# LIE GROUPS AS PERMUTATION GROUPS: ULAM'S PROBLEM IN THE NILPOTENT CASE

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ABSTRACT. Ulam asked whether every connected Lie group can be represented on a countable structure. This is known in the linear case. We establish it for the first family of non-linear groups, namely in the nilpotent case. Further context is discussed to illustrate the relevance of nilpotent groups for Ulam's problem.

### 1. Introduction

Cayley's principle states that every group is a permutation group. In particular, every finite group can be faithfully represented in a symmetric group Sym(n) for some  $n \in N$ .

For infinite groups, the situation becomes immediately more interesting. We shall investigate it at the light of the following notion:

**Definition.** A group is **countably representable** if it admits a faithful action on some countable set.

In other words, *G* is countably representable if it can be realised as a subgroup of  $Sym(\aleph_0)$ . This is also equivalent to admitting a faithful linear representation on some countable vector space.

Of course the question of countable representation only arises for groups whose cardinality does not exceed the continuum cardinal  $c = 2^{\aleph_0}$  since this is the size of Sym( $\aleph_0$ ). On the other hand the question is non-trivial only for uncountable groups.

This leaves us with a very rich landscape of groups to investigate, because the groups of size c include all (non-discrete) Polish groups, thus in particular all (non-discrete) locally compact second countable groups, and still more particularly all (non-trivial) connected Lie groups.

At first sight, the familiar Lie groups have no obvious countable representation; such a representation would be non-continuous, even non-measurable, and indeed in Solovay's model [Sol70] they do not have any. But in November 1935, Schreier and Ulam proved the following (see Section 2 for the elementary argument):

**Proposition 1** (Schreier–Ulam). *The group* **R** *is countably representable.* 

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#### NICOLAS MONOD

This is Item 95 in the translated Scottish Book [Ula58], although we only found it as a question in the original version [Scottish].

Ulam later asked a question that remains open to this day:

**Problem** (Ulam). *Is every connected Lie group countably representable?* 

(See [Ula60, II.7] and [Ula64, V.2].)

This question is really the key to the wider world of locally compact groups, because the solution to Hilbert's fifth problem leads to the following fact (see Section 4):

**Proposition 2.** If the answer to Ulam's Problem is affirmative, then every locally compact second countable group is countably representable.

On the ominous side, some of the most familiar Polish groups spectacularly fail to be countably representable. For instance, it follows from the Rosendal–Solecki automatic continuity theorem [RS07] that any action of  $G = \text{Homeo}(S^1)$  or  $G = \text{Homeo}(\mathbb{R})$  on a countable set is trivial on the neutral component of G (which is of index two in this case). This was generalised to arbitrary compact manifolds by Mann [Man16]. Another very familiar group without any non-trivial action on a countable set is the unitary group of the infinite-dimensional separable Hilbert space, by Tsankov's automatic continuity [Tsa13].

In the positive direction, Proposition 1 can be generalised to show that in the abelian case there is no obstruction at all beyond the obvious size restriction (see Section 4 for two proofs):

**Proposition 3** (de Bruijn [dB64]). Every abelian group is countably representable as long as its cardinality does not exceed c.

Returning to Lie groups and Ulam's Problem, there is a much stronger result than Proposition 1, namely:

**Theorem 4** (Thomas [Tho99]). Every linear Lie group is countably representable.

This was first established in §2 of [Tho99]; the proof was rediscovered by Kallman [Kal00] and later Ershov–Churkin [EC04]; see Section 3 for the argument.

Therefore it remains to settle Ulam's Problem for non-linear Lie groups. The first example is Birkhoff's reduced Heisenberg group [Bir36], a central extension of  $\mathbf{R}^2$  with center  $\mathbf{S}^1$ . More generally, a connected nilpotent Lie groups is non-linear unless it is a semi-direct product of a *simply connected* group by a torus [HN12, 16.2.7]. Thus nilpotent Lie groups include a large family of non-linear groups; we can answer Ulam's question in this case:

## **Theorem 5.** Every nilpotent connected Lie group is countably representable.

It is fair to ask whether this result just follows formally from the case of abelian groups by taking successive extensions and invoking a general stability property of the class of countably representable groups. It turns out

that there is no such general stability: McKenzie [McK71] has constructed a nilpotent group M which is not countably representable, but is a central extension

$$0 \longrightarrow \mathbf{Z} \longrightarrow M \longrightarrow B \longrightarrow 0$$

where *B* is the free Z/2Z-module of rank c. Of course this group is not a Lie group, but it is nilpotent and of size c. This shows in particular that countable representability is not preserved by central extensions (see Section 5). A close variant of this group was considered by Churkin [Chu05]. As we shall see, these obstructions to representability still hold for the corresponding extensions with finite center Z/2Z instead of Z, and for certain extensions of the form

