A TYPE I CONJECTURE AND BOUNDARY REPRESENTATIONS OF HYPERBOLIC GROUPS

PIERRE-EMMANUEL CAPRACE, MEHRDAD KALANTAR, AND NICOLAS MONOD

ABSTRACT. We establish new results on the weak containment of quasi-regular and Koopman representations of a second countable locally compact group G associated with non-singular G-spaces. We deduce that any two boundary representations of a hyperbolic locally compact group are weakly equivalent. We also show that non-amenable hyperbolic locally compact groups with a cocompact amenable subgroup are characterized by the property that any two proper length functions are homothetic up to an additive constant. Combining those results with the work of L. Garncarek on the irreducibility of boundary representations of discrete hyperbolic groups, we deduce that a type I hyperbolic group with a cocompact lattice contains a cocompact amenable subgroup. Specializing to groups acting on trees, we answer a question of C. Houdayer and S. Raum.

1. INTRODUCTION

1.1. Type I groups. According to L. Auslander and C. Moore [AM66, p. 1],

"one of the fundamental questions one can raise about any locally compact group or C^* -algebra is that of determining when it is type I."

The notion of type I, hailing from the very origins of operator algebras and representation theory [MvN36], can be seen as a rigorous way to define the class of groups for which unitary representations can be classified in any meaningful manner. That notion is of fundamental importance, although the definition may seem technical at first sight; we refer to the recent book [BdlH20] for a detailed account and a review of known result. Let us merely mention here that, by a celebrated result of E. Thoma [Tho64, Tho68], a discrete group is type I if and only if it is virtually abelian.

In the non-discrete case, the current state of the art is not nearly as complete, despite numerous results ensuring that various important families of groups are type I and thus showing that the type I class is much richer than in the discrete case. What is completely lacking, in contrast to Thoma's theorem, is a definite structural consequence of type I. We venture the following.

Conjecture A. Every second countable locally compact group of type I admits a cocompact amenable subgroup.

The structural property contemplated in this conjecture is a strong one. For instance, hyperbolic unimodular groups with a cocompact amenable subgroup are described in detail in [CCMT15]. In the wide setting of non-positively curved (CAT(0)) groups, a classification is obtained in [CM15] using also [CM09, CM13]. These descriptions are in terms of semi-simple groups (Lie or non-Archimedean) and tree automorphism groups, and it turns out that those classes include most known examples of type I groups with a trivial amenable radical.

(The study of type I among amenable groups is also interesting and very incomplete, but Conjecture A is obviously irrelevant there.)

P.-E.C. is a F.R.S.-FNRS senior research associate.

MK is supported by a Simons Foundation Collaboration Grant (# 713667).

A more partial motivation for Conjecture A is that *Gelfand pairs* (G, K), which are characterised by a very tame K-spherical dual for a compact subgroup K < G, admit a cocompact amenable subgroup P < G, indeed even an "Iwasawa decomposition" G = KP [Mon20].

Conjecture A is further discussed in Section 6. We show that this conjecture would follow from the simplicity of the C^* -algebra $C^*_{\lambda_{G/H}}(G)$, where H is the stabilizer of a generic point in the Furstenberg boundary of G (see Proposition 6.3 and Remark 6.10).

The bulk of this article is devoted to a completely different line of attack towards Conjecture A, in a geometric set-up that fosters a rich interplay between the dynamical and geometrical aspects of boundary theory. Namely, one of the conclusions of this paper is that Conjecture A holds for all hyperbolic locally compact groups which admit a uniform lattice.

Theorem B. Let G be a hyperbolic locally compact group admitting a uniform lattice.

If G is type I, then G has a cocompact amenable subgroup.

As hinted to above, this result leads to a rather precise description of G thanks to [CCMT15, Theorem D]:

Corollary C. Let G be a hyperbolic locally compact group admitting a uniform lattice. Recall that G admits a unique maximal compact normal subgroup W.

If G is type I, then the quotient group G/W satisfies exactly one of the following descriptions:

- (i) G/W is the group of isometries of a rank one symmetric space of noncompact type, or its identity component, which has index at most 2. In particular G/W is a simple Lie group of rank one.
- (ii) G/W is a closed subgroup of the automorphism group of a locally finite nonelementary tree T, acting without inversions and with exactly two orbits of vertices, and acting 2-transitively on the set ∂T of ends.
- (iii) G/W is trivial or isomorphic to \mathbf{Z} , \mathbf{R} , $\mathbf{Z} \rtimes \{\pm 1\}$ or $\mathbf{R} \rtimes \{\pm 1\}$.

In all cases, it follows that G has an Iwasawa decomposition G = KP, where K is a compact subgroup and P an amenable closed subgroup.

It is tempting to believe that Corollary C could lead to a necessary and sufficient structural characterisation of type I among hyperbolic locally compact group admitting a uniform lattice. Indeed, every group as in (i) or (iii) is known to be type I; as for case (ii), these groups are conjectured to be type I since [Neb99] (C. Nebbia's conjecture in loc. cit. is formulated for regular trees, and predicts the formally stronger statement that a group as in (ii) is CCR).

