Spaces of non-positive curvature and groups
Setting Def A metric on a

point (X, d) where X is a non-empty set and d: X \times X \rightarrow \mathbb{R} s.t.

1. d(x, y) = 0 \iff x = y
2. d(x, y) \leq d(x, z) + d(y, z) \forall x, y, z \in X.

d is called distance or metric.

Sometimes we write d_x for d.
Ex.

Prove (i) \( d(x, y) = d(y, x) \)

(ii) \( d(x, y) \geq 0 \) \( \forall x, y \in X \).

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Recall: (i) A sequence \((x_n)\) converges to \(x\)

if \( \lim_{n \to \infty} d(x_n, x) = 0 \). Notation: \( \lim_{n \to \infty} x_n = x \).

(ii) \((x_n)\) is Cauchy if

\( \forall \epsilon > 0 \ \exists N \in \mathbb{N} : \forall n, m \geq N : d(x_n, x_m) < \epsilon \).

(iii) \( X \) is complete if every Cauchy sequence converges.
(iv) If $X$ is any metric space, it admits a completion $\tilde{X}$. From now on, unless stated otherwise, all spaces are assumed complete.
Terminology: A map \( f : X \to Y \) between metric spaces is \textbf{isometric} if \( \forall x, x' \in X: \)
\[
d(x, x') = d_Y(f(x), f(x')).
\]

Then \( f \) is injective.

Example: If \( X \) is a subset of \((Y, d_Y)\), define \( d_X = d_Y|_{X \times X} \) and \( f = \text{inclusion} \).
Def. $f : X \to Y$ is an **isometry** if it is a bijective isometric map; note that $f^{-1} : Y \to X$ is also isometric.

Def. An **interval** is a (non-Empty) connected subset of $\mathbb{R}$. (Usually $[a, b]$.)

We endow it with the usual metric of $\mathbb{R}$, i.e. $d(x, y) = |x - y|$. 
If $\gamma$ is a geodesic in a metric space $X$, then $I \subset X$, where $I$ is an interval. 

Warning: we often confuse $\gamma$ and $\gamma(I) \subset X$. Note: given $\gamma(I)$ there are almost two isometric images; $I \mapsto \gamma(I)$. 

If $I = [a, b]$, we often specify geodesic segment; if $I = \mathbb{R}$, we often say geodesic line; if $I = [a, +\infty)$, geodesic ray.
Def. $X$ is called a **geodesic metric space** if $\forall x, y \in X \exists$ geodesic connecting $x$ to $y$.

i.e. $\forall x, y \in X \exists \gamma: [a, b] \to X$

1. th. $\gamma(a) = x$
2. th. $\gamma(b) = y$.

Rem. 1. Then $|b - a| = d(x, y)$.
2. Isometric $\Rightarrow$ continuous.
3. Hence, geodesic metric spaces are (arcwise) connected.
Given $x, y, z \in \mathbb{R}$, where $d : \mathbb{R}^2 \to \mathbb{R}$ is a metric function, there exist $\overline{x}, \overline{y}, \overline{z} \in \mathbb{R}^2$ such that

- $d(x, y) = d_{\mathbb{R}^2}(\overline{x}, \overline{y})$,
- $d(x, z) = d_{\mathbb{R}^2}(\overline{y}, \overline{z})$,
- $d(y, z) = d_{\mathbb{R}^2}(\overline{z}, \overline{y})$.

Def $(\overline{x}, \overline{y}, \overline{z})$ is called a comparison triangle (in $\mathbb{R}^2$) for $(x, y, z)$. (any metric space)
Rem - For 1 or 2 points, trivial.
- For 4 points or more: **wrong**.
  (even replacing $\mathbb{R}^2$ by $\mathbb{R}^n$).

Proof of Lemma Take $\mathbf{x} = (0,0) \in \mathbb{R}^2$

To show: the two circles do intersect.
To avoid:

1) $d(x, y) > d(x, z) + d(y, z)$

2) $d(x, y) > d(x, z) + d(y, z)$ impossible!

3) $d(x, y) > d(x, z) + d(y, z)$ impossible.

c.f. (2).
Def: A metric space (complete) is called CAT(0) (a non-positively curved) if:

\[ \forall x, y \exists m \forall z : \]

\[ d^2(x, m) \leq \frac{d^2(x, x) + d^2(y, y)}{2} - \frac{1}{4} d^2(x, y) \]
Lemma \( R^n \in \text{CAT}(0) \) (for the usual distance).

Proof: Let \( x, y \in R^n \). Let \( u := \frac{x + y}{2} \). Let \( z \in R^n \).

WLOG \( z = 0 \). (Translations preserve distance and def. of \( u \)).

\[
\begin{align*}
  d^2 (z, u) &= \|u\|^2 = \frac{1}{4} \|x + y\|^2 = \frac{1}{4} \left( \|x\|^2 + \|y\|^2 + 2 \langle x, y \rangle \right) \\
  \frac{1}{2} (d^2 (z, x) + d^2 (z, y)) - \frac{1}{4} d^2 (x, y) &= \frac{1}{4} \left( 2 \|x\|^2 + 2 \|y\|^2 - \|x - y\|^2 \right) = \\
  &= \frac{1}{4} \left( \|x\|^2 + \|y\|^2 + 2 \langle x, y \rangle \right). \square
\end{align*}
\]

Remark: The \( \text{CAT}(0) \) inequality is an \text{EQUALITY} in \( R^n \)!
Lemma: If $X$ is CAT(0), then it has unique midpoints. 
($\forall x, y \exists ! m: d(x, m) = d(m, y)$)

Proof: Given $x, y \in X$, take $m$ as in the axiom.

$X$ is CAT(0) for $\varepsilon = x$: 

$d^2(x, m) \leq \frac{d^2(x, x) + d(x, y)}{2} - \frac{1}{4} d^2(x, y) = \frac{1}{4} d^2(x, y)$

$\Rightarrow d(x, m) \leq \frac{1}{2} d(x, y)$. 

Likewise:

$d(y, m) \leq \frac{1}{2} d(x, y)$

$d(x, y) \leq d(x, m) + d(m, y) \leq d(x, y)$.

