# ANALYSIS ON GROUPS:
# MEANS AND DUAL SPACES

In this set, exercises 6 and 11 are the most important ones.
In the lectures we proved that the space of means $\mathscr{M}(X)$ on a set $X$ is compact. Using the same argument, prove the following result, known as the Banach-Alaoglu Theorem:

**Exercise 1.** Let $\mathcal{E}$ be a Banach space. Prove that the unit ball $(\mathcal{E}^*)_1 := \{\eta \in \mathcal{E}^* \mid \|\eta\| \leq 1\}$ is compact in the weak-$*$ topology.

Let $X$ be a set. You may suppose that $X$ is countable if you like. In this note we will consider the following "function spaces" on $X$:

(i) The space $c_0(X)$ of all functions $f\colon X \to \mathbb{R}$ vanishing at infinity; that is, such that for every $\varepsilon > 0$ there is a finite set $K \subseteq X$ such that $f(x) < \varepsilon$ for all $x \in X \setminus K$. This is a Banach space with the uniform norm $\|f\|_\infty := \sup\{|f(x)| \mid x \in X\}$.

(ii) The space $\ell^1(X)$ of all functions $f\colon X \to \mathbb{R}$ such that $\sum_{x\in X}|f(x)| < +\infty$. This is a Banach space with the 1-norm: $\|f\|_1 := \sum_{x\in X}|f(x)|$.

(iii) And the space $\ell^\infty(X)$ of all functions $f\colon X \to \mathbb{R}$ such that there is an $M \in \mathbb{R}_+$ such that $|f(x)\| \leq M$ for all $x \in X$. This is a Banach space with the uniform norm as well: $\|f\|_\infty := \sup\{|f(x)| \mid x \in X\}$.

**Exercise 2.** Show that $\ell^1(X) = c_0(X)^*$ and $\ell^\infty(X) = \ell^1(X)^*$. (The equalitites should be read as "canonically isomorphic": to do the exercise, the first step should be to figure out how an element of, say $\ell^1(X)$ acts as a continuous linear functional on $c_0(X)$.

**Exercise 3.** Recall that a Banach space $\mathcal{E}$ is called *reflexive* if the canonical embedding $\iota\colon \mathcal{E} \to \mathcal{E}^{**}$ is an isomorphism. Show that $\mathcal{E}$ is reflexive if and only if the unit ball $(\mathcal{E})_1$ is compact in the weak topology. Conclude that, at least when $X$ is countable, neither $c_0(X)$ nor $\ell^1(X)$ is reflexive.

**Exercise 4.** Let $\mathcal{E}$ be a *separable* Banach space. Prove that the weak-$*$ topology on $(\mathcal{E}^*)_1$ is metrizable. (Hint: recall the general construction that, given a sequence of *uniformly bounded* metrics $(d_n)$ on a set $Y$, we get a new metric defined by $d(y,z) := \sum_n 2^{-n} d_n(y,z)$.) In particular, every sequence in $(\mathcal{E}^*)_1$ has a weak-$*$-convergent subsequence.

For completeness, let's recall the following definition from the lectures:

**Definition 5** (the space of means)**.** Let $X$ be a set and denote by $\mathscr{P}(X)$ the set of all subsets of $X$. A *mean* on $X$ is a function $m\colon \mathscr{P}(X) \to [0,1]$ such that $m(X) = 1$, and such that for all $A_1, \ldots, A_n \in \mathscr{P}(X)$, satisfying $A_i \cap A_j = \emptyset$ whenever $i \neq j$, we have

$$m(\cup_i A_i) = \sum_i m(A_i).$$

**Exercise 6.** Let $X$ be a set, and consider the subspace $\mathcal{E}_0 \leq \ell^\infty(X)$ spanned by indicator functions $\mathbb{1}_A, A \subseteq X$, that is,

$$\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases},$$

and $\mathcal{E}_0 := \{t_1.\mathbb{1}_{A_1} + \cdots + t_n.\mathbb{1}_{A_n} \mid t_i \in \mathbb{R}, A_i \subseteq X\}$. Let $m \in \mathscr{M}(X)$. Show that $m$ induces a linear functional, which we also denote $m$, on $\mathcal{E}_0$ by

$$m\left(\sum_i^{fin.} t_i.\mathbb{1}_{A_i}\right) := \sum_i t_i.m(A_i),$$

and that this functional extends by continuity to all of $\ell^\infty(X)$. That is, given a mean $m$ on $X$, the above formula gives rise to an element in $\ell^\infty(X)^*$.

**Exercise 7.** Let $X$ be a set and $m \in \mathscr{M}(X)$. Show that the linear functional $m \in \ell^\infty(X)^*$ constructed in the previous exercise is *positive*, in the sense that $m(f) \geq 0$ whenever $f$ is a positive function, and that $\|m\| \leq 1$. (Presumably you already did the norm part above. Maybe you even did both parts...)

From this point on, we will use the terms '$m$ is a mean on $X$' and '$m$ is a positive functional of norm one in $\ell^\infty(X)$' synonymously. The latter may seem the more complicated one, but in many ways it is the easier to work with of the two. That's because Banach spaces are awesome!

**Exercise 8.** Here is something slightly more tricky, included just for fun: Let $X$ be a set; show that any (complex-) linear map $\varphi\colon \ell^\infty(X, \mathbb{C}) \to \mathbb{C}$ such that $\varphi(\mathbb{1}_X) = 1$ and $\varphi(f) \geq 0$ whenever $f \geq 0$, is bounded with norm $\|\varphi\| = 1$.

**Definition 9** (Følner sets)**.** Let $\Gamma$ be a group. A sequence of *finite* subsets $F_n \subseteq \Gamma$ is called a (left-) Følner sequence if for every $g \in \Gamma$ we have

$$\frac{\sharp(g.F_n \Delta F_n)}{\sharp F_n} \to 0, \quad \text{as}\, n \to +\infty.$$

**Exercise 10.** In the lectures we proved that if a group $\Gamma$ has a Følner sequence, then $\Gamma$ is amenable. Go through the proof and fill in *all* the details.

Bonus: In the lectures we showed that the space of means $\mathscr{M}(X)$ is compact using the language of *nets*. You might want to try to write down the argument using just open sets instead.

**Exercise 11.** In the group $\Gamma := \mathbb{Z}$, consider the sequence of subsets given by $F_n := \{1, \ldots, n\}, n \in \mathbb{N}$. Observe that $(F_n)_{n\in\mathbb{N}}$ is a Følner sequence, and show that the associated sequence of means $(m_{F_n})_{n\in\mathbb{N}} \subseteq \ell^\infty(X)^*$ has no (weak-$*$-)convergent subsequence.

– HDP, February 25, 2015