As we shall discuss below (see Remark 5.6), the assumption on the existence of a uniform lattice can probably be relaxed with further work. Two points should be emphasized in that relation:

On the one hand, this assumption incidentally ensures that G is unimodular, which rules out *amenable* non-elementary hyperbolic groups [CCMT15, Theorem 7.3]. Those can indeed fail to be type I and rule out the most naive converse to Conjecture A, see Proposition 5.8 for an example.

On the other hand, if as we suspect we can replace the assumption on uniform lattices by the weaker unimodularity assumption, then this stronger version of Corollary C would imply a posteriori that G does contain a uniform lattice. Indeed, this follows from the Borel–Harish-Chandra theorem [BHC62] in case (i), from the Bass–Kulkarni theorem [BK90] in case (ii), and is clear in case (iii).

One of the upshots of Corollary C is that the totally disconnected case is reduced to the setting of tree automorphism groups. Taking a step back, we note that the class of compactly generated closed subgroups G of the automorphism group of any locally finite tree T provides a broad natural source of examples of hyperbolic locally compact groups. Indeed, the group G then acts cocompactly on the smallest G-invariant subtree T' of T (see [CDM11, Lemma 2.4]), so that G is quasi-isometric to T'. In particular G is hyperbolic. By specializing Theorem B and Corollary C to this setting, we obtain the following.

Corollary D. Let T be a locally finite tree and $G \leq \operatorname{Aut}(T)$ be a closed nonamenable subgroup acting minimally on T. If G is type I, then the G-action on the set of ends ∂T is 2-transitive.

This strengthens Theorem A from [HR19], establishing the weaker property that the *G*-action on *T* is *locally* 2-*transitive*, i.e. the stabilizer of every vertex of valency ≥ 3 is 2-transitive on the set of incident edges. Corollary D contributes to the characterization of the type I property among closed subgroups of the automorphism group of a tree by solving the second part of Problem 1 in [HR19]. It should also be noted that C. Houdayer and S. Raum establish a stronger result than their Theorem A mentionned above (see [HR19, Theorem C]), which establishes the same conclusion under the hypothesis that the group von Neumann algebra L(G) is amenable, which is formally weaker than the type I condition.

1.2. Boundary representations. Glimm's theorem [Gli61] characterizes the type I property for G by the fact that any two weakly equivalent irreducible unitary representations of G are equivalent. For this reason, a substantial part of this paper is a contribution to the unitary representation theory of hyperbolic locally compact groups. An important source of unitary representations of such a group G is provided by the so-called **boundary representations**, which are the Koopman representations κ_{ν} associated with a quasi-invariant probability measure ν supported on the Gromov boundary ∂G . We shall prove the following.

Theorem E. Any two boundary representations of a non-amenable hyperbolic locally compact group are weakly equivalent.

For a detailed account of notion of weak containment and weak equivalence of unitary representations, we refer to [BdlHV08, Appendix F]. Let us simply recall here that two unitary representations π_1, π_2 of a locally compact group G are weakly equivalent if their C^* -kernel coincide; in other words the respective representations of $C^*(G)$ corresponding to π_1, π_2 have the same kernel. For a discrete non-elementary hyperbolic group G, it is well-known that every boundary representation of G is weakly contained in the regular representation (see [Ada94, Theorem 5.1] and [Kuh94]). Using that the kernel of the G-action on ∂G is the amenable radical R(G), which is finite, the assertion of Theorem E for a discrete hyperbolic group directly follows from the fact that G/R(G) is C^* -simple, i.e. its the reduced C^* -algebra is simple, see [dlH88] and [BKKO17, Theorem 6.5]. For non-discrete hyperbolic groups, the latter property fails: indeed, every rank one connected simple Lie group with finite center is hyperbolic, but no such group is C^* -simple (see [dlH07, App. G]).

The proof of Theorem B combines Theorem E with several other ingredients that we now proceed to describe. In trying to apply Glimm's characterization of the type I condition, it is obviously useful to have a large supply of irreducible representations of G at one's disposal, together with a good understanding of their classification up to equivalence. In the setting of Theorem B, these representations will be the boundary representations associated with *Patterson–Sullivan measures*.

Specifically, consider a locally compact hyperbolic group G. For the purpose of Theorem B, we can readily reduce ourselves to the case where G is totally disconnected. In particular, it admits a Cayley–Abels graph X on the vertex set G/U, where U is any given compact open subgroup. More precisely, the edge structure of the graph X depends on the choice of a compact generating set for G, or equivalently of a suitable word metric on G; we will crucially use this freedom of choice.

Since X is a locally finite hyperbolic graph, its Gromov boundary is a compact G-space and supports a canonical family of Patterson–Sullivan measures. For the time being, recall simply that any choice of a base-point in X determines a measure ν on $\partial X \cong \partial G$ and that any other point gives an equivalent measure; in particular, ν is quasi-invariant under G and defines a Koopman representation κ_{ν} of G on $L^2(\partial G, \nu)$, that we call a **PS-representation**.

Our strategy consists in first using Theorem E to apply Glimm's criterion to these representations and then establishing a number of geometric consequences culminating in the fact that G acts transitively on its boundary ∂G . To this end, we rely on the groundbreaking work of L. Garncarek [Gar14], extending earlier results of Bader–Muchnik [BM11], and solving an important case of the conjecture that they formulated in loc. cit. The fact that Garncarek's results are stated and proved for *discrete* hyperbolic groups explains our auxiliary hypothesis that G admits a uniform lattice. On the one hand, Garncarek proves that PS-representations are all irreducible. On the other hand, he shows that two such representations are equivalent if and only if the underlying word metrics are **roughly similar**, which means that they are homothetic up to an additive constant. Applying these results to the lattice yields them a fortiori for G, since the representations are already defined on G.

