ANALYSIS ON GROUPS:
THE HAHN-BANACH THEOREM(S)

A *topological vector space* is by definition a vector space $\mathcal{E}$ with a (Hausdorff) topology $\mathscr{S}$ such that the vector space operations are continuous: that is, the maps $\mathcal{E} \times \mathcal{E} \to \mathcal{E} \colon (\xi, \eta) \mapsto \xi + \eta$ and $\mathfrak{k} \times \mathcal{E} \to \mathcal{E} \colon (t, \xi) \mapsto t.\xi$, where $\mathfrak{k}$ is a base field, are continuous. For us, unless explicitly stated otherwise, $\mathfrak{k}$ will be the real numbers $\mathbb{R}$. Sometimes it might also be the complex numbers $\mathbb{C}$, and generally it doesn't really matter. Note that formally, 'a topological vector space' is a *pair* $(\mathcal{E}, \mathscr{S})$, but that following usual conventions we generally just refer to $\mathcal{E}$ when no confusion can arise.

Exercise 1. Maybe you already know the following concepts, however it's always good to make sure. In each case, write out explicitly the definition and make sure you understand all the words in it:

- (i) The definition of a (semi-)norm on a vector space.
- (ii) Banach space. Hilbert space.
- (iii) The definition of continuous / bounded linear maps on Banach spaces.
- (iv) The dual norm on $\mathcal{E}^*$, where $\mathcal{E}$ is a normed space. Show that $\mathcal{E}^*$ is a Banach space. When $\mathcal{E}$ is a Hilbert space the situation is special. Why?
- (v) Recall Banach's inversion theorem. Recall also the more general results known as the Open Mapping Theorem and the Closed Graph Theorem.

The following exercise is very basic. But it is also very important.

Exercise 2. Let $\mathcal{E}, \mathcal{F}$ be topological vector spaces and suppose that $\dim \mathcal{E} < +\infty$. Show that *every* linear map $\varphi \colon \mathcal{E} \to \mathcal{F}$ is continuous. Conclude that on every finite dimensional vector space, there is a unique vector space topology.

Theorem 3 (Hahn-Banach Extension Theorem)**.** *Let $\mathcal{E}$ be a normed vector space, let $\mathcal{E}_0 \leq \mathcal{E}$ be a closed subspace, and let $\varphi \in \mathcal{E}_0^*$. Then there is a continuous linear functional $\tilde{\varphi} \colon \mathcal{E} \to \mathbb{R}$ such that $\tilde{\varphi}_{|\mathcal{E}_0} = \varphi$, and $\|\tilde{\varphi}\| = \|\varphi\|$.*

Exercise 4 (special case)**.** Prove the Hahn-Banach Extension Theorem in the special case where $\mathcal{E}$ is a Hilbert space.

Exercise 5 (Proof)**.** This exercise outlines the proof of the Hahn-Banach Extension Theorem in general.

- (i) Show, by taking the completion of $\mathcal{E}$, that we may as well assume from the start that $\mathcal{E}$ is a Banach space.
- (ii) Let $\xi_1 \in \mathcal{E} \setminus \mathcal{E}_0$, and let $\mathcal{E}_1 = \overline{\text{span}}(\mathcal{E}_0 \cup \{\xi_1\})$ be the (closed) subspace generated by $\mathcal{E}_0$ and ξ_1. Show that $\mathcal{E}_1 \cong \mathcal{E}_0 \oplus_1 \mathbb{R}.\xi_1$, where the norm on the right-hand side is given by
$$\|(\xi_0, t.\xi_1)\| := \|\xi_0\| + t \cdot \|\xi_1\|, \quad \xi_0 \in \mathcal{E}_0.$$
- (iii) Conclude that the theorem is true in the special case that $\mathcal{E} = \mathcal{E}_1$.
- (iv) Let $\mathscr{E}_0$ be the set of closed subspaces of $\mathcal{E}$ containing $\mathcal{E}_0$ and consider the set $\mathscr{C}$ of pairs $(\mathcal{F}, \psi)$ where $\mathcal{F} \in \mathscr{E}_0$ and $\psi_{|\mathcal{E}_0} = \varphi$. We define a partial ordering on $\mathscr{C}$ by:
$$(\mathcal{F}_1, \psi_1) \prec (\mathcal{F}_2, \psi_2) \Leftrightarrow \left(\mathcal{F}_1 \subseteq \mathcal{F}_2, \text{and } (\psi_2)_{|\mathcal{F}_1} = \psi_1\right).$$
  By the Hausdorff maximality principle[1] it follows that there is a maximal element $(\mathcal{F}_m, \psi_m)$ in $\mathscr{C}$. Show that $\mathcal{F}_m = \mathcal{E}$, and hence conclude that the theorem holds.

[1]It is equivalent to Zorn's Lemma and (hence) the axiom of Choice. You should look it up if you don't know it.

A subset $V \subseteq \mathcal{E}$ is called *convex* if $t.\xi + (1-t).\eta \in V$ whenever $\xi, \eta \in V$ and $t \in [0,1]$.

Exercise 6. Recall the definition of a locally convex topological vector space. How is local convexity related to semi-norms?

Hint: given any convex neighbourhood V of 0 in the (real) locally convex space $\mathcal{E}$, such that $V = -V$, you can define a semi-norm by $p(\xi) := \inf\{t \in \mathbb{R}_+ \mid t^{-1}.\xi \in V\}$.

Exercise 7. Let $\mathcal{E}$ be a topological vector space and let $p\colon \mathcal{E} \to \mathbb{R}$ be a continuous semi-norm. Show that the set $\mathcal{F} := \{\xi \in \mathcal{E} \mid p(\xi) = 0\}$ is a closed subspace, and that p induces a norm on the quotient space $\mathcal{E}_p := \mathcal{E}/\mathcal{F}$.

Exercise 8. Using the Hahn-Banach theorem, show that for any locally convex topological vector space $\mathcal{E}$ and any $\xi \in \mathcal{E} \setminus \{0\}$, there exists a $\varphi \in \mathcal{E}^*$ such that $\varphi(\xi) \neq 0$. (The terminology for this kind of thing is that '$\mathcal{E}^*$ separates points on $\mathcal{E}$'. Explain this terminology.)

Exercise 9 (Hahn-Banach Separation Theorem)**.** (i) Let $\mathcal{E}$ be a (real) locally convex space and let $V = -V$ be a convex neighbourhood of the identity. Let $\xi \in \mathcal{E} \setminus V$. Show that there is a $c > 0$ and a continuous linear functional $\varphi \in \mathcal{E}^*$ such that for all $\eta \in V$:

$$\varphi(\eta) < c \leq \varphi(\xi).$$

(ii) Show that the same conclusion holds with any convex open set in place of V. (This is often called the Hahn-Banach Separation Theorem.)

– HDP, February 17, 2015