 $0 \longrightarrow \mathbf{R} \longrightarrow H \longrightarrow \mathbf{R} \longrightarrow 0.$ 

In fact, Churkin conjectured that his group is not a subgroup of any separable (Hausdorff) topological group [Chu05, Rem. 3]. We confirm this:

**Theorem 6.** Any homomorphism from McKenzie's or Churkin's nilpotent group to any separable Hausdorff topological group is trivial on the center.

The same statement holds for homomorphisms to Lindelöf Hausdorff topological groups.

This is a strengthening of not being countably representable since  $Sym(\aleph_0)$  is Polish, in particular *both* separable and Lindelöf.

We note that Lindelöf groups include e.g. all compact groups and more generally any countable product of  $\sigma$ -compact groups, see [Com84, 8.1(ii)].

Besides nilpotent groups, perhaps the best known example of a nonlinear Lie group is Cartan's example [Car36, SV] of the universal cover of  $SL_2(\mathbf{R})$ , or already its double cover, the metaplectic group. There, the methods of the present note seem to fall short:

**Problem.** Let  $G = \mathbf{SL}_n(\mathbf{R})$  and let  $\widetilde{G}$  be its universal cover. Is  $\widetilde{G}$  countably representable?

Recall that  $\widetilde{G}$  is a central extension of G by  $\mathbb{Z}/2\mathbb{Z}$  when  $n \ge 3$  and by  $\mathbb{Z}$  when n = 2. We observe below (Remark 14) that the approach used in the previous cases cannot work for these groups.

**Remark 7.** Ulam did not explicitly assume connectedness, but this is only a matter of terminology. Under the convention that Lie groups are second countable, they have at most countably many connected components and hence we are immediately reduced to the connected case (Lemma 11 below).

If on the other hand no such restriction is made, then we can trivially obtain disconnected Lie groups that are not countably representable by considering a suitably large group as zero-dimensional Lie group.

#### NICOLAS MONOD

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#### 2. Preliminary observations

We call a subgroup H < G cocountable if it is of countable index in *G*, i.e. if the coset space G/H is countable.

Then *G* is countably representable if and only if it admits a sequence of cocountable subgroups  $G_n < G$  ( $n \in \mathbb{N}$ ) with trivial intersection  $\bigcap_n G_n$ . Indeed, given such a sequence, *G* is represented as a permutation group of the countable disjoint union  $\bigsqcup_n G/G_n$  and the converse follows by taking  $G_n$  to be the point stabilizers in any countable representation. Furthermore, there is no loss of generality in assuming that the sequence  $G_n$  is decreasing, as is seen by considering finite intersections.

We shall often use tacitly the fact that the intersection of a cocountable group with any subgroup remains cocountable in the latter.

Since the (direct) product of any family of symmetric groups acts faithfully on the disjoint union of the corresponding sets, we have:

**Lemma 8.** The class of countably representable groups is closed under finite and countable products.  $\Box$ 

The class being closed under passing to subgroups, this further implies:

**Lemma 9.** The class of countably representable groups is closed under taking inverse limits of countable inverse systems.  $\Box$ 

The Schreier–Ulam 1935 proof that **R** is countably representable was probably as follows; this argument can also be found in [KS56, p. 65].

*Proof of Proposition 1.* By Lemma 8, the countable power  $\mathbf{Q}^{\mathbf{N}}$  is countably representable. Since this group is uniquely divisible, it is a **Q**-vector space. Its dimension is  $\mathfrak{c}$  and therefore it is isomorphic to **R**.

We will establish countable representability for general abelian groups in Section 4 but already record the following:

**Corollary 10.** The circle group  $S^1$  is countably representable.

*Proof.* Viewing again **R** as a **Q**-vector space, we split off **Q** and obtain an isomorphism  $\mathbf{R}/\mathbf{Z} \cong \mathbf{Q}/\mathbf{Z} \oplus \mathbf{R}$ ; the result now follows from Proposition 1 (and Lemma 8).