In summary, relying on Glimm's and Garncarek's theorems, our strategy consists of the following two steps.

The first one, of analytic flavour, is to prove Theorem E. It implies that all PS-representations (associated to any word metric) are weakly equivalent. We shall establish such a weak equivalence in a more general context for rather more general groups G, as stated in Corollary H below.

The second step, of geometric flavour, is to establish that G admits indeed a cocompact amenable subgroup when any two word metrics on G are roughly similar. This requires new results on hyperbolic groups, entering Theorem J below.

1.3. Weak containment of Koopman and quasi-regular representations. The first remaining step for the proof of Theorem E relies on new results on weak containment of unitary representations in a broad context of locally compact transformation groups, regardless of any hyperbolicity assumption. In order to present their statements, we recall that a locally compact group H is called **regionally elliptic** if every compact subset of H is contained in a compact subgroup. If H is σ -compact, this is equivalent to requiring that H is a countable ascending union of compact subgroups. A **tdlc group** is a locally compact group which is totally disconnected.

Theorem F. Let G be a second countable tdlc group and (X, ν) be a standard probability space endowed with a measurable G-action, such that ν is quasi-invariant under G. Let κ be the Koopman unitary representation of G on $L^2(X, \nu)$. Then there is a co-null set $Y \subseteq X$ such that for every $x \in Y$ with regionally elliptic stabilizer G_x , the quasi-regular representation λ_{G/G_x} is weakly contained in the Koopman representation κ .

Let us point out that for a discrete group, every Koopman representation weakly contains the quasi-regular representation associated with almost every point stabilizer (see [DG17, Proposition 7]). We do not know whether this result holds in the non-discrete case; Theorem F provides a special case where this is indeed true.

The relevant weak containment in the opposite direction is established by the following statement, which crucially relies on the work of C. Anantharaman-Delaroche [AD03].

Theorem G. Let G be a second countable locally compact group, and let X be a minimal compact G-space. Let ν be a G-quasi-invariant Radon probability measure on X such that $L^2(X,\nu)$ is separable. Assume that the G-action on (X,ν) is amenable in the sense of Zimmer, and let $X_1 \subseteq X$ denotes the conull subset consisting of those $x \in X$ such that G_x is amenable. Then the following assertions hold.

- (i) For all $x, y \in X_1$, the quasi-regular representations λ_{G/G_x} and λ_{G/G_y} are weakly equivalent.
- (ii) For all $x \in X_1$, the quasi-regular representation λ_{G/G_x} weakly contains the Koopman representation κ_{ν} .

We emphasize that, if the G-action on X is topologically amenable, then $X_1 = X$. By combining a topological version of Theorem F, recorded as Theorem 3.1 below, with Theorem G, we obtain the following consequence.

Corollary H. Let G be a second countable tdlc group and X be a minimal compact G-space equipped with a G-quasi-invariant Radon probability measure ν such that $L^2(X,\nu)$ is separable. Suppose that for some $x \in X$, the stabilizer G_x is regionally elliptic.

If the G-action on (X, ν) is amenable in Zimmer's sense, then the Koopman representation κ of G on $L^2(X, \nu)$ is weakly equivalent to the quasi-regular representation λ_{G/G_y} for any $y \in X$ such that G_y is amenable.

Thus in particular, in the setting of Corollary H, the weak equivalence class of the Koopman representation κ is independent of ν .

The regionally elliptic hypothesis appearing in Theorem F and Corollary H may seem rather restrictive (although the special case where $G_x = \langle e \rangle$ is interesting in its own). Nonetheless, it turns out that the results of this section are general enough to be applied to the action of a hyperbolic locally compact group G on its Gromov boundary $X = \partial G$: that action is indeed topologically amenable (see [Ada96, Theorem 6.8] and [Kai04]). Moreover, we establish new results on the algebraic structure of amenable subgroups of G which ensure that the regionally elliptic hypothesis of Corollary H is automatically satisfied in the context of Theorem E (see Section 4.1 below), using a reduction to the tdlc case that relies on [CCMT15].

1.4. Characterizing non-amenable hyperbolic groups with a cocompact amenable subgroup. As mentioned above, the proof of Theorem B requires identifying the non-amenable hyperbolic groups with a cocompact amenable subgroup with those hyperbolic groups for which any two word metrics are roughly similar. This is ensured by Theorem J, which supplements [CCMT15, Theorem D and Theorem 8.1]. Given a point ξ in the Gromov boundary of a hyperbolic locally compact group G, we denote by G^0_{ξ} the kernel of the Busemann homomorphism $\beta_{\xi} : G_{\xi} \to \mathbf{R}$ (see Section 4.1 below). **Theorem J.** Let G be a non-amenable hyperbolic locally compact group and (X, d) be a proper geodesic metric space on which G acts continuously, properly and cocompactly by isometries.

The following conditions are equivalent.

