitly is part of the exercise!)

$$(g.\xi - \xi)(h) \geq t_k + \cdots + t_l, \quad l = \max\{k' \mid h \in g.A_{k'}\}$$

Show an analogous result for $h \notin g.A_k$, and deduce in detail the conclusion used in the lecture:

$$\sum_k t_k \cdot \sharp(g.A_k \Delta A_k) \leq \|g.\xi - \xi\|_{\ell^1\Gamma}.$$

Exercise 4. Show, using the results of the lecture, that if $\Gamma \curvearrowright \ell^1\Gamma$ almost has invariant vectors, then so does $\Gamma \curvearrowright \ell^p\Gamma$ for all $p \in [1, \infty)$.

Definition 5 (Property T)**.** A group Γ is said to have property T if every unitary representation $\Gamma \curvearrowright^\pi \mathcal{H}$ almost having invariant vectors, in fact has a non-zero invariant vector. That is, whenever we have a sequence $(\xi_i)_i \subseteq \mathcal{H}$ of unit vectors (i.e. $\|\xi_i\|_\mathcal{H} = 1$) such that $\|g.\xi_i - \xi_i\|_\mathcal{H} \to 0$ as $i \to \infty$, for all $g \in \Gamma$, then in fact there is a $\xi \in \mathbb{H}$ with $\|\xi\|_\mathcal{H} = 1$ such that $g.\xi = \xi$ for all $g \in \Gamma$.

Exercise 6. Let Γ be a (countable) group. Show that if Γ is amenable and has property T, then Γ is finite. By the way, make sure you understand why every finite group has property T. Hopefully you already understand why every finite group is amenable :o

Property T was defined by David Kazhdan in 1967. It's super important. The prototype example of a group with property is $SL_3(\mathbb{Z})$. The proof that this in fact does have property T is not easy. One of Kazhdan's motivations for introducing the definition, was the following observation:

Exercise 7. Let Γ be a (countable) group with property T. Show that Γ is finitely generated. (Hint: consider representations on $\ell^2(\Gamma/\Lambda)$ for $\Lambda \leq \Gamma$.)

Exercise 8. Let $N \trianglelefteq \Gamma$ such that Γ/N is *infinite* amenable. Show that Γ does not have property T. Thus in particular, free non-abelian groups do not have property T.

– HDP, March 24, 2015