Another basic hereditary property relies on the technique of induced actions:

**Lemma 11.** If G admits a countably representable cocountable subgroup, then G is itself countably representable.

*Proof.* Suppose that the group *G* admits a cocountable subgroup  $G_0 < G$  which acts faithfully on a countable set *X*. Define  $Y = (G \times X)/G_0$ , where  $G_0$  acts diagonally on  $G \times X$ , from the left on *G*. Then *Y* is countable since it admits a bijection with  $(G_0 \setminus G) \times X$ . We endow *Y* with a *G*-action by letting *G* act by right multiplication on *G* and trivially on *Y*, noting that this action on  $G \times X$  descends indeed to *Y*. The resulting action is faithful.

### 3. Proof of Theorem 5

We begin by recalling Thomas's proof for linear Lie groups:

Let *K* be the field of Puiseux series over  $\mathbf{Q}$  endowed with its valuation  $v: K \to \mathbf{Q} \cup \{\infty\}$ ; let *V* be its valuation ring. Then  $\mathbf{SL}_n(V)$  is a cocountable subgroup of  $\mathbf{SL}_n(K)$ , see [Tho99, Thm. 2.5]. The argument for that is a refinement of the fact that *discrete* valuations give rise to an action on the Bruhat–Tits building, which is countable: here *v* is not discrete on *K*, but *K* is the countable union of the subfields  $\overline{\mathbf{Q}}((t^{1/q}))$  of Laurent series on  $t^{1/q}$ , on each of which *v* is discrete.

By the Newton–Puiseux theorem, *K* is algebraically closed. Since moreover *K* has characteristic zero and cardinality  $\mathfrak{c}$ , there is a field isomorphism  $K \cong \mathbb{C}$ . We denote by  $V_{\mathbb{C}}$  the image in  $\mathbb{C}$  of the valuation ring of *K* and by  $L_{\mathbb{C}}$  the corresponding maximal ideal. Define the congruence subgroup  $\mathbf{SL}_m(V_{\mathbb{C}}; L_{\mathbb{C}})$  to be the kernel of the reduction map  $\mathbf{SL}_m(V_{\mathbb{C}}) \to \mathbf{SL}_m(\overline{\mathbb{Q}})$ . Then  $\mathbf{SL}_m(V_{\mathbb{C}}; L_{\mathbb{C}})$  is a cocountable subgroup of  $\mathbf{SL}_m(\mathbb{C})$  since  $\mathbf{SL}_m(\overline{\mathbb{Q}})$  is countable.

At this point it follows already that  $PSL_m(C)$  is countably representable (since it is simple); finally, any linear Lie group can be embedded into some  $PSL_m(C)$ .

Now we give a more explicit expansion of this argument in order to be able to adapt it for some non-linear groups.

Define the decreasing sequence of ideals  $L_n < V_C$  by  $\{x : v(x) > n\}$  under the chosen isomorphism  $K \cong C$ . Thus the decreasing sequence of groups  $SL_m(V_C; L_n)$  has trivial intersection.

# **Lemma 12.** The congruence subgroups $SL_m(V_C; L_n)$ are cocountable in $SL_m(C)$ .

*Proof.* As recalled above,  $SL_m(V_C)$  is cocountable in  $SL_m(C)$  by Thm. 2.5 in [Tho99]. Therefore it suffices to show that  $SL_m(V_C;L_n)$  is cocountable in  $SL_m(V_C)$ , i.e. that the ring  $V_C/L_n$  is countable. This is the case because any

given element of this ring is represented by a *finite* sum

$$\sum_{j=0}^{qn} \lambda_j t^{j/q}$$

for some choice of denominator  $q \in \mathbf{N}_{>0}$  and elements  $\lambda_j \in \overline{\mathbf{Q}}$  indexed by  $j \in \mathbf{N}$ ; here *t* is the formal variable of Puiseux series.

We will need the following additional observation:

**Lemma 13.** Let C < C be a finitely generated subgroup and let  $z \in C$  be an element not in C. Then there is n such that  $L_n$  does not meet the coset z + C.

*Proof.* Viewing everything in K rather than in C, the claim is that v is uniformly bounded over z + C; note that this coset does not contain 0. Letting D be the group generated by C and z, it suffices to prove the following claim:

For every finitely generated subgroup D < K, the valuation v is uniformly bounded over the non-zero elements of D.