- (i) G has a cocompact amenable subgroup.
- (ii) The G-action on the Gromov boundary ∂X is 2-transitive.
- (iii) For all $\xi \in \partial X$, we have $G_{\xi}^0 \neq G_{\xi}$.
- (iv) For some continuity point $\eta \in \partial X$ of the stabilizer map, we have $G_n^0 \neq G_\eta$.
- (v) There is a constant K such that, for every hyperbolic element $\gamma \in G$, there exists a hyperbolic element $\gamma' \in G$ with asymptotic displacement length $|\gamma'|_{\infty} \leq K$ such that γ and γ' share the same pair of fixed points in ∂X .
- (vi) For any word metric d' on G with respect to a compact generating set, each orbit map $G \to X$ is a rough similarity.

The equivalence between (i) and (ii) is taken from [CCMT15]. A special instance of the equivalence between (ii) and (iii) has been observed for specific families of groups of tree automorphisms by C. Ciobotaru in her PhD thesis [Cio14, Proposition 2.2.11]. The proof of Theorem B uses the implications (iv) \Rightarrow (i), (v) \Rightarrow (i) and (vi) \Rightarrow (i), which are all new.

1.5. Epilogue on boundary representations. We finish by recording another result, established along the way, that has its own interest. We recall that a C^* -algebra A is called **CCR** if $\pi(A)$ consists of compact operators for every irreducible representation π of A ([Dix96, Definition 4.2.1]). A is called **GCR** if every non-zero quotient of A contains a non-zero CCR closed two-sided ideal ([Dix96, Definition 4.3.1]).

We say a unitary representation π of a locally compact group G is CCR (resp. GCR) if the C*-algebra $C^*_{\pi}(G) := \pi(C^*(G))$ is CCR (resp. GCR). In particular, if π is an irreducible CCR representation, then $C^*_{\pi}(G)$ consists of compact operators, but this need not be the case if π is not irreducible (indeed, a normal operator need not be compact). It is thus important to underline that the boundary representation κ in the following theorem is arbitrary, and need not be irreducible a priori.

Theorem K. Let G be a non-amenable hyperbolic locally compact group admitting a uniform lattice. For any boundary representation κ of G, the following assertions are equivalent.

- (i) $C^*_{\kappa}(G)$ contains a non-zero CCR closed two-sided ideal.
- (ii) κ is GCR.
- (iii) κ is CCR.
- (iv) $C^*_{\kappa}(G)$ entirely consists of compact operators.
- (v) G has a cocompact amenable subgroup.

The implications (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i) in Theorem K are tautological, while the implication (v) \Rightarrow (iv) follows from the fact that if G has a cocompact amenable subgroup P, then ∂G can naturally be identified with G/P so that κ becomes equivalent to the quasi-regular representation $\lambda_{G/P}$. That the quasi-regular representation defined by a cocompact subgroup satisfies the condition (iv) follows by general principles (compare Proposition 5.1).

The key implication is (i) \Rightarrow (v). The formally weaker implication (iv) \Rightarrow (v) is much more straightforward, and can be established without requiring that G has a cocompact lattice by invoking Propositions 4.1 and 6.5 below (see also [Neb99] in the special case of groups acting on trees).

CONTENTS

1. Introduction	1
2. Preliminaries	7
2.1. Koopman unitary representations	7
2.2. URS and Fell's continuity theorem	9
3. Koopman and quasi-regular representations	10
3.1. The Koopman representation weakly contains a quasi-regular	
representation	10
3.2. The Koopman representation is weakly contained in a quasi-regular	
representation	11
4. Hyperbolic locally compact groups	12
4.1. The structure of relatively amenable subgroups	13
4.2. Ballistic boundary points	15
4.3. Roughly similar word metrics	21
5. Boundary representations of hyperbolic groups and the type I property	24
5.1. Quasi-regular representations defined by cocompact subgroups	24
5.2. Weak equivalence of boundary representations	25
5.3. Type I hyperbolic groups	26
6. Conjectures and relation to C^* -simplicity	29
Acknowledgements	
References	33

2. Preliminaries

In this section we gather some general facts which we will use in proofs of our results.

2.1. Koopman unitary representations. Throughout the paper, by a representation of a locally compact group G we always mean a continuous unitary representation. The most commonly used representations in this work are the Koopman unitary representations κ_{ν} associated to measurable actions of locally compact groups G on G-quasi-invariant σ -finite measure spaces (X, ν) . Recall that κ_{ν} is defined by

$$(\kappa_{\nu}(g)\xi)(x) := \sqrt{\frac{dg\nu}{d\nu}(x)}\,\xi(g^{-1}x)$$

for all $g \in G$, $\xi \in L^2(X, \nu)$ and ν -a.e. $x \in X$.

Some care should be taken with regards to the continuity of the action, which is not guaranteed in this generality (see [BdlHV08, Remark A.6.3]). The representation κ_{ν} is a continuous, for instance, when G is σ -compact and $L^2(X,\nu)$ is separable ([BdlHV08, Proposition A.6.1] and [SvN50, Theorem 2]). Except in some places in Section 6, we always work with continuous actions of G on metrizable locally compact spaces X. When ν is a Radon measure on X, then $L^2(X,\nu)$ is separable. Also, note that all locally compact hyperbolic groups are σ -compact. In particular, the continuity issue will only be relevant in some places in the last section, where we consider general boundary actions. However, in that case, considering those measures ν whose L^2 -spaces are separable will suffice for our purposes, and therefore again we will not have any continuity issue for the Koopman representations.