$\Rightarrow d(x, m) = \frac{1}{2} d(x, y)$ 

$\Rightarrow \exists$ mid-point.
Let now \( m' \) be any midpoint for \( x, y \).

\[ \text{CAT}(0) \text{ for } t = m': \]

\[ d^2(m, m') \leq \frac{d^2(m, x) + d^2(m, y)}{2} - \frac{1}{4} d(x, y) = \]

\[ = \frac{1}{2} \left( \frac{1}{4} d^2(x, y) + \frac{1}{4} d^2(x, y) \right) - \frac{1}{4} d^2(x, y) = 0. \]

\[ \implies m' = m. \]

\( \square \)

**COR** Complete CAT(0) spaces are geodesic and even uniquely geodesic. \( \square \)
1. The CAT(0) inequality makes sense without completeness. However, incomplete CAT(0) spaces are traditionally supposed geodesic.

2. We write, given $x, y \in X$, $[x, y] \subseteq X$ for the (image of the) unique geodesic from $x$ to $y$.

3. Given $x, y, t \in X$, we write $\Delta(x, y, t) := [x, y] \cup \{y, t\} \cup [t, x] \subseteq X$. 

\[ \xymatrix{ x \ar[rr]^{y} & & } \]
Prop Let $\times$ be CAT(0); $x, y \in X$ and
in the midpoint of $x, y$.

\[ \forall z: \quad d(z, t) \leq \frac{1}{2} (d(z, x) + d(z, y)) \]

Proof

\[ d^2(w, z) \leq \frac{d^2(z, x) + d(z, y)}{2} \]

\[ d(x, y) \geq d(y, t) - d(x, t) \]

\[ \leq \frac{1}{4} \left(2 d^2(z, x) + 2 d^2(z, y) - d^2(y, t) - d^2(x, t) + 2 d(y, t) d(x, t) \right) \]
\[ \begin{align*}
&= \frac{1}{4} \left( d(x, z) + d(z, \eta) \right)^2 \\
&= \left( \frac{d(x, z) + d(z, \eta)}{2} \right)^2
\end{align*} \]
**Theorem (The comparison theorem)**

Let $x, y, z \in X$ (complete cat $CW$).

Consider the comparison map,

$$\Delta(x, y, z) \rightarrow \Delta(\bar{x}, \bar{y}, \bar{z}) \subseteq R^2$$

$p \mapsto \bar{p}$.

Then $\forall p, q \in \Delta(x, y, z)$,

$$d(x, q) \leq d(\bar{p}, \bar{q})$$
NON-EXAMPLE \[ X = S^2 \] with spherical distance.
Proof Start with special case of $p = 2$.

The inequality is non-trivial only if $q \in [x, y]$.

We know the inequality if $q$ is the midpoint of $x, y$ (then it is the $Cr(\omega)$ inequality).

Define $E_0 \in X$ recursively by $E_0 = [x, y]$.

$E_{n+1} = \{ \text{midpoint } (a, b) : a, b \in E_n \}$.

This is an increasing sequence, its union is dense in $[x, y]$.

So it suffices to prove this case for $q \in E_n$ (for $\forall n$).

Ok for $E_0$, $E_1$. 
Induction step: suppose \( \alpha \) for \( E_n \). Let \( q = \text{mid}(a, b) \in E_{n+1} \) for some \( a, b \in E_n \).

\[
\tilde{p} = \frac{1}{\alpha} \tilde{q} = \frac{1}{\alpha} \tilde{b}
\]

Let \( \tilde{p}, \tilde{a}, \tilde{b} \in R^2 \) be a coxaoson \( \Delta \) for \( \Delta(p, a, b) \). No reason that \( \tilde{p}, \tilde{a}, \tilde{b} \) should be a coxaoson triagle? WLOG \( \tilde{p} = \tilde{b} \).
Inductive assertion: \[ d_X(p, a) \leq d_{\mathbb{R}^2}(\bar{p}, \bar{a}) \]
\[ d(\bar{p}, \bar{a}) \leq d_{\mathbb{R}^2}(\bar{p}, \bar{a}) \]

\( \text{CAT}(0) \Rightarrow \)
\[ d_X(p, q) \leq d_{\mathbb{R}^2}(\bar{p}, \bar{q}) \leq d_{\mathbb{R}^2}(\bar{p}, \bar{q}) \]

\( \text{algebra in } \mathbb{R}^2 \) since:
\[ d(c, \bar{c}) = d(\bar{a}, \bar{b}) \]
\[ d(\bar{p}, \bar{a}) \leq d(\bar{p}, \bar{a}) \]
\[ d(\bar{p}, \bar{b}) \leq d(\bar{p}, \bar{b}) \]

\( \text{special case} \)
General case:

\[ \vec{p} = \vec{x} \]
\[ \vec{y} = \vec{z} \]

Case \( \mathbb{R}^2 \):

Comp. \( \Delta \) for \((x, y, \vec{p})\) in \((\vec{x}, \vec{y}, \vec{p}) \in \mathbb{R}^2 \)

WLOG \( \frac{x}{x} = \frac{y}{y} \), \( d(\vec{x}, \vec{p}) \leq d(I, \vec{p}) \) by special case.
Now \( d_x(p, q) \leq d_{\mathbb{R}^2}(\vec{p}, \vec{q}) \) (by special case).

\[
\leq d_{\mathbb{R}^2}(\vec{p}, \vec{q})
\]

by algebra in \( \mathbb{R}^2 \).
An example of CAT(0) space:

Consider the "tripod" $T = \left( \mathbb{R}_+ \cup \mathbb{R}_+ \right) \cup \mathbb{R}_+$.

$$T \overset{def}{=} \left( \mathbb{R}_+ \times \{1,2,3\} \right) / \left( (0,1) \sim (0,2) \sim (0,3) \right).$$

$$d((t,i), (s,j)) = \begin{cases} |t-s| & \text{if } i = j \\ \Delta + \ell & \text{if } i \neq j \end{cases}$$
Only non-trivial case (up to permutation):

To show:

\[(\beta + \gamma)^2 \leq \frac{(\beta + \alpha)^2 + (\beta + \gamma + \delta)^2}{2}\]

\[-\frac{1}{4} (\alpha + \gamma + \delta)^2\]

\[\forall \alpha, \beta, \gamma, \delta \in \mathbb{R}_+\]
§ Encounter with the III types of isomorphisms.

