This problem set still has some preparatory stuff and Banach spaces. Starting next week, all problems will be directly related to the class!

Exercise 1 \( (\text{Useful for Ex. 3 of Set 1...}) \)
Let \( X \) be a set and define \( \ell^1(X) = \{ f : X \to \mathbb{R} : \|f\|_1 < \infty \} \), where \( \|f\|_1 = \sum_{x \in X} |f(x)| \).
Make sure that you really understand the definition of \( \sum_{x \in X} |f(x)| \) and deduce that \( \sum_{x \in X} f(x) \) makes sense for \( f \in \ell^1(X) \). Give a formal definition of \( \sum_{x \in X} f(x) \).

Exercise 0
Prove the Alaoglu Theorem:
Given any Banach space \( V \), the set
\[
B = \{ \lambda \in V^* : \|\lambda\| \leq 1 \}
\]
is weak-* compact.

\text{Hint: apply the Tychonoff theorem to some huge product of compact spaces.}
\text{Note: B is usually called the “closed unit ball”, so maybe a first step is to clarify in your mind in which topologies } B \text{ is closed...}

Exercise 1
For \( j \in \mathbb{Z} \), consider the mean \( \delta_j \in \mathcal{M}(\mathbb{Z}) \).
We define a sequence \( \{\mu_n\} \) in \( B \) by
\[
\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_j.
\]
Although \( \mathcal{M}(\mathbb{Z}) \) is compact, prove that \( \{\mu_n\} \) has no convergent subsequence.

Exercise 1’
We define \( \delta_j(i) \) by 1 if \( i = j \) and 0 if \( i \neq j \). We consider \( \delta_j \) as an element of \( \ell^1(\mathbb{Z}) \subseteq \ell^\infty(\mathbb{Z})^* \).
We endow the closed unit ball \( B \) of \( \ell^\infty(\mathbb{Z})^* \) with the weak-* topology and we define a sequence \( \{\mu_n\} \) in \( B \) by
\[
\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_j.
\]
Although \( B \) is compact by Ex. 0, prove that \( \{\mu_n\} \) has no convergent subsequence.

Exercise 2
Let \( X \) be a set and \( m \in \ell^\infty(X)^* \). Consider the following properties:
\begin{enumerate}
  \item \( m(f) \geq 0 \) if \( f \geq 0 \).
  \item \( m(1_X) = 1 \), where \( 1_X \) is the constant function 1.
  \item \( \|m\| = 1 \).
\end{enumerate}
Show that any two of these properties imply that all three hold.