The following basic facts are well-known. We record them for easy reference in few places later in the paper.

Lemma 2.1. Let G be a σ -compact locally compact group and X be a locally compact G-space. If ν and ν' are two equivalent σ -finite Radon measures on X which are quasi-invariant under G, then the Koopman representations of G on $L^2(X,\nu)$ and $L^2(X,\nu')$ are unitary equivalent.

Proof. It is straightforward to see that the map $T : L^2(X, \nu) \to L^2(X, \nu')$ defined by $T\xi = \sqrt{\frac{d\nu}{d\nu'}}\xi$ is a unitary that intertwines κ_{ν} and $\kappa_{\nu'}$.

Lemma 2.2. Let G be a locally compact group and X be a locally compact G-space. Let ν be a Borel probability measure on X which is quasi-invariant under G. Then for any Borel measure μ on G we have $\mu * \nu \sim \nu$.

Proof. Let Y be a Borel subset of X. If $\nu(Y) = 0$, then $\nu(gY) = 0$ for all $g \in G$ by quasi-invariance, and therefore $\mu * \nu(Y) = \int_G \nu(g^{-1}Y) d\mu(g) = 0$.

Conversely, assume $\mu * \nu(Y) = 0$. Then $\nu(g^{-1}Y) = 0$ for μ -a.e. $g \in G$. Hence, $\nu(Y) = 0$ by quasi-invariance.

We recall from [Dix96, §13.7] that a **positive definite measure** on a locally compact group G is a measure μ on G such that $\int_G f * \tilde{f} d\mu \ge 0$ for all $f \in C_c(G)$, where $C_c(G)$ denotes the set of continuous compactly supported complex valued functions on G and $\tilde{f}(g) = \overline{f(g^{-1})}$. (The convolution is with respect to a left Haar measure.) Every such measure μ defines a unitary representation π_{μ} of G constructed as follows. The formula $\langle f, g \rangle_{\mu} = \int_G \tilde{g} * f d\mu$ turns $C_c(G)$ into a pre-Hilbert space; we define the Hilbert space \mathscr{H}_{μ} to be its separated completion. The representation π_{μ} is induced by the representation s on $C_c(G)$ defined by $(s(g)f)(x) = f(g^{-1}x)\Delta_G(g)^{1/2}$, where Δ_G is the modular function of G.

For example, for the Dirac mass δ_e at the neutral element, we have $\pi_{\delta_e} \cong \lambda_G$, whereas the representation associated with a right Haar measure on G is the trivial representation. More generally, we have the following result of Blattner.

Theorem 2.3. Let G be locally compact group and $H \leq G$ be a closed subgroup. Let α_H be a left Haar measure on H, viewed as a measure on G, and define a mesure μ on G be setting $d\mu = \sqrt{\frac{\Delta_G}{\Delta_H}} d\alpha_H$, where Δ_G and Δ_H are the modular functions of G and H respectively. Then μ is positive definite and π_{μ} is equivalent to the quasi-regular representation $\lambda_{G/H} := \operatorname{Ind}_H^G(\mathbf{1})$.

Proof. We refer to [Bla63, Theorem 1]; note that Blattner uses the opposite convention from Dixmier regarding convolution. \Box

We endow the set of measures on G with the **vague topology**, which is the topology of pointwise convergence on $C_c(G)$ (see [Bou65, Chapter III, §1, no. 9]). In particular, a sequence $(\mu_n)_n$ of measures on G **vaguely converges** to a measure μ if for every function $f \in C_c(G)$, the sequence $(\int_G f d\mu_n)_n$ converges to $\int_G f d\mu$.

Lemma 2.4. Let G be a lcsc group and \mathscr{M} be a set of measures on G. Then \mathscr{M} is relatively compact for the vague topology if and only if for each compact subset $K \subseteq G$, there is a constant M_K such that $|\mu|(K) \leq M_K$ for all $\mu \in \mathscr{M}$.

Proof. We refer to Proposition 15 in [Bou65, Chapter III, §1, no. 9].

The following continuity principle is of fundamental importance for Theorem F. It is related to Fell's continuity theorem recalled in Section 2.2 below.

Lemma 2.5. Let G be a lcsc group and (μ_n) a sequence of positive definite measures on G. If $\mu_n \to \mu$ vaguely, then μ is also positive definite and $\pi_{\mu_n} \to \pi_{\mu}$ in Fell's topology.

9

Proof. The fact that μ is also positive definite is obvious. The definition of Fell convergence needs only to be checked on a dense subset of vectors (see e.g. [BdlHV08, Lemma F.1.3]), so we check it using the canonical image of $C_c(G)$ in the Hilbert spaces $\mathscr{H}_{\pi_{\mu_n}}$ and $\mathscr{H}_{\pi_{\mu}}$ respectively. Explicitly, given $f \in C_c(G)$ and a compact subset $Q \subset G$, it suffices to show that the matrix coefficient function

$$g \longmapsto \langle f, \pi_{\mu_n}(g)f \rangle_{\mu_n} = \mu_n((s(g)f)^{\sim} * f)$$

converges to $\mu((s(g)f)^{\sim} * f)$ uniformly over $g \in Q$. The vague convergence assumption implies that this holds pointwise for each g. However, the set $\{(s(g)f) * f : g \in Q\}$ is compact in $C_c(G)$ for the topology of uniform convergence on compact subsets because the orbital map $g \mapsto s(g)f$ is continuous. So we only need to recall that the vague convergence of a *sequence* is actually uniform on compact subsets of $C_c(G)$, see Proposition 17(ii) in [Bou65, Chapter III, §1, no. 10].