Let $\mathbb{X}$ be a (complete CAT(0)) space. Let $g \in \text{Isom}(\mathbb{X})$.

**Def.** The displacement length of $g$ in is $\inf_{x \in \mathbb{X}} d(gx, x)$.

**Def.** $g$ is called **elliptic** if it fixes a point.
- $g$ is called **hyperbolic** if $|g| > 0$ and the inf is a **min.**
- $g$ is called **parabolic** if $\inf_{x \in \mathbb{X}} d(gx, x)$ is achieved.
Def: $g$ is **ballistic** if $|g| > 0$.

|            | $|g| > 0$ | $|g| = 0$ |
|------------|-----------|-----------|
| min.       | hyperbolic| elliptic  |
| not achieved | parabolic | parabolic |
§ Center, balls, convexity

Def. A subset $C \subseteq X$ is called **convex** if $[x, y] \subseteq C \quad \forall x, y \in C$.

Note 1: $(C, d_{C \times C})$ is uniquely geodesic if (and only if) it is convex.

(Recalling that $X$ is uniquely geodesic).

2) $C$ is also $\text{CAT}(0)$ (if closed).
Recall \( B(x, r) := \{ y : d(x, y) < r \} \) and \( \overline{B}(x, r) := \{ y : d(x, y) \leq r \} \).

(If \( x \) is geodesic, \( \overline{B}(x, r) = \overline{B}(x, 2r) \).)

**Corollary** \( \overline{B}(x, 2r) \) is convex in any \( \text{CAT}(0) \) space.

**Proof** Follows from \( d(z, m) \leq (d(z, x) + d(x, y))/2 \) whenever \( m = \text{mid}(x, y) \). \( \square \)

**Remark** \( C \subseteq X \) is convex \( \Longleftrightarrow \text{mid}(x, y) \in C \) for all \( x, y \in C \).
Non-Example

$S^2$

$\bar{\Omega}(N, \frac{2\pi}{3})$
Def  Let $A \subseteq X$ be bounded, non-empty.

There exists $x \in X$ such that $A \subseteq B(x, \rho_x(A))$ and for any $a, a' \in A$, $d(a, a') < +\infty$.

The radius of $A$ is

$$\rho_x(A) = \inf \{ r \mid \exists x \in X : A \subseteq B(x, r) \}$$

A point $x \in X$ is called a center (or circumcenter) of $A$ if $A \subseteq \overline{B}(x, \rho_x(A))$.

Ex  If $A = \{a, b\}$ then $\text{mid}(a, b)$ is the unique center.
THM (The center lemma)

1. Any bounded non-empty subset of a (complete) CAT(0) space admits a center, and it is unique.
2. If moreover $A$ is convex closed, then its center belongs to $A$. 
Proof of existence. Consider \( \mathcal{A} \subseteq X \) bounded.

Let \( (x_n) \) be a decreasing sequence converging to \( p_x(A) \) with \( n > p_x(A) \).

\( \forall n \exists x_n \in X : A \subseteq B(x_n, 2n) \). \( \circledast \)

Claim: Any sequence \( (x_n) \) satisfying \( \circledast \) is Cauchy.

Let \( m = m_{p,q} := \min \{ x_p, x_q \} \).

For any \( a \in A \), apply \( \Delta + (0) \) to \( (a, x_p, x_q) \).
\[ d^2(a, m) \leq \frac{1}{4} (d^2(a, x_p) + d^2(a, x_q)) - \frac{1}{4} d^2(x_p, x_q). \]

\[ d^2(x_p, x_q) \leq 2 \lambda_p^2 + 2 \lambda_q^2 - 4 d^2(a, m). \]

But \( \text{sgn} \ d(a, m) \geq \Psi_x(\varepsilon) \).

\[ d^2(x_p, x_q) \leq 2 \lambda_p^2 + 2 \lambda_q^2 - 4 \rho^2(x). \]

\[ \Rightarrow \text{claim ok.} \]
Let now \( c = \lim_{n \to \infty} x_n \).

forall \( a \in A: d(c, a) = \lim_{n \to \infty} d(x_n, a) \leq \rho(u) \leq \lambda_n \).

Existence OK.

Uniqueness: if \( c, c' \) are centers, define

\[
x_n = \begin{cases} c, & \text{for } n \text{ even} \\ c', & \text{for } n \text{ odd} \end{cases}
\]

Candy \( a \Rightarrow c = c' \) ✓

Part 2: stay tuned.
Moreover, preserves the boundedness of $X$. For $A \subseteq \mathbb{R}^2$, we define the unique point $x \in X$ such that $x \in A$.
\( F_x : g x = x \).
1. $g$ is elliptic. $\exists x : g^2 x = x$.

2. Every $g$-orbit is bounded. $\forall x : \exists \varphi x | n \in \mathbb{Z}$ bounded.

3. Some $g$-orbit is bounded. $\exists x : \exists \varphi x | n \in \mathbb{Z}$ bounded.

4. $g$ preserves some non-empty bounded set. $\exists \emptyset \neq A \subseteq X$ bounded: $g A = A$. 
1. \( g \) is elliptic. \( \exists x_0: g \cdot x_0 = x_0. \)
\[ d(g^k x, x_0) = d(x, g^{-k} x_0) = d(x, x_0) \]

2. Every \( g \)-orbit is bounded. \( \forall x: \{ g^n x \mid n \in \mathbb{Z} \} \) bounded.

3. Some \( g \)-orbit is bounded.
\[ \exists x: \{ g^n x \mid n \in \mathbb{Z} \} \) bounded. \]

4. \( g \) preserves some non-empty bounded set.
\[ \exists \emptyset \neq A \subseteq X \) bounded: \( g A = A. \)
Proof of Thm: To do: $4 \implies 0$.

Let $\emptyset \neq A \in X$ bounded with $gA = \lambda$.

Let $C_A \in X$ be the center of $A$.

Note that $gC_A$ is a center of $gA$.

So $gC_A$ is a center of $A$.

