Analysis on groups

Problem Set 12 8 December 2021

**Exercise 0**
Check that the entire proof of the (Hausdorff–) Banach–Tarski paradox holds more generally in $\mathbb{R}^n$ for all $n \geq 3$.
What goes wrong for $n = 2$?

**Exercise 1 (Flash-back to Reiter)**
Let $G$ be a group acting on a set $X$; remind yourself of the path of reasoning showing that $(R_2)$ is equivalent to amenability.

Prove that the following condition is equivalent to $(R_2)$:
For every $S \subseteq G$ and every $\varepsilon > 0$ there is $v \in \ell^2(X)$ with
\[
\left\| \frac{1}{|S|} \sum_{s \in S} sv \right\|_2 > (1 - \varepsilon)\|v\|_2.
\]

How about $(R_p)$? Pay special attention to $p = 1$.

**Exercise 2**
We denote by $SO(3)$ the group of $3 \times 3$ orthogonal (real) matrices with determinant one. We denote by $SU(2)$ the group of $2 \times 2$ unitary (complex) matrices with determinant one.

i) Make sure that you understand these definitions and the fact that $SO(3)$ is isomorphic to the group of rotations around the origin in $\mathbb{R}^3$.

ii) Understand the statement and the proof of the fact that there is a surjective homomorphism $SU(2) \to SO(3)$. What is the kernel?

*Hint for finding the homomorphism: try to use complex numbers to describe real vectors.*