We shall also need the following basic fact.

Lemma 2.6. Let G be a locally compact group and X be a locally compact G-space. For each $x \in X$ and every compact subset $K \subset G$ with $K \cap G_x = \emptyset$, there exists an open neighbourhood α of x such that $g\alpha \cap \alpha = \emptyset$ for all $g \in K$.

Proof. Suppose it is not the case. Then for each open neighbourhood α of x, there exists $g_{\alpha} \in K$ and $x_{\alpha} \in \alpha$ with $g_{\alpha}(x_{\alpha}) \in \alpha$. The compactness of K ensures that (g_{α}) subconverges to an element $g \in K$ with g(x) = x, contradicting the hypothesis that $K \cap G_x = \emptyset$.

2.2. URS and Fell's continuity theorem. We recall Fell's continuity theorem:

Theorem 2.7 (Fell 1964). The quasi-regular representation $\lambda_{G/H}$, viewed as map from the Chabauty space of closed subgroups H < G of a given locally compact group G to the Fell space of equivalence classes of unitary G-representations, is continuous.

Proof. This is contained in Theorem 4.2 of [Fel64].

The relation with Lemma 2.5 is that one can embed the Chabauty space into the space of Radon measures on G by assigning to H < G a suitably normalized Haar measure on H, viewed as a measure on G. When both modular functions Δ_G and Δ_H are trivial on H, this directly gives a proof of Theorem 2.7; we spell out the arugment for completeness:

Proposition 2.8. Let G be a lcsc group and (H_n) be a sequence of unimodular closed subgroups, all contained in $\text{Ker}(\Delta_G)$, that converges to $H \leq G$ in the Chabauty topology. Then λ_{G/H_n} converges to $\lambda_{G/H}$ in Fell's topology.

We note that the assumptions on the modular functions are satisfied in the setting of Theorem F since it is concerned with regionally elliptic subgroups H. Accordingly, the proof of that theorem will invoke Lemma 2.5 rather than the full generality of Theorem 2.7.

Proof of Proposition 2.8. Let V be a compact identity neighbourhood in G. For each closed subgroup $J \leq G$, we fix the left Haar measure α_J on J such that $\alpha_J(J \cap V) = 1$. We view α_J as a measure defined on G and supported on J. The map $J \mapsto \alpha_J$ defines a homeomorphism of the Chabauty space $\mathbf{Sub}(G)$ onto its image, which is endowed with the vague topology, see [Bou63a, Chapter VIII, §3 and §6]. Moreover, the set of unimodular closed subgroups is Chabauty closed by [Bou63a, Chapter VIII, §3, Theorem 1]. It follows that H is unimodular. For each *n* define the measure μ_n by $d\mu_n = \sqrt{\Delta_G} d\alpha_{H_n}$, and define μ by $d\mu = \sqrt{\Delta_G} d\alpha_H$, where Δ_G is the modular function of *G*. Since $H_n \leq \text{Ker}(\Delta_G)$ by hypothesis, we have $H \leq \text{Ker}(\Delta_G)$, and it follows $\mu_n = \alpha_{H_n}$, and $\mu = \alpha_H$. Therefore, we deduce from Theorem 2.3 that $\pi_{\alpha_{H_n}}$ (resp. π_{α_H}) is equivalent to λ_{G/H_n} (resp. $\lambda_{G/H}$). By hypothesis, we know that α_{H_n} vaguely converges to α_H . Hence, the required conclusion follows from Lemma 2.5.

Let us now consider a locally compact group G. A uniformly recurrent subgroup (or URS) is a minimal G-invariant closed subset of $\operatorname{Sub}(G)$. We recall from [GW15] that every minimal compact G-space X yields a URS called the stabilizer URS, defined as the unique URS contained in the closure of the set of stabilizers $\{G_x \mid x \in X\}$. It is denoted by $\mathcal{ST}_G(X)$. If $\operatorname{Sub}(G)$ is metrizable (which is automatic if G is second countable), then a classical semi-continuity argument ensures that X contains a dense G_{δ} -set of points x such that the stabilizer map $x \mapsto G_x$ is continuous (see [Kur28, Theorem VII]). The set of those continuity points is denoted by X_0 . For $x \in X_0$, we have $G_x \in \mathcal{ST}_G(X)$.

By Fell's continuity theorem, if \mathcal{Y} is a URS of G, then for any two $Y_1, Y_2 \in \mathcal{Y}$, the representations $\lambda_{G/Y_1}, \lambda_{G/Y_2}$ are weakly equivalent. In particular \mathcal{Y} yields a canonical quotient of the maximal C^* -algebra of G, defined by

$$C^*(\mathcal{Y}) = C^*_{\lambda_{G/Y}}(G),$$

where Y is an arbitrary element of \mathcal{Y} . This fact has recently been observed in the case of a discrete group G by T. Kawabe [Kaw17] and G. Elek [Ele18].