Since centers are unique, $gC_A = C_A$.

\[ \square \]

$D$ open half-disc in $\mathbb{R}^2$
§ Gluing geodesics

Lemma Let \( a, b, c \) (distinct) \( \in X \).
Suppose \( \exists u \in J_a, b \). \( v \in J_b, c \) s.t. \( d(u, b) + d(b, v) = d(u, v) \).
Then \( [a, b] \cup [b, c] = [a, c] \).
Proof To show: 
\[ \forall x \in [a,b] \forall y \in [b,c] : \]
\[ d(x,y) = d(x,b) + d(b,y). \]

We need to distinguish cases according to the relative positions of \( u, x \in [a,b] \), and \( v, y \in [b,c] \).
Case 1: \( x \in [a, b], \quad y \in [b, c] \).

- \( d(x, b) = d(u, b) - d(u, x) \)
- \( d(y, b) = d(v, b) - d(v, y) \)
- \( d(z, b) = d(u, b) - d(u, z) \)

\[
d(x, b) + d(b, y) = d(u, b) + d(b, v) - d(u, x) - d(v, y)
\]

\[
d(u, v) = d(u, x) - d(u, x) - d(v, y)
\]

Thus all 3 are inequalities. Write equalities in particular, \( d(x, z) = d(x, b) + d(b, y) \). \( \square \).
Case 2: \( u \in \{x, b\}, \ \nu \in \{\nu, y\} \).

If, for a contradiction,
\[ d(u, \nu) \neq d(x, b) + d(b, \nu) \]
then contradiction. Consider the comparison \( \Delta \) for \((x, y, b)\). Note that \((\tilde{x}, \tilde{y}, \tilde{b})\) is not degenerate.

\[
d(\tilde{u}, \tilde{v}) < d(\tilde{u}, \tilde{b}) + d(\tilde{b}, \tilde{v})
\]

Comp. Then \( d(u, \nu) \leq d(\tilde{u}, \tilde{v}) < \frac{1}{2} d(\tilde{u}, \tilde{b}) + d(\tilde{b}, \tilde{v}) = d(u, b) + d(b, \nu). \)
THM If \( g \) is hyperbolic, then it admits an axis, i.e., a geodesic line \((\equiv \mathbb{R})\) invariant under \( g \).

Moreover, \( 1g1 \) is achieved on this axis.

\[
\text{Rem Let } \sigma : \mathbb{R} \rightarrow X \text{ be this geodesic line. Then }\]

\[
g(\sigma(t)) = \sigma(t + 1g1). 
\]
Proof: \( \exists x_0 \in X : d(gx_0, x_0) = |g| = \inf d(gx, x) \).

Define \( x_n := g^n x_0 \ (n \in \mathbb{Z}) \).

Let \( l = \bigcup \{ x_n, x_{n+1} \} \subseteq X \).

Note: \( gx = l \). To show: \( l \) is a geodesic.

Enough to check inductively that \( \{ x_0, x_n \} \cup \{ x_n, x_{n+1} \} \)

is a geodesic (i.e., \( \{ x_0, x_{n+1} \} \)).

(Indeed, applying \( g^{-1} \), which is isometric, yields the result for \( \{ x_{n+1}, x_n \} \ \forall n \geq 0 \).)
Enough to deduce: $[x_{n-1}, x_n] \cup [x_n, x_{n+1}] = [x_{n-1}, x_{n+1}]$.

By the lemma, enough to find $u \in Jx_{n-1}, x_n$ and $v \in Jx_n, x_{n+1}$ s. t. $d(u, v) = d(u, x_n) + d(x_n, v)$.

Take $u = \text{mid}(x_{n-1}, x_n)$; $v = \text{mid}(x_n, x_{n+1})$.

Note: $gu = v \Rightarrow d(u, v) \geq |g| 1$.

If (by contr.) $d(u, v) < d(u, x_n) + d(x_n, v) = \\
\frac{1}{2}d(x_{n-1}, x_n) + \frac{1}{2}d(x_n, x_{n+1}) = |g| 1$ absurd.

$d(x_0, x_1) = d(x_0, gx_0) = |g| 1$
So we proved that $g$ admits an axis $l$. Since $g$ is an isom of $\mathbb{R}$ and $d(g, x_n) = |g| \forall n$, we conclude that, on $\mathbb{R}$, $g/l$ is a translation by $|g|$. 

\[\square\]
\textbf{Gluing CAT(0) Spaces}

Let \( X, Y \) be metric spaces, \( p \in X, \ q \in Y \).

Define \((X \cup Y)_{p=q} = Z\)

Define \(d_Z\) on \(Z\) by \(d_{Z} |_{X^2} = d_{X}, \ d_{Z} |_{Y^2} = d_{Y}\)

and \(d_{Z}(x,y) = d_{X}(x,p) + d_{Y}(q, y)\).

Check that it is a distance!
THM If $X, Y$ are complete $\text{CAT}(0)$ then
so $\exists Z = X \cup Y$.

Proof Check $\text{CAT}(0)$ inequality.

Up to symmetries, there are 2 non-trivial cases:

1. $x, y \in X$, $z \in Y$.
To show:

\[ 4d^2(z,w) \leq 2d^2(z,x) + 2d^2(x,y) - d^2(x,y). \]

i.e.

\[ 0 \leq 2(\alpha + d(p,x))^2 + 2(\alpha + d(p,y))^2 - d^2(x,y). \]

If \( \alpha = 0 \): OK since \( X \in \text{CAT}(0) \).

\[
\frac{\partial}{\partial \alpha} (\text{RHS}) = 4(\alpha + d(p,x)) + 4(\alpha + d(p,y)) - 0
\]

\[
= 4 \left( d(p,x) + d(p,y) - 2d(p,w) \right) \geq 0. \quad \text{Prop} \checkmark
\]
Case 2. \( x \in X; \ y, z \in Y \)

Subcase: \( m = \text{mid}(x, y) \in X \).

Base: \( m = p \).