The group *D* is free abelian of finite rank  $r \ge 1$ . We shall prove by induction on *r* that *v* takes at most r + 1 distinct values on *D*. The base case r = 1 holds because v(px) = v(x) for all  $x \in K$  and all non-zero  $p \in \mathbb{Z}$ ; this gives the two values v(x) and  $v(0) = \infty$  on *D* in rank one.

We turn to the induction step for  $r \ge 2$ . Let  $x_1, ..., x_r$  be a basis of *D*. We order the basis so that for some  $1 \le s \le r$  we have

$$v(x_1) = \dots = v(x_s) < v(x_j) \quad \forall j > s.$$

Given any element  $x = p_1 x_1 + \dots + p_r x_r$  in D (with  $p_i \in \mathbb{Z}$ ), write  $x' = p_1 x_1 + \dots + p_s x_s$  and consider its residue  $\omega \in \overline{\mathbb{Q}}$  at  $v(x_1)$ , i.e. the coefficient of  $t^{v(x_1)}$  in x'. We have  $v(x) = v(x') = v(x_1)$  unless  $\omega$  vanishes. However,  $\omega = 0$  defines a subgroup  $D_0$  of D; this subgroup is still free abelian but of smaller rank since  $x \mapsto \omega$  defines a non-trivial torsion-free quotient of D. Therefore, the induction hypothesis limits at r the number of distinct values on  $D_0$ , which completes the induction step once we add the value  $v(x_1)$ .

(In fact we will only use the case where *C* is cyclic, but this is still rank r = 2 in the above proof.)

End of proof of Theorem 5. Let *G* be a nilpotent connected Lie group. We refer to [HN12, §11.2] (especially Thm. 11.2.10 therein) for background on the structure of *G*. We recall in particular the following. The fundamental group of *G* is a free abelian group of finite rank  $d \ge 0$ . Moreover, denoting by  $\widetilde{G}$  the universal cover of *G*, there is a discrete central subgroup  $\Gamma < \widetilde{G}$ isomorphic to  $\pi_1(G)$  with  $\widetilde{G} \to \widetilde{G}/\Gamma \cong G$  implementing the covering morphism. In addition,  $\Gamma$  is a lattice in a central subgroup  $Z \cong \mathbb{R}^d$  of  $\widetilde{G}$ , so that  $T = Z/\Gamma$  is a central *d*-dimensional torus in *G*.

Our proof is by induction on *d*. The base case d = 0 holds by Thomas's result since *G* is then simply connected and hence linear [HN12, 16.2.7].

The case d = 1 is the main case because the general case is easily reduced to it; hence we treat it separately. Since *Z* is connected, the quotient  $H = G/T \cong \tilde{G}/Z$  is simply connected. Therefore, by the case d = 0, there is a countable family of cocountable subgroups  $H_n < H$  such that every non-trivial element of *H* is outside some  $H_n$ . Taking pre-images gives cocountable subgroups  $G_n < G$  such that every element of *G not in T* lies outside  $G_n$  for some *n*. It therefore suffices to produce another sequence of cocountable subgroups  $K_n < G$  such that every non-trivial element of *T* lies outside some  $K_n$ .

Since  $\widehat{G}$  is simply connected, it is linear; more precisely, by the Lie– Kolchin theorem, we can identify it with a Lie subgroup of the upper unitriangular subgroup  $U_m$  of  $\mathbf{SL}_m(\mathbf{R})$  for some m. We define  $\widehat{K}_n < \widetilde{G}$  by taking the intersection of  $\widetilde{G}$  with the congruence subgroup  $\mathbf{SL}_m(V_{\mathbf{C}}; L_n)$  under such an embedding; thus  $\widehat{K}_n$  is cocountable in  $\widetilde{G}$  by Lemma 12. Let  $K_n < G$  be the image of  $\widehat{K}_n$  in G, which is cocountable.

Let us reformulate for  $\widehat{K}_n < \widetilde{G}$  the desired property of  $K_n$ : we need to show that every element  $\zeta \in Z < \widetilde{G}$  which is not in  $\Gamma$  lies outside the group  $\Gamma \widehat{K}_n$  for some *n*. Indeed,  $\Gamma \widehat{K}_n$  is the pre-image of  $K_n$  in  $\widetilde{G}$ . Equivalently, we need to prove that  $\zeta \Gamma \cap \widehat{K}_n = \emptyset$ .

