Exercise 0
We saw the following facts in the proof of this week’s theorem. Provide your own precise proofs:

(i) For absolutely convergent series: \( \left( \sum_{k=1}^{\infty} t_k \right)^n = \sum_{k_1, \ldots, k_n=1}^{\infty} t_{k_1} \cdot \cdots \cdot t_{k_n} \). Likewise for convolutions.

(ii) Let \( \vartheta, \alpha, \eta \in \text{Proba}(G) \). If \( \alpha \) is \((B, \epsilon)\)-invariant and if the support of \( \vartheta \) is contained in \( B \), then

\[ \| \vartheta \ast \alpha \ast \eta - \alpha \ast \eta \|_1 < \epsilon. \]

Exercise 1
Let \( G \) be a countable amenable group. Prove that there is a probability measure \( \mu \) on \( G \) such that

for all \( g \in G \) we have

\[ \lim_{n \to \infty} \| g \mu^{*n} - \mu^{*n} \| = 0 \]

and also

\[ \lim_{n \to \infty} \| \mu^{*n} g - \mu^{*n} \| = 0. \]

To do this, prove that there is \( \mu \) such that \( \mu = \tilde{\mu} \) and the first limit holds, recalling that \( \tilde{\mu}(g) = \mu(g^{-1}) \). (Why is this sufficient?)

This exercise is just an excuse to make you go through all the details of the theorem proved in class this week. Thus, make sure that you justify in detail every step and every inequality! You are not allowed to write “... and then we proceed like in class...”

Exercise 2 (optional; cultural facts about rotations)
We denote by \( \text{SO}(3) \) the group of \( 3 \times 3 \) orthogonal (real) matrices with determinant one. We denote by \( \text{SU}(2) \) the group of \( 2 \times 2 \) unitary (complex) matrices with determinant one.

(i) Make sure that you understand these definitions and the fact that \( \text{SO}(3) \) is isomorphic to the group of rotations around the origin in \( \mathbb{R}^3 \).

(ii) Understand the statement and the proof of the fact that there is a surjective homomorphism \( \text{SU}(2) \to \text{SO}(3) \). What is the kernel?

Hint for finding the homomorphism: try to use complex numbers to describe real vectors.