To show: \( 4 \frac{d^2(z, m)}{d^2(x, y)} \leq 2 \left( \frac{d^2(z, x)}{d^2(x, y)} + \frac{d^2(z, y)}{d^2(x, y)} \right) - \frac{d^2(x, y)}{(d(z, p) + d(p, x))^2} \frac{4}{d^2(x, p)} \\
\leq 2 \left( \frac{d^2(z, x) + d^2(z, y)}{d^2(x, y)} \right) - \frac{d^2(x, y)}{(d(z, p) + d(p, x))^2} \frac{4}{d^2(x, p)} \\
i.e., \ 4 \frac{d^2(z, p)}{d^2(x, y)} \leq 2 \frac{d^2(z, x)}{d^2(x, y)} + 2 \frac{d^2(z, y)}{d^2(x, y)} + \frac{4}{d^2(x, p)} d(z, p) d(p, x) \\
\frac{2}{d^2(x, p)} + \frac{d^2(z, y)}{d^2(x, y)} - \frac{4}{d^2(x, p)} d(z, p) d(p, x) \leq d^2(z, y) \).
The triangle inequality states that:

\[ d(x, p) - d(x, y) \leq d(y, p) \]
To show (general case):

\[ 4d^2(\tau, m) \leq 2(d^2(\tau, x) + d^2(\tau, y)) - d^2(x, y) \]

\[ \alpha := d(p, m). \text{ Ok if } \alpha = 0. \]

Enough to show that

\[ 2\alpha (2d^2(\tau, x) + 2d^2(\tau, y) - d^2(x, y) - 4d^2(\tau, m)) \]

\[ \geq 0 \]

To show:

\[ 2d(\tau, x) \geq d(x, y) + 2d(\tau, m) \]

become \( x, m, \tau \) on a geodesic.
Eagle. Any (combinatorial) graph can be considered as a metric space, provided it is connected, by identifying every edge with $[0, 1] \subseteq \mathbb{R}$. This space is $\text{CAT}(0) \Rightarrow$ the graph is simply connected.

\[ \Rightarrow \text{it is a "tree".} \]
Ex. of CAT(0) spaces
Let $M$ be a Riemannian manifold. Then

**FACT** $M$ is CAT(0) $\iff$

1. $M$ is simply connected.
2. The (sectional) curvatures of $M$ is $\leq 0$.

Ex. Hyperbolic $n$-space:

More generally:

$\mathbb{H}^n$ Lobachevsky

Symmetric spaces (of non-compact type) Mimobushi group on $\mathbb{R}^{n+1}$

Bunke-Tits buildings $\leftrightarrow$ Alg. groups $/$ $\mathbb{Q}_p$
Observation: If $X, Y$ are CAT($0$), then so is $X + Y$ with the distance

$$d = \sqrt{d_x^2 + d_y^2}.$$ 

Because the CAT($0$) inequality is a linear condition w.r.t. the squares of distances.
§ An example of a parabolic isometry

Let $X = \ell^2(\mathbb{Z}) := \{(a_n)_{n \in \mathbb{Z}} : \sum_n a_n^2 < +\infty\}$

Then, every finite subset of $X$ is contained in an isometric copy of $\mathbb{R}^n$ (for some $n \in \mathbb{N}$).

$\implies X$ is in CAT(0). ($d(a,b) := \|a - b\|$).
Define $\tau : X \to X$ by $(\tau a)_n = a_{n-1}$ ("shift" or "rotation").

This is elliptic: $a = 0$ is fixed!

Lemma: $0$ is the only fixed pt. of $\tau$.

Proof: if $\tau a = a$ then $a_n$ is constant in $n$.

But $\sum a_n < +\infty \implies a_n \to 0$.

Another idea: in $a \to a + \delta_0$

($\delta_0, n = 90^\circ$ if $n > 0$)

This is hyperbolic:

$d(a, a + \delta_0) = 1$.

$\| \delta_0 \|$.
Let $T \in Is(X)$ be defined by $T\alpha = Ta + S_0$. 

**THM** $T$ is parabolic.

**Ex** Prove it by studying $|T|$.

**Proof of THM**. 

Step I: $T$ is not elliptic. 

Step II: $T$ has no axis.
Step 1: Suppose for contradiction: \( \exists a \in X: T_a = a \)

\[
(Ta)_n = (Ta + \delta_0)_n = a_{n-1} + \delta_0, n = a_n
\]

\( n = 0: \quad a_{-1} = a_0 - 1 \)

\( n = -1: \quad a_{-2} = a_{-1} \). In general, \( n < 0 \), by induction:

\[ a_n = a_0 - 1 \]

Since \( \|a\| < +\infty \), it follows:

\[ a_n = 0 \quad \forall n < 0 \]. Back to \( n = 0: \quad a_0 = 1 \).

\( n = 1: \quad a_0 = a_1 \). More generally, \( n > 0 \), by induction:

\[ a_n = a_0 = 1 \] contradicts \( \|a\| < +\infty \). \( \square \) Step 1.
Step II: Suppose for contradiction $T$ has an axis.

This means: \( T : \mathbb{R} \rightarrow X \) isometry

With: \( \forall t \in \mathbb{R} : Tl(t) = l(t+c) \)

(in fact we know \( c = 1 \| l \| > 0 \)).

\( l \) is of the form \( l(t) = u + t v \)

with \( u \in X \), \( v \in X \) with \( \| v \| = 1 \).

\( \| l(t) - l(s) \| = |t-s| \).
\[ T(k) = \nabla (u + tk) + \delta_0 = \ell(h+c) \]

Take limit \( \frac{1}{t} (\ldots) \) as \( t \to \infty \):

\[
\lim_{t \to \infty} \frac{1}{t} \frac{\nabla (u + tku)}{\nabla (u) + 1} = 0.
\]

\[
= \nabla (u) = 0.
\]

\[ \Rightarrow \text{Lemma: } \tau = 0 \]

Conclude \( \| \tau \| = 1 \)
Projections, I

Let $X$ be a complete CAT(0) space, $C \subseteq X$ a closed convex subset ($\neq \emptyset$).

Goal: Define a projection $\pi_C = \pi_C : X \to C$.

Theorem: For all $x \in X$ there is a unique closest point $p \in C$.

Definition: $\pi_C(x) := p$. 
Proof. Let \((p_n)\) be a sequence in \(C\) st.
\[
\lim_{n \to \infty} d(x, p_n) \to \inf_{y \in C} d(x, y).
\]
Claim \((p_n)\) is Cauchy.

Note: ① Claim \(\Rightarrow\) existence: \(p := \lim_{n \to \infty} p_n \in \overline{C}\) closed

② Claim \(\Rightarrow\) convergence:
if \(p, q\) minimize \(d(x, y)\) over \(y \in \overline{C}\)
then take \(p_n = \begin{cases} p & \text{if } n \text{ even} \\ q & \text{if } n \text{ odd} \end{cases} \to p = q.

