Analysis on groups

Problem Set 7  3 November 2021

Exercise 0
Let \( G \) be a group, \( f \in \ell^\infty(G) \) and \( h \in \ell^1(G) \). Show that the following are equal for all \( x \in G \) (and pay attention to absolute convergence).

\[
\sum_{y \in G} f(xy^{-1}) h(y), \quad \sum_{y \in G} f(y) h(y^{-1}x), \quad \sum_{\{y,z \in G : yz=x\}} f(y) h(z).
\]

Exercise 1
Let \( f,g,h \) be functions on a group \( G \) and let \( x,y \in G \).

(i) Verify that \( f \ast (g \ast h) = (f \ast g) \ast h \) holds (as soon as all these sums are absolutely convergent).

(ii) Compute \( \delta_x \ast \delta_y, \delta_x \ast f \) and \( f \ast \delta_x \).

(iii) Find a formula for \( (f \ast g)^\vee \), where for any function \( h \) we write \( h^\vee(x) := h(x^{-1}) \).

(iv) Let \( 1 \leq p \leq \infty \) and suppose \( f \in \ell^p(G), g \in \ell^1(G) \). Verify \( \|g \ast f\|_p \leq \|f\|_p \cdot \|g\|_1 \). Deduce \( \|f \ast g\|_p \leq \|f\|_p \cdot \|g\|_1 \). (You might want to separate the case \( p = \infty \).)

(v) For \( f \in \ell^1(G) \), write \( \Sigma f := \sum_{x \in G} f(x) \). Assuming \( f,g \in \ell^1(G) \), prove \( \Sigma(f \ast g) = \Sigma f \Sigma g \). Deduce that \( f,g \in \text{Prob}(G) \) implies \( f \ast g \in \text{Prob}(G) \).

Exercise 2
On the group \( \mathbb{Z} \) with the probability measure \( \mu = \frac{1}{3} \delta_{-1} + \frac{2}{3} \delta_1 \), determine all \( \mu \)-harmonic functions.

Exercise 3
In class, we found a function \( f : F_2 \to \mathbb{R} \) on \( F_2 = \langle a,b \rangle \) that is \( \mu \)-harmonic for

\[
\mu = \frac{1}{4}(\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}}).
\]

Find your own \( \mu \)-harmonic function on \( F_2 \) for the same measure \( \mu \); try not to choose simply a linear combination of \( f \) and \( 1_G \).