The lower central series  $U_m^i$  of  $U_m$  is defined by  $U_m^0 = U_m$  and  $U_m^{i+1} = [U_m^i, U_m]$ . Concretely, the (p, q)-entry of a matrix in  $U_m^i$  vanishes when  $p < q \le p + i$ . Consider the largest *i* such that  $\zeta \in U_m^i$ ; since  $\zeta$  is non-trivial,  $i \le m - 2$ . Moreover, the evaluation map  $\epsilon \colon U_m \to \mathbf{R}$  at a (p, p + i + 1)-entry is a group homomorphism on  $U_m^i$ , namely one of the canonical coordinates of  $U_m^i/U_m^{i+1} \cong \mathbf{R}^{m-i-1}$ . By maximality of *i*, we can choose *p* such that  $\epsilon(\zeta)$  is non-zero. Recalling that *Z* has dimension d = 1, it follows that  $\epsilon$  restricts to an isomorphism  $\epsilon|_Z \colon Z \to \mathbf{R}$ . In conclusion,  $z = \epsilon(\zeta)$  lies outside  $C = \epsilon(\Gamma)$ . By Lemma 13, there is *n* such that z + C does not meet  $L_n$ . This implies that  $\zeta\Gamma$  does not meet  $\widehat{K}_n$ , as was to be shown.

Finally, we perform the induction step for  $d \ge 2$ . We choose two distinct elements  $\gamma_1, \gamma_2$  of some basis of  $\Gamma$ , write  $\Gamma_i = \langle \gamma_i \rangle$  for the group generated by  $\gamma_i$  and  $Z_i < Z$  for the one-parameter subgroup containing  $\gamma_i$ . The quotient  $T_i = Z_i/\Gamma_i$  is a central circle group in *G* and thus we can form  $H_i = G/T_i$ . Then the fundamental group of  $H_i$  is isomorphic to  $\Gamma/\Gamma_i$ , e.g. because  $\widetilde{G}/Z_i$ is a universal cover of  $H_i$  since  $Z_i$  is connected.

Thus the fundamental group of  $H_i$  has rank d-1. By the induction hypothesis, there are two countable families  $H_{i,n}$  of cocountable subgroups of  $H_1$  respectively  $H_2$  such that any non-trivial element of  $H_i$  lies outside  $H_{i,n}$  for some n. Taking pre-images, we obtain cocountable subgroups  $G_{i,n} < G$  such that every  $g \notin T_i$  lies outside some  $G_{i,n}$ . Since  $T_1 \cap T_2$  is trivial, this completes the induction step.

**Remark 14.** Consider the question whether the double cover  $\widetilde{G}$  of  $G = SL_3(\mathbf{R})$  is countably representable. Since G is countably representable, it

#### NICOLAS MONOD

all amounts to finding a cocountable subgroup of  $\widetilde{G}$  which meets trivially the kernel of the covering map. Equivalently, the question is:

Does G itself contain a cocountable subgroup that can be lifted to  $\widetilde{G}$ ?

This cannot be proved by using as above a cocountable congruence subgroup  $G_n = G \cap \mathbf{SL}_3(V_{\mathbf{C}}; L_n)$ . Indeed we claim that  $G_n$  does not lift to  $\widetilde{G}$ .

The reason is as follows. The set  $1 + L_n$  meets **R** densely because  $L_n \cap \mathbf{R}$  is a cocountable subgroup of **R**. Therefore,  $1 + L_n$  contains a negative real number x. It follows that  $G_n$  contains the diagonal matrices with entries  $(x, x^{-1}, 1)$  and  $(x, 1, x^{-1})$  (noting that  $1 + L_n$  is a multiplicative subgroup of  $V_{\mathbf{C}}^{\times}$ ). The classical relation between Steinberg symbols and central extensions of G, as explained e.g. on p. 95 of Milnor's book [Mil71], shows that these two diagonal matrices do not admit commuting lifts in  $\tilde{G}$ . This is because the Steinberg symbol for  $\tilde{G}$  is the Hilbert quadratic residue symbol (see p. 104 in [Mil71]), but x < 0 means that the Hilbert symbol is non-trivial. This excludes a minori any lift of the group  $G_n$ .

### 4. Auxiliary proofs

The reduction from general locally compact second countable groups to connected Lie groups is standard:

*Proof of Proposition 2.* Let *G* be a locally compact second countable group. Denoting the neutral component of *G* by  $G_0$ , the quotient  $G/G_0$  is locally compact and totally disconnected; therefore, van Dantzig's theorem [vD31, p. 18] implies that  $G/G_0$  contains a compact-open subgroup *K*. We denote by  $G_1 < G$  the pre-image of *K* in *G*; thus  $G_1/G_0$  is compact.