**Proof of claim**  
\[ \mu := \text{midpoint } (p_n, p_m) = (\mu_n, \mu_m) \]

If \( C \) convex then \( \mu \in C \)

\[ \Rightarrow d(x, \mu) \geq \inf_{y \in C} d(x, y) =: I \]

- **CAT(0)**:  
  \[ 4d^2(x, \mu) \leq 2d^2(x, p_n) + 2d^2(x, p_m) \]

\[ 4I^2 \leq \quad \quad \quad 2I^2 \]

\[ d^2(p_n, p_m) \leq \left( 2d^2(x, p_n) + 2d^2(x, p_m) - 4I^2 \right) \to 0 \quad \text{as } n, m \to \infty \]
§ Angles.

Def. Given $a, b, c$ in $X$, the \textbf{comparison angle} $\overline{\angle}_a (b, c)$ is the Euclidean angle at $\overline{a}$ in $\Delta(a, b, c) \subseteq \mathbb{R}^2$ (i.e. $\overline{\Delta}^\mathbb{R}^2 (b, c)$).

Note. This is in $[0, \pi]$. 

Rem. Well-defined if $b, c \neq a$. 

Prop/Def: Let $a, b, c \in X$ (call $(X, \tau)$); $b, c \neq a$.

Let $\sigma: I \to X$ be good. segments from $a \to b$

$a \to c$

when $I = [0, d(a,b)]$, $J = [0, d(a,c)]$.

The **Alexandrov angle** is

$$\Delta_a(b,c) := \lim_{s, t \to 0} \overrightarrow{X_a}(\sigma(s),\; \tau(t)) .$$

This limit exists.
Proof

\[
\cos \Delta \omega (\tau (t), \tau (t)) = \frac{s^2 + t^2 - 1}{2s \pm t}.
\]

To show: $s \neq t$.

In fact, $\cos s \neq t \not< 0$.

This quotient decreases.

\[\alpha^2 = x^2 + y^1 - 2 \beta \gamma \cos \Delta \text{ in } \mathbb{R}^2\]

i.e. the angle decreases.
Let then \( 0 < s' \leq s, \ 0 < t' \leq t \).

In \( \mathbb{R}^3 \), \( \odot \) would be constant.

Apply the comparison theoer to \( \alpha, \sigma(x), \tau(t) \):

\[
\Delta_{\alpha}^{R^3}(\overline{\sigma(x)}, \overline{\tau(t)}) \leq d(\overline{\sigma(x)}, \overline{\tau(t)})
\]

But \( \Delta_{\alpha}^{R^3}(\overline{\sigma(x)}, \overline{\tau(t)}) = \Delta_{\alpha}^{R^3}(\overline{\sigma(x)}, \overline{\tau(t')}) \).

Conclusion \( \odot \odot \geq \odot \odot, \)

\( \odot \) because the only

\[
\square
\]
**Cor** \( \mathbb{X}_a(b, c) \leq \mathbb{X}_c(b, c) \)

**Ex 0.** If \( X = \mathbb{R}^n \), then \( X = \mathbb{F} \).

**Ex 1.** Take \( a, b, c \) generic in the tripod. Then \( \mathbb{X}_a(b, c) = 0 \).

(But \( \mathbb{X}_c(b, c) \) could e.g. be \( \pi / 3 \).)
**COR** Let \( b, c, q \neq a \). Then
\[
\Delta_a(b, q) \leq \Delta_a(b, c) + \Delta_a(c, q).
\]

Proof True in \( \mathbb{R}^2 = \text{True for } \overline{E} \).

Since \( \overline{E} \) is a limit of \( \overline{E} \), true \( \triangle \). \( \square \)
Lemma: Let $a, b, c, q$ be distinct. If $\angle (a, c)$ and $\angle (a, q) \geq \pi/2$ then $d(c, q) \geq d(a, b)$.

Proof: True in $\mathbb{R}^2$: 

$d(c, q) \geq d(a, b)$ because if $c, q$ cross this ship.
Consider caps \( \vec{a}, \vec{b}, \vec{c} \) and \( (\vec{\bar{a}}, \vec{\bar{c}}, \vec{\bar{c}}) \).

Since \( d(a, q) = d(\bar{a}, \bar{q}) \), we can drop WLOG \( \vec{x} \) and \( \vec{y} \).

\[
\begin{align*}
\text{cor.} & \quad 
\Delta_{\vec{a}}(\vec{b}, \vec{c}) = \Delta_{\vec{a}}(\vec{b}, \vec{q}) + \Delta_{\vec{c}}(\vec{q}, \vec{c}) \\
& \geq \Delta_{\vec{a}}(b, q) + \Delta_{\vec{a}}(q, c) \\
& \geq \frac{\pi}{2}.
\end{align*}
\]
1st case \[ \Rightarrow \ \text{if } d(\bar{b}, \bar{b}) = d(a, b) \]

\[ d(\bar{c}, \bar{q}) \]

where \( d(\cdot, \cdot) \) denotes some metric or distance function.
§ Projections, II

Naturality: Let $g$ be an element of $X$. Then

$$\Pi g C (g x) = g \Pi C (x).$$

Indeed: The RHS is in $g C$ and minimizes distance to $g x$. Thus uniquely defines $\Pi g C (g x)$.

Reg. Lilevra: $g g A = g C A$. 

Theorem: Let $C \subseteq X$ be $\neq \emptyset$, closed, convex; $\pi : X \rightarrow C$.

1. $\forall x \in C \exists \pi(x) \in \overline{B}(x, \pi(x)) = \{ y \in C : d(x, y) \leq \frac{1}{2}\}$.

2. $\forall x, y \in C, y \neq \pi(x)$:
   \[ d(x, \pi(x)) \leq \frac{1}{2}\]
   \[ d(x, y) \geq \frac{1}{2}\]
   \[ \Rightarrow d(x, y) \geq \frac{1}{2}\]

3. $\forall x, y \in X$:
   \[ d(\pi(x), \pi(y)) \leq d(x, y)\]
   \[ \Rightarrow \text{continuous.} \]
   \[ \{ \pi \text{ is } 1\text{-Lipschitz}. \]
Proof of (1).
Let \( y \in \{ x, \pi(x) \} \) to show: \( \pi(y) = \pi(x) \).

