Exercise 1 (Alexandrov’s Lemma)
This exercise is purely in the Euclidean plane! Let \( A, B, B', C \in \mathbb{R}^2 \). Consider the two triangles \( \triangle \equiv \triangle(A, B, C) \) and \( \triangle' \equiv \triangle(A, B', C) \) and suppose that \( B, B' \) are on different sides of the axis determined by \( A, C \). Let \( \alpha, \beta, \gamma \) (resp. \( \alpha', \beta', \gamma' \)) be the angles of \( \triangle \) (resp. \( \triangle' \)) at the vertices \( A, B, C \) (resp. \( A, B', C \)). Suppose \( \gamma + \gamma' \geq \pi \).
Consider furthermore the triangle \( \triangle : = \triangle(A, B, B') \) such that \( d(A, B) = d(A, B') \), \( d(A, B') = d(A, C) + d(C, B') \).

Let \( \overline{C} \) be the point of the segment \( \overline{B, B'} \) such that \( d(B, C) = d(B, C') \). Denote the corresponding angles as in the figure below.

Prove \( \alpha \geq \alpha' \), \( \beta \geq \beta' \) and \( d(A, C) \geq d(A, C') \).
Note that the inequalities are strict unless \( \gamma + \gamma' = \pi \).

Exercise 2
Let \( X \) be a complete metric space admitting midpoints. We say that \( X \) satisfies the **four points condition** if for all \( x_1, y_1, x_2, y_2 \in X \) there is \( x_1, y_1, x_2, y_2 \in \mathbb{R}^2 \) such that
\[
d(x_i, y_j) = d(x_i, y_j) \quad \text{pour} \quad i, j \in \{1, 2\},
\]
\[
d(x_1, x_2) \leq d(x_1, x_2),
\]
\[
d(y_1, y_2) \leq d(y_1, y_2).
\]

Prove that \( X \) is CAT(0) IFF it satisfies the four points condition.

Hints: You will probably consider some sort of “comparison quadrangle”. For \( \Rightarrow \), distinguish cases according to whether this quadrangle is convex or not. In that second case, use Alexandrov’s lemma.

Exercise 3
(i) Prove that every isometry of \( f \in Is(\mathbb{R}^n) \) is of the type \( f(x) = Ax + b \), where \( b \in \mathbb{R}^n \) and \( A \) is an orthogonal matrix (i.e. \( A^tA = I \)).
(ii) Deduce that \( \mathbb{R}^n \) has no parabolic isometries. *Hint: If you wish, just do the case \( n = 2 \).*
(iii) Optional and unrelated to (ii): prove that every isometric map \( f : \mathbb{R}^n \to \mathbb{R}^n \) is an isometry.