Since *G* is second countable, we can choose a countable base  $\mathscr{U}$  of neighbourhoods  $U \subseteq G_1$  of the identity in  $G_1$ . By the solution of Hilbert's fifth problem (see e.g. [MZ55, 4.6]), there is a compact normal subgroup  $N_U \triangleleft G_1$  contained in *U* such that  $G_1/N_U$  is a Lie group. Note that  $G_1/N_U$  has finitely many connected components; therefore, since we assumed that the answer to Ulam's Problem is affirmative,  $G_1/N_U$  is countably representable.

On the other hand,  $G_1$  is the inverse limit of the system  $(G_1/N_U)_{U \in \mathscr{U}}$ . Since  $\mathscr{U}$  is countable, Lemma 9 implies that  $G_1$  is countably representable. Finally, we use again that G is second countable to deduce that  $G/G_1$  is countable since  $G_1$  is open in G. Applying Lemma 11, we conclude that G itself is countably representable.

Next, we propose two proofs that every abelian group is countably representable, unless its cardinality is too large. Yves Cornulier informed me that the second proof below was already known to de Bruijn, see Thm. 4.3 in [dB64].

Analytic proof of Proposition 3. Let *G* be an abelian group with  $|G| \le c$ . We consider *G* as a discrete group, so that its Pontryagin dual  $\widehat{G} = \text{Hom}(G, \mathbf{S}^1)$  is a compact group. Then the topological weight of  $\widehat{G}$ , namely the smallest

cardinality of a base for its topology, is also  $\leq c$ , see [HM06, 7.76]. It now follows from Engelking's theorem [Eng65, Thm. 10] that  $\widehat{G}$  is separable; the fact that Engelking's theorem applies to compact groups is due to the fact that they are *dyadic spaces*, see e.g. [HM06, 10.40]. We can therefore choose a sequence of characters  $f_n: G \rightarrow S^1$  that is dense in  $\widehat{G}$ .

Since  $S^1$  is countably representable by Corollary 10, there is a sequence of morphisms  $h_m$  from  $S^1$  to countable quotients  $h_m(S^1)$  such that the intersection of the kernels of all  $h_m$  is trivial. It now suffices to verify that for every non-trivial  $g \in G$  there is a pair (n, m) such that  $h_m(f_n(g))$  is nontrivial.

To this end, Pontryagin duality ensures that there is a character  $f \in G$  such that f(g) is non-trivial. By density of  $(f_n)$ , we can fix n such that  $f_n(g)$  is non-trivial in  $\mathbf{S}^1$ , and it remains only to choose  $h_m$  in terms of  $f_n(g)$ .  $\Box$ 

The above use of dyadic spaces and of Engelking's general theorem is convenient but perhaps an overkill. It has been observed already in the 1940s that continum powers of separable spaces are separable (the Hewitt– Marczewski–Pondiczery theorem [Hew46, Mar47, Pon44]); this applies to  $(\mathbf{S}^1)^G$ . The fact that this still holds for the subspace  $\widehat{G} \subseteq (\mathbf{S}^1)^G$  is a very special case of Engelking's generalization of the Hewitt–Marczewski–Pondiczery theorem.

Algebraic proof of Proposition 3. Let *G* be an abelian group with  $|G| \le c$ . The injective hull *D* of *G* contains *G*, and its construction shows that *D* still has size  $\le c$  (see e.g. the proof of Thm. 17 in [Gri70, III]). It therefore suffices to consider divisible abelian groups such as *D*. By the structure theory of divisible abelian groups (see e.g. [Fuc70, Thm. 23.1]), *D* is a sum of a **Q**-vector space and of *p*-torsion parts, where *p* ranges over all primes. It suffices to consider each piece individually because of Lemma 8. The **Q**-vector space is handled by the argument of Proposition 1 since it has dimension at most *c*.

The *p*-torsion part is of the form  $\bigoplus_{\kappa} \mathbb{Z}(p^{\infty})$  for some cardinal  $\kappa$ , where  $\mathbb{Z}(p^{\infty})$  is the Prüfer *p*-group (see again [Fuc70, Thm. 23.1]); upon enlarging it we can assume  $\kappa = \mathfrak{c} = 2^{\aleph_0}$ . It is therefore contained in  $\mathbb{Z}(p^{\infty})^{\aleph_0}$ , see Ex. 3 in [Fuc70, §23]. We conclude again by Lemma 8 since  $\mathbb{Z}(p^{\infty})$  is countable.