Enough to show: \( \forall z \in \mathbb{C}: d(y, \pi(x)) \leq d(y, z) \).

We know:
\[
d(x, \pi(x)) \leq d(x, z)
\]
1. geodesic
\[
d(x, y) + d(y, \pi(x)) \Rightarrow d(y, \pi(x)) \leq \left( d(x, z) - d(x, y) \right)
\]
\[
\leq d(y, z) + d(z, x) \leq d(x, z)
\]
Proof of (2). Suppose for contradiction that $x \notin \pi(x)$ and $y \in \pi(x)$.

For $p, q \in [x, \pi(x)]$ and $y \in \pi(x)$, we have $\frac{\pi(p, q)}{\pi(x)} < \frac{1}{2}$.

By (1), $\pi(p) = \pi(x)$.

Consider $\Delta$ for $(p, q, \pi(x))$.

For $z$ close enough to $\pi(x)$ on $\partial C$, $d(p, z) \leq d(\bar{p}, z) < d(p, \pi(x)) = d(p, x)$.
Proof of (3).

By (2), $\Delta_{\pi(z)}(x, \pi(y))$ and $\Delta_{\pi(y)}(y, \pi(z))$ are $\geq \frac{\pi}{2}$.

We conclude by the lemma on 4 points and 2 angles. This settles the generic case of 4 distinct points; the other cases are simpler.
Corollary: Let $A$ be a closed, convex, bounded set. Then $A \supseteq c_A$.

Proof: Let $p := \pi_A(c_A)$. Claim: $p$ is a center for $A$.

Note: claim $\Rightarrow p = c_A$.

To show: $\forall a \in A: d(a, p) \leq s_A$.

But $d(a, p) \leq d(a, c_A) \leq s_A$ by def. of $c_A$. $\square$
Corollary. \( g \circ \iota \in \mathcal{C}(X) \) and \( C \subseteq X \) closed, convex, \( \# \neq \emptyset \) and \( g \mid C = C \).

Then \( |g| = |g \mid C| \).

Proof. By def., \( |g| \leq |g \mid C| \). (infimum).

For \( \geq \), let \( \varepsilon > 0 \). \( \exists x \in X : d(gx, x) < |g| + \varepsilon \).

\( \pi(gx) = g \pi_C(x) \) because \( C = g \mid C \).

\[
\begin{align*}
&d(g \pi_C(x), \pi_C(x)) \\
&\leq d(gx, x) < |g| + \varepsilon \\
&\leq |g| + \varepsilon \min_{x \in C} \pi_C(x) \in C
\end{align*}
\]
Corollary \( g \in Is(X) \) has no fixed point.

If \( g \) preserves a geodesic line, then \( g \) is hyperbolic and this line is an axis.

(i.e. \( 1g1 \) is realized on this line).

Proof: Let \( L \) be this line. Previous corollary \( \Rightarrow 1g1 = 1g0c1 \). But \( g/c \) is a translation \((C = \mathbb{R})\). \( \Box \)
The length 1g₁, part II.

Theorem Let g ∈ Is(X). For all x ∈ X:

\[ 1g₁ = \lim_{n \to \infty} \frac{1}{n} d(g^n(x), x). \]

Proof (i) The limit exist, in fact it is

\[ \inf_{n} \frac{1}{n} d(g^n(x), x). \]

To show for (ii): \( a_n := d(g^n(x), x) \) is subadditive.

\[ a_{m+n} = d(g^{m+n}(x), x) \leq d(g^{m+n}(x), g^m(x)) + d(g^m(x), x) = d(g^m(x), x) + d(g^n(x), x) = a_m + a_n. \]
\[(ii) \lim_{n \to \infty} \frac{1}{n} d(\gamma^n, x) \leq 1 \text{ if and limit indep. of } x.\]

Indeed, \( \lim \inf_{n} \leq \frac{1}{n} d(\gamma^n, x) \quad \forall x. \)

The link does not dep. on \( x: \)

\[
\lim_{n \to \infty} \frac{1}{n} d(\gamma^n, y) \leq \lim_{n \to \infty} \frac{1}{n} \left( d(\gamma^n \gamma, \gamma^n \gamma) + d(\gamma^n \gamma, x) \right) = d(\gamma \gamma, y) = \lim_{n \to \infty} \frac{1}{n} d(\gamma^n \gamma, x) \quad \Rightarrow \text{ limit indep. of } x.
\]

Take \( \inf \) of \( \bigcirc \) over \( x \in X \)

\[ \Rightarrow \text{ limit } \leq 1. \]
Finally: \( \lim_{n \to \infty} \geq 1 \).

\[ \lim_{n \to \infty} z^g = \lim_{n \to \infty} g^n z. \]

If \( p \) is the midpoint of \( z^g, g^n z \),

\[ d(g^n p, p) \leq \frac{1}{2} d(g^{-n}, g^n z) = \frac{1}{2} d(g^{2n} z, z) = \frac{1}{2} d(g^n z, z). \]

\[ \Rightarrow \frac{1}{n} d(g^n p, p) \leq \frac{1}{2n} d(g^{2n} z, z). \]
For a contradiction, assume: \( \lim_{x \to \infty} d(g^x z, x) < 1 \).

Fix some \( x_0 \in X \). \( \exists N \forall n \geq N : \frac{1}{n} d(g^nx_0, x_0) < 1 \).

Take \( 2^k \geq N \) for some \( k \).

Let \( x_k \) be the midpoint \( p = \text{mid}(x_0, g^{-2^k}x_0) \).

