Abstract Analysis on Groups

Problem Set 2 27 September 2023

**Exercise 0 (Functional Analysis)**
Prove the **Banach–Alaoglu Theorem**:
Given any Banach space $V$, the set

$$B = \{ \lambda \in V^* : \| \lambda \| \leq 1 \}$$

is weak-* compact.

*Hint: apply the Tychonoff theorem to some huge product of compact spaces.*

*Note: $B$ is usually called the “closed unit ball”, so maybe a first step is to clarify in your mind in which topologies $B$ is closed...*

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**Exercise 1**
For $j \in \mathbb{Z}$, consider the mean $\delta_j \in \mathcal{M}(\mathbb{Z})$.
We define a sequence $\{\mu_n\}$ in $B$ by

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_j.$$

Although $\mathcal{M}(\mathbb{Z})$ is compact, prove that $\{\mu_n\}$ has no convergent subsequence.

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**Exercise 2**
(i) Prove that the image of the Dirac mass map $\delta: X \to \mathcal{M}(X)$ is a discrete subspace.

*From now on, we identify (abusively) this image with $X$. Define $\beta X$ to be the closure of $X$ in $\mathcal{M}(X)$.***

(iii) Give a precise proof that $\beta X \neq X$ when $X$ is infinite.

(iii) Prove $\beta X = \{ \mu \in \mathcal{M}(X) : \mu(A) \in \{0,1\} \forall A \subseteq X \}$.  
(iv, optional) Prove that $\beta N$ is a separable compact space but is not second countable.

*Hint: think about Ex. 1 above; also, if needed, use Ex. $\infty$ below.*

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**Exercise $\infty$ (Topology, optional)**
Recall the definitions of “second countable”, “metrizable” and “separable”. Prove the following.

(i) A metrizable space is second countable iff it is separable.

(ii) A compact space is second countable iff it is metrizable.

(iii) A compact space $K$ is metrizable iff the Banach space $C(K)$ (with sup-norm) is separable.