Exercise 0
Let $G$ be a group, $f \in \ell^\infty(G)$ and $h \in \ell^1(G)$. Show that the following are equal for all $x \in G$ (and pay attention to absolute convergence).

$$\sum_{y \in G} f(xy^{-1}) h(y), \quad \sum_{y \in G} f(y) h(y^{-1}x), \quad \sum_{\{y,z \in G : yz = x\}} f(y) h(z).$$

Exercise 1
Let $f, g, h$ be functions on a group $G$ and let $x, y \in G$.

(i) Verify that $f \ast (g \ast h) = (f \ast g) \ast h$ holds (as soon as all these sums are absolutely convergent).

(ii) Compute $\delta_x \ast \delta_y$, $\delta_x \ast f$ and $f \ast \delta_x$.

(iii) Find a formula for $(f \ast g)\check{}$, where for any function $h$ we write $h\check{}(x) := h(x^{-1})$.

(iv) Let $1 \leq p \leq \infty$ and suppose $f \in \ell^p(G)$, $g \in \ell^1(G)$. Verify $\|g \ast f\|_p \leq \|f\|_p \cdot \|g\|_1$. Deduce $\|f \ast g\|_p \leq \|f\|_p \cdot \|g\|_1$. (You might want to separate the case $p = \infty$.)

(v) For $f \in \ell^1(G)$, write $\Sigma f := \sum_{x \in G} f(x)$. Assuming $f, g \in \ell^1(G)$, prove $\Sigma (f \ast g) = (\Sigma f) (\Sigma g)$. Deduce that $f, g \in \text{Prob}(G)$ implies $f \ast g \in \text{Prob}(G)$.

Exercise 2
In class, we found a function $f : F_2 \to \mathbb{R}$ on $F_2 = \langle a, b \rangle$ that is $\mu$-harmonic for

$$\mu = \frac{1}{4} (\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}}).$$

Find your own $\mu$-harmonic function on $F_2$ for the same measure $\mu$; try not to choose simply a linear combination of $f$ and $1_G$.

Exercise 3
On the group $\mathbb{Z}$ with the probability measure $\mu = \frac{1}{3} \delta_{-1} + \frac{2}{3} \delta_1$, determine all $\mu$-harmonic functions.