Convexity + Thelin (previous page) \[ \frac{1}{2^{k-1}} d(g^{2^k-1}x_k, x_k) \leq \frac{1}{2^k} d(g^{2^k}x_0, x_0) < 1 \]

Repeat \( k \) times: gives \( x_k \in X \): \[ \frac{1}{2} d(g^{2^k}x_0, x_0) < 1 \] Absurd.
\[ \text{Cor} \quad |g^n| = 1 n \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad A \quad n \in \mathbb{Z} \]

Proof \quad \begin{align*}
|g^\text{'}| &= \lim_{n \to \infty} \frac{1}{n} d(g, x, x) = \lim_{p \to \infty} \frac{1}{p} d(g^p, x) \\
&= \lim_{p \to \infty} \frac{1}{p} d(g^p, x) = n |x| \quad \blacksquare.
\end{align*}

\[ \text{Cor} \quad |gh^{-1}| = |g| \cdot |h| \]

If \quad \begin{align*}
|gh^{-1}| &= \lim_{n \to \infty} \frac{1}{n} d(gh^{-1}, x, x) \\
d(g, y, y) &= \|y = h^{-1}.
\end{align*}

In general \quad |gh| \neq |g| \cdot |h|. \quad \blacksquare
§ Convexity. Recall: if $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$, then $f$ is called convex if $\forall x, y \in I \forall t \in [0, 1]$:

$$f \left( (1-t)x + ty \right) \leq (1-t)f(x) + t f(y)$$

\[ \text{Diagram:} \quad q \quad p \]
Def a function $f : X \to \mathbb{R}$ is \underline{convex} if \underline{$\forall$ geodesic $\gamma : I \to X$ that}

In,

funk $f : I \to \mathbb{R}$ in \underline{convex}. (Sub-level set)

Rem If $f$ is convex, then the set $\{ x \in X | f(x) \leq c \}$ is convex for all $c \in \mathbb{R}$ (Sub-level set).

(Please verify.)
Def: $\gamma: I \to X$ is an **affine geodesic**

if $I \subset \mathbb{R}$ and $\gamma(0) = \gamma(1)$ are geodesics

$\frac{1}{n} I \to X$

not a geodesic: more general.

Observation: $f: X \to \mathbb{R}$ is convex

and $f$ is convex "on" any affine geod.

i.e. $f \circ \gamma$ is convex $\forall$ affine geod. $\gamma$.

And $f \circ \gamma$ is convex $\forall$ affine geod $\gamma: [0, 1] \to X$. 
Prop: The metric $d$ of a $\text{CAT}(0)$ space $X$ is convex.

Interpretation 1: $d : X \times X \rightarrow \mathbb{R}$ is convex, i.e. $\forall$ (aff.) geodesic $\gamma : I \rightarrow X$, $d \circ \gamma : I \rightarrow \mathbb{R}$ is a convex function.

Interpretation 2: $\forall$ geodesics $\gamma_1, \gamma_2 : I \rightarrow X$, the function $I \rightarrow \mathbb{R}$ $t \mapsto d(\gamma_1(t), \gamma_2(t))$ is convex. $\mathbb{Q} \Rightarrow \mathbb{R}$.
\[ \sigma_1 : [0, 1] \rightarrow X \]

**Proof of prop.** To show: given all good, \( \sigma_i : [0, 1] \rightarrow X \)

**WLOG** the function \( f \mapsto d(\sigma_1(1), \sigma_2(t)) \)

is convex (\( [0, 1] \rightarrow \mathbb{R} \)).

Let \( \tau : [0, 1] \rightarrow X \) be an affine good from \( \tau(0) \) to \( \tau(1) \).
Step 0. It suffices to show: \( \forall b \in \mathbb{R}, \mu: \) 
\[
   d(\\sigma_1(t), \\sigma_2(t)) \leq (1-t) d(\sigma_1(0), \sigma_2(0)) + t d(\sigma_1(1), \sigma_2(1)).
\]

i.e. Step 0: it suffices to take \( \alpha = 0, \gamma = 1 \).
Step 1: Special case $\Sigma_1(0) = \Sigma_2(0)$

To show: $\forall \theta \in [0, \frac{\pi}{2}] : (x = 0, y = 1)$

$$d(\Sigma_1(t), \Sigma_2(t)) \leq t d(\Sigma_1(1), \Sigma_2(1))$$

$$\left[ + (1-t) d(\Sigma_1(0), \Sigma_2(0)) \right] = 0$$

True by comparison + Thales
General case

To show (by \( d \nu = 0 \)):

\[
\forall t \in [0, 1]:
\]

\[
d(\bar{\sigma}_1 (t), \bar{\sigma}_2 (t)) \leq (1-t) \ d(\bar{\sigma}_1 (0), \bar{\sigma}_2 (0)) + t \ d(\bar{\sigma}_1 (1), \bar{\sigma}_2 (1))
\]

But

\[
\leq d(\bar{\sigma}_1 (1), \tau (1)) + d(\tau (1), \bar{\sigma}_2 (1))
\]

\[
\leq t \ d(\bar{\sigma}_1 (1), \tau (1)) + (1-t) \ d(\bar{\sigma}_1 (0), \bar{\sigma}_2 (0))
\]

\( \text{case 1} \)

\( \bar{\sigma}_1 (1) \)

\( \bar{\sigma}_2 (1) \)

\( \bar{\sigma}_1 (0) \)

\( \bar{\sigma}_2 (0) \)

\( \tau (1) \)

\( \text{case 2} \)
\textbf{Cor 1} \quad \forall x \in X : \quad d(x, \cdot) \quad \text{is convex.}

[Special case where $\mathcal{T}$ is constant.]


\textbf{Cor 2} \quad \text{for } G \subseteq X \neq \emptyset \quad \text{convex closed.}

Then \quad d(G, \cdot) \quad \text{is a convex function,}

\begin{align*}
\text{where} \quad d(G, x) : &= \inf_{c \in G} d(c, x) \\
&= d(x, \overline{G}(x)).
\end{align*}
Proof: Enough to show: \( V : \mathbb{C} \to X, \)
\[ \forall t \in \mathbb{C}, \pi(\sigma(t)) = \pi(t), \]
where \( \sigma(t) \) is the trajectory defined by the differential equation:
\[ \frac{d\sigma(t)}{dt} = \mathbb{I} - d(\sigma(t), \mathcal{C}). \]

The condition is satisfied if the distance from \( \sigma(t) \) to \( \mathcal{C} \) remains constant:
\[ d(\sigma(t), \mathcal{C}) = \text{constant}. \]

Proposition: \( \pi(t) \) is a good flow, from \( \pi(\sigma(t)) \) to \( \pi(t) \), where \( t \) remains in \( \mathbb{C} \).
Ex. Take $g$ is Tsom ($X$).

Then $x \mapsto d(gx, x)$

$X \rightarrow \mathbb{R}$ is convex.