# 5. On McKenzie's and Churkin's examples

Churkin [Chu05] defines a group *G* by generators and relations as follows. Let *I* be an index set of cardinality c.

$$G = \left\langle a_i, b_i, c \ (i \in I) : \text{all generators commute, except } [a_i, b_i] = c \ (\forall i) \right\rangle$$

Earlier, McKenzie defined an almost identical group M with the same notation as above but imposing in addition that all  $a_i$  and  $b_i$  have order two (proof of Thm. 4 in [McK71]). We shall work with *G*, but the arguments work equally well for *M*.

Observe that there is an epimorphism from *G* to a free abelian group *A* of continuum rank  $\mathfrak{c}$ . Indeed define *A* to be free on a set  $\{\bar{a}_i, \bar{b}_i : i \in I\}$  and map  $a_i \mapsto \bar{a}_i, b_i \mapsto \bar{b}_i$ . The kernel of this epimorphism is generated by *c*. In particular, *G* is nilpotent and of size  $\mathfrak{c}$ . (In the case of *M*, replace *A* by the free  $\mathbb{Z}/2\mathbb{Z}$ -module *B* on  $\{\bar{a}_i, \bar{b}_i : i \in I\}$ .)

**Theorem 15** (McKenzie, Churkin). Every homomorphism from G or M to  $Sym(\aleph_0)$  factors through A, respectively B.

Since we found library access to [Chu05] difficult, we reproduce its beautiful argument:

*Churkin's proof of Theorem* 15. Let *X* be a countable set with a *G*-action. We need to show that *c* fixes any given  $x \in X$ . Consider the map  $I \to X$ ,  $i \mapsto a_i x$ . Since *I* is uncountable, it contains an uncountable subset *J* such that  $a_i x$  is constant over  $i \in J$ . For the same reason, there is an uncountable subset  $K \subseteq J$  such that  $b_i x$  is constant over  $i \in K$ . Choose three distinct indices i, j, k all in *K*. Since both  $a_j^{-1}a_i$  and  $b_k^{-1}b_i$  fix x, so does  $[a_j^{-1}a_i, b_k^{-1}b_i]$ . On the other hand, this commutator is *c* because  $a_j$  commutes with  $a_i, b_k, b_i$  and  $b_k$  commutes with  $a_i, a_i, b_i$ . The same proof works for *M*.

To deduce that *G* is not countably representable, it remains of course to prove that the subgroup generated by *c* in *G* is not trivial. Unfortunately the argument given in [Chu05], aiming to map *G* onto the integral Heisenberg group, does not work — and indeed Theorem 15 rules out witnessing the non-triviality of *c* in any countable quotient.

There is nonetheless a direct argument. Consider the central extension

$$0 \longrightarrow \mathbf{Z} \longrightarrow E \longrightarrow A \longrightarrow 0$$

given by the following two-cocycle (factor set)  $f: A \times A \rightarrow \mathbf{Z}$ . Denote by  $\alpha_i: A \rightarrow \mathbf{Z}$  the coefficient map of  $\bar{a}_i$  and likewise  $\beta_i$  for  $\bar{b}_i$ . Then define  $f(x, y) = \sum_{i \in I} \alpha_i(x)\beta_i(y)$ , noting that the sum is finite for each  $x, y \in A$ . This is a normalised two-cocycle and hence defines an extension *E* as above (see e.g. Ex. IV.3.8 and V.6.5 in [Bro82] for the connection between **Z**-bilinear maps and two-cocycles on abelian groups). Explicitly, recall that the underlying set for *E* is  $\mathbf{Z} \times A$  with multiplication

$$(n,x)\cdot(m,y)=\big(n+m+f(x,y),x+y\big).$$

The presentation of *G* implies that there is an epimorphism  $G \to E$  mapping *c* to  $(1_{\mathbb{Z}}, 0_A)$  and compatible with the epimorphisms  $G \to A$  and  $E \to A$ . The latter property ensures that this map  $G \to E$  is actually an isomorphism  $G \cong E$ . By construction,  $(1_{\mathbb{Z}}, 0_A)$  is non-trivial in *E* and indeed has infinite order.

