**Exercise 0.** Let $X$ be a (Hausdorff) locally compact space and write $C^+_c(X)$ for the subset of functions $f \geq 0$ in $C_c(X)$. Recall that a linear map $\mu: C_c(X) \to \mathbb{R}$ is **positive** if $\mu(f) \geq 0$ for all $f \in C^+_c(X)$.

(i) Show that such a positive $\mu$ is a measure (i.e. that it is continuous).

(ii) Let $\nu: C^+_c(X) \to \mathbb{R}_{\geq 0}$ be additive ($\nu(f + f') = \nu(f) + \nu(f')$) and homogeneous ($\forall t > 0, \nu(tf) = t\nu(f)$). Prove that there exists a measure which coincides with $\nu$ on $C^+_c(X)$ and observe that it is unique.

**Notation and definitions.** Let $(V, \langle \cdot, \cdot \rangle)$ be a Hilbert space and write $\mathcal{L}(V)$ for the space of all continuous linear maps $\alpha: V \to V$. Seeing $\alpha$ in $V^V$, we can endow $\mathcal{L}(V)$ with the product of the norm topology on each factor $V$; this is called the **strong operator topology** (SOT). For instance, $\alpha_n \to \alpha$ means simply $\|\alpha(v) - \alpha(v)\| \to 0$ for all $v \in V$.

Likewise, the **weak operator topology** (WOT) is the product of the weak topology on each factor $V$.

**Exercise 1.** Prove that the weak and norm topology agree on the unit sphere $S \subseteq V$. First, give an example where this fails for the unit ball of $V$.

**Hints.** In $V = \ell^2(\mathbb{N})$, let $\delta_n \in V$ be the sequence which is 1 at $n$ and 0 otherwise. Consider the sequence (of sequences!) $\{\delta_n\}_{n \in \mathbb{N}}$. Meditate why the first and second part of the exercise do not contradict each other.

**Exercise 2.** Prove that SOT and WOT agree on the orthogonal group $O(V) \subseteq \mathcal{L}(V)$. First, give an example to show that SOT $\neq$ WOT in general.

**Exercise 3.** Prove that $O(V)$ is a topological group for SOT and WOT (keep Ex. 2 in mind, of course).