Remark 2.9. The fact that the set X_0 of continuity points of the stabilizer map $X \to \mathbf{Sub}(G)$ is a dense G_{δ} holds more generally, for the same reason, if there is a closed metrizable subset $\mathcal{M} \subseteq \mathbf{Sub}(G)$ such that $G_x \in \mathcal{M}$ for all $x \in X$. This happens notably in the case where the quotient of G by the kernel W of the G-action on X is second countable: indeed the closed subset $\{H \in \mathbf{Sub}(G) \mid H \geq W\} \subseteq \mathbf{Sub}(G)$ is homeomorphic to $\mathbf{Sub}(G/W)$, and is thus metrizable.

3. Koopman and quasi-regular representations

This section is devoted to the proofs of Theorems F and G stated in the Introduction. We shall occasionally use the shorthand $\pi \prec \sigma$ to denote that π is weakly contained in σ .

3.1. The Koopman representation weakly contains a quasi-regular representation. In the setting of Theorem F, we invoke the existence of a compact model, i.e. a metrizable compact space X' with a continuous G-action, and a quasiinvariant probability measure ν' on X', such that the G-spaces (X, ν) and (X', ν') are measurably isomorphic. This well-known fact follows from [Var63, Theorem 3.2], which affords the metrizable compact G-space X' and a G-equivariant isomorphism of Borel spaces $\phi: X \to X'_0$, where X'_0 is a G-invariant Borel subset. The measure ν' can then be defined by setting $\nu' = \phi_*(\nu)$. Given the existence of compact models, Theorem F stated in the introduction directly follows from the topological version of that result.

Theorem 3.1. Let G be a second countable tdlc group, X be a locally compact Gspace, and ν a G-quasi-invariant σ -finite Radon measure on X such that $L^2(X,\nu)$ is separable. Let κ be the Koopman representation of G on $L^2(X,\nu)$.

If $x \in X$ is a point belonging to the support of ν and such that the stabilizer G_x is regionally elliptic, then the quasi-regular representation λ_{G/G_x} is weakly contained in the Koopman representation κ .

Proof. We choose a decreasing sequence U_n of relatively compact open subsets of G forming a neighbourhood basis at e. We further choose an increasing sequence of compact subsets K_n covering G. Finally, let Q_n be an increasing sequence of compact subgroups of G_x covering G_x . Note that K_n and Q_n ultimately cover any compact subsets of G and G_x , respectively. Define $K'_n = K_n \setminus U_n G_x$; this is a compact set. Therefore, by Lemma 2.6, there is a neighbourhood α_n of x such that $K'_n \alpha_n \cap \alpha_n = \emptyset$.

For each n, we now choose an open neighbourhood β_n of x such that $g^{-1}\beta_n \subseteq \alpha_n$ holds for all $g \in Q_n$. This is possible since Q_n is compact and G_x fixes x. The characteristic function $\mathbf{1}_{\beta_n}$ is a non-negative element of $L^1(X, \nu)$ which is non-zero since x is in the support of ν . Now the element

$$\xi_n = \kappa_{\nu}(\mathbf{1}_{Q_n})\mathbf{1}_{\beta} = \int_{Q_n} \kappa_{\nu}(g)\mathbf{1}_{\beta} \, dg \in L^1(X,\nu)$$

is non-negative, non-zero, supported in α , and Q_n -invariant.

We now define measures μ'_n and μ_n on G as follows. Let $d\mu'_n(g) = (\kappa(g)\xi_n, \xi_n)dg$ for some choice of right Haar measure on G and $\mu_n = \mu'_n/\mu'_n(U_0)$. Note that μ_n is invariant under right translation by Q_n , and vanishes on K'_n by the choice of ξ_n .

We first claim that for any compact set $\Omega \subset G$, there is m such that for all $n \geq m$ we have

$$\Omega \subseteq K'_n \cup U_n Q_m.$$

Indeed, take *m* large enough to have $\Omega \subseteq K_m$ and $G_x \cap U_0^{-1}\Omega \subseteq Q_m$. Then any $\omega \in \Omega \setminus K'_n$ can be written $\omega \in uh$ for some $u \in U_n$ and $h \in G_x$. This *h* is in $U_n^{-1}\Omega$ and hence in Q_m , so that $\omega \in U_n Q_m$ as claimed.

Next, we claim that the sequence $\mu_n(\Omega)$ remains bounded for any given compact set $\Omega \subset G$. Fix m as in the first claim and consider any $n \geq m$. Since the sequence U_m is decreasing and a basis of relatively compact neighbourhoods, we can assume $\overline{U_m} \subseteq U_0$ after possibly increasing m. Since the first claim implies in particular $\Omega \subseteq K'_n \cup U_m Q_m$, it suffices to bound $\mu_n(U_m Q_m)$ independently of n. By compactness of $\overline{U_m}Q_m$, there are $q_1, \ldots, q_r \in Q_m$ (independent of n) such that $U_m Q_m$ is in the union of the r translates U_0q_i . Since each μ_n is (right) Q_m -invariant, it follows $\mu_n(U_m Q_m) \leq r \mu_n(U_0)$ for all n, and the claim follows.