This confirms that *G* is not countably representable and further implies that for all  $n \ge 2$  the quotient of *G* by  $\langle c^n \rangle$  gives an extension of *A* by  $\mathbb{Z}/n\mathbb{Z}$  which is not countably representable either.

The above explicit construction can also be used as follows. The cocycle f extends by bilinearity to a **Q**-valued cocycle on the free **Q**-module on  $\{\bar{a}_i, \bar{b}_i : i \in I\}$ , which is isomorphic to **R**. We therefore obtain an extension

$$0 \longrightarrow \mathbf{Q} \longrightarrow F \longrightarrow \mathbf{R} \longrightarrow 0$$

containing Churkin's group. Defining  $H = F \times \mathbf{R}$ , this further implies the following curiosity, in marked contrast to our Theorem 5:

**Corollary 16.** There exists a nilpotent group H given by a central extension

$$0 \longrightarrow \mathbf{R} \longrightarrow H \longrightarrow \mathbf{R} \longrightarrow 0$$

which is not countably representable.

Finally, we present the proof of Theorem 6, establishing that *G* and *M* do not admit any faithful representation to any Hausdorff topological group that is separable or Lindelöf. This contains Theorem 15 since  $Sym(\aleph_0)$  is a Polish group.

*Proof of Theorem* 6. We keep the notation introduced above for Churkin's group G and for its explicit construction as central extension E (the proof for M is the same).

We claim that if *G* is covered by countably many left translates of some subset  $W \subseteq G$ , then the central generator *c* is a commutator of two elements of the product set  $W^{-1}W$ .

Indeed there is an uncountable subset  $J \subseteq I$  such that all  $a_j$  with  $j \in J$  belong to a fixed translate of W, say rW for some  $r \in G$ . Likewise there is  $K \subseteq J$  infinite and  $s \in G$  such that  $b_k \in sW$  for all  $k \in K$ . As in Theorem 15, we observe that  $c = [a_j^{-1}a_i, b_k^{-1}b_i]$  whenever i, j, k are distinct. Taking all three indices in K, both  $a_j^{-1}a_i$  and  $b_k^{-1}b_i$  belong to  $W^{-1}W$ , confirming the claim.

Consider now a group homomorphism  $\pi: G \to H$  to some separable or Lindelöf topological group H. It suffices to show that  $\pi(c)$  belongs to every neighbourhood  $U \subseteq H$  of the identity in H.

To that end, choose a symmetric neighbourhood V of the identity in H such that  $V^{16} \subseteq U$ . There exists a sequence  $\{h_n\}$  in H such that  $\{h_nV\}$  covers H. Indeed, in the Lindelöf case this holds by definition of that property. In the separable case, we can take any sequence  $\{h_n\}$  that is dense in H since then for any  $h \in H$  the sequence  $h_n^{-1}h$  must meet V.

In particular,  $\pi(G)$  is covered by  $\{h_nV\}$  and upon extracting we assume that moreover  $h_nV$  meets  $\pi(G)$  for every *n*. We then choose  $g_n \in G$  with  $\pi(g_n) \in h_nV$ . It now follows

$$h_n V \subseteq \pi(g_n) V^{-1} V = \pi(g_n) V^2.$$

Thus  $\pi(G)$  is covered by the sequence  $\pi(g_n)V^2$  and we conclude that *G* is covered by countably many left translates of  $\pi^{-1}(V^2)$ .

We now apply the initial claim to  $W = \pi^{-1}(V^2)$  and deduce that *c* is a commutator of two elements of  $W^{-1}W = \pi^{-1}(V^4)$ . Therefore,  $\pi(c)$  is the product of four elements of  $V^4$  and thus belongs indeed to  $V^{16} \subseteq U$ , completing the proof.

The reader will have noticed that this proof is inspired by two sources: Churkin's proof above and the use of the Steinhaus property made in the separable case by Rosendal and Solecki in Prop. 2 of [RS07].

**Remark 17.** Our proof makes almost no distinction between the separable and the Lindelöf cases, but they are formally independent. On the one hand, there obviously are non-separable Lindelöf groups since there is no limit to the cardinality of a compact group (while separable regular spaces have cardinality  $\leq 2^{c}$  [HNV04, a-3]). The reverse question is more delicate and was notably asked by Wilansky in 1968 for normal groups [ST74]. An example conditional to the continuum hypothesis was provided in [HJ76], and then an unconditional example in [HvM85].

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