This problem set is not representative of “Analysis on groups”. We just want to collect some useful tools from topology and functional analysis. Then we will move on!

Exercise -1
Read http://en.wikipedia.org/wiki/Net_(mathematics) or use the library to understand the concept of nets, also known as generalized sequences. Understand the connection to open and closed sets, to continuous maps, to compactness. In particular, make sure to understand subnets.

The bottom line is that nets behave like sequences, but indexed by a larger set than \( \mathbb{N} \), in the sense that all first-year ideas connecting sequences and continuity, open/closed sets, etc., hold — except they hold for general situations, not just first-year analysis. This would be false for sequences. Test: is a subnet of a sequence a sequence?

Exercise 0
Make sure that you know the following notions, using, if needed, the library and Wikipedia:

- the dual \( V^* \) of a normed vector space \( V \),
- the weak topology on \( V \) and the weak-* topology on \( V^* \),
- the norm of a linear map between normed vector spaces,
- “the” Hahn–Banach theorem for normed spaces (see e.g. the section “important consequences” on the English Wikipedia).

Exercise 1
Let \( V \) be a Banach space. Define \( \iota : V \to V^{**} \) by \( \iota(v)(f) = f(v) \). Prove:

(i) \( \iota \) is well-defined, linear and isometric.
(ii) \( \iota \) is a homeomorphism onto its image when \( V \) has the weak topology and \( V^{**} \) the weak-* topology.
(iii) \( \iota(V) \) is weak-* dense in \( V^{**} \).
(iv) If \( \lambda \in V^{**} \) is weak-* continuous as a map \( V^* \to \mathbb{R} \), then \( \lambda \in \iota(V) \).

Exercise 2
Let \( V, W \) be Banach spaces.

(i) Given a continuous linear map \( \alpha : V \to W \), define \( \alpha^* : W^* \to V^* \) and check that \( \|\alpha^*\| = \|\alpha\| \).

(ii) Let \( \beta : W^* \to V^* \) be a continuous linear map. Prove that \( \beta = \alpha^* \) for some \( \alpha : V \to W \) if and only if \( \beta \) is continuous for the weak-* topologies. Hint: use Ex. 1(iv).

Exercise 3
Let \( X \) be a set and define \( \ell^1(X) = \{ f : X \to \mathbb{R} : \|f\|_1 < \infty \} \), where \( \|f\|_1 = \sum_{x \in X} |f(x)| \).

(i) Make sure that you really understand the definition of \( \sum_{x \in X} |f(x)| \) and deduce that \( \sum_{x \in X} f(x) \) makes sense for \( f \in \ell^1(X) \). Give a formal definition of \( \sum_{x \in X} f(x) \).

(ii) Given \( a \in \ell^1(X) \) and \( b \in \ell^\infty(X) \), we can define the number \( \langle a, b \rangle = \sum_{x \in X} a(x)b(x) \). (BTW: why?) Use this to construct an isometric isomorphism between \( \ell^\infty(X) \) and \( \ell^1(X)^* \).

Additional question: why does it not similarly identify \( \ell^1(X) \) with \( \ell^\infty(X)^* \)?