Upon passing to a subsequence, it follows from Lemma 2.4 that (μ_n) vaguely converges to a positive definite measure μ . Since $\mu_n(U_0) = 1$ for all n, we have $\mu(U) = 1$. Since $\mu_n(\Omega \cap K'_n) = 0$ for all n, it follows that μ is supported on G_x . Moreover, we observe that μ is (right) G_x -invariant since $G_x = \bigcup_n Q_n$ and since μ_n is (right) Q_n -invariant for all n. Hence, by [Bou63a, Chapter VIII, §5.1, Lemma 1], we infer that μ is a Haar measure on G_x . Note that G_x is unimodular since it is regionally elliptic. Moreover, since every element of G_x is contained in a compact subgroup, we see that G_x is contained in Ker (Δ_G) , where Δ_G is the modular function of G. Therefore, we have $\sqrt{\Delta_G} d\mu = d\mu$, so that the representation π_{μ} is equivalent to the quasi-regular representation λ_{G/G_x} by Theorem 2.3.

Finally, we notice that the unitary representation π_{μ_n} is contained in κ for all n, since π_{μ_n} is the representation associated with the positive definite function $g \mapsto (\kappa(g)\xi_n, \xi_n)$ via the GNS construction. From Lemma 2.5, we deduce $\pi_{\mu} \prec \kappa$, which completes the proof.

3.2. The Koopman representation is weakly contained in a quasi-regular representation. In this section we prove Theorem G. We will need the following general fact which should be of independent interest.

Proposition 3.2. Let G be a second countable locally compact group, and X a minimal locally compact G-space. Assume $x \in X$ is such that the stabilizer G_x is amenable. Then $\lambda_{G/G_x} \prec \lambda_{G/G_y}$ for every $y \in X$.

Proof. Let $y \in X$ be arbitrary. Since the *G*-action on *X* is minimal, there is a sequence (g_n) in *G* such that $(g_n y)$ converges to *x*. By passing to a subsequence, we may assume that $G_{g_n y}$ Chabauty converges to some subgroup $H < G_x$. By Fell's continuity theorem (Theorem 2.7), the quasi-regular representation $\lambda_{G/G_{g_n y}}$ converges to $\lambda_{G/H}$. Since $\lambda_{G/G_{g_n y}} = \lambda_{G/g_n G_y g_n^{-1}}$ is equivalent to λ_{G/G_y} for all *n*, we deduce $\lambda_{G/H} \prec \lambda_{G/G_y}$.

Since G_x is amenable, the trivial representation $\mathbf{1}_{G_x}$ of G_x is weakly contained in $\lambda_{G_x/H}$ (see [Eym72, n° 2, § 4]). Inducing up to G, we deduce $\lambda_{G/G_x} \prec \lambda_{G/H}$. We have seen that the latter is weakly contained in λ_{G/G_y} . Thus the proof is complete. \Box

Proof of Theorem G. Denote by $X_1 \subseteq X$ the conull subset consisting of those points $x \in X$ whose stabilizer G_x is amenable. Since the G-action on X is minimal by hypothesis, the assertion (i) follows from Proposition 3.2.

We shall now prove the assertion (ii).

Fix a regular probability measure μ on G in the class of the Haar measures. Then for every $x \in X$, the measure $\mu_x := p_*^x(\mu)$ is quasi-invariant, where $p^x : G \to G/G_x$ is the canonical projection.

We endow the product $G \times X$ with a G-action defined by $g \cdot (h, x) = (hg^{-1}, gx)$. This action preserves the class of the measure $\mu \times \nu$.

Let now $Y \subset X \times X$ be the orbit equivalence relation for the *G*-action on *X*. We endow *Y* with the *G*-action defined by $g \cdot (x, y) = (gx, y)$. Consider the Borel map $p: G \times X \to Y$ defined by p(g, x) = (x, gx). Observe that *p* is *G*-equivariant. We define $\nu_Y = p_*(\mu \times \nu)$, so that ν_Y is quasi-invariant under *G*. Moreover, we have $\nu_Y = \int_X (\delta_x \times \mu_x) d\nu(x)$, where we have identified the *G*-orbit of *x* with G/G_x . This implies that the Koopman representation κ_{ν_Y} is equivalent to $\int_X \lambda_{G/G_x} d\nu(x)$.

Since the action of G on (X, ν) is amenable in the sense of Zimmer, we have $\nu(X_1) = 1$. In view of the assertion (i), it follows that $\int_X \lambda_{G/G_x} d\nu(x)$ is weakly equivalent to λ_{G/G_z} for each $z \in X_1$. Therefore κ_{ν_Y} is weakly equivalent to λ_{G/G_z} for any $z \in X_1$.

Now, consider the projection $q: Y \to X$ onto the first coordinate. The map q is Borel measurable and G-equivariant. Moreover, it follows from the definitions that $q_*(\nu_Y) = \mu * \nu$. By Lemma 2.2 we have $\mu * \nu \sim \nu$. Since the action of G on $(X, \mu * \nu)$ is amenable in the sense of Zimmer (see [ADR00, Proposition 3.3.5], and [Ada96, Theorem 6.8] or [Kai04]), it follows from [AD03, Proposition 4.3.2 & Theorem 3.2.1] that κ_{ν_Y} weakly contains the Koopman representation of G on $L^2(X, \mu * \nu)$. Therefore, using Lemma 2.1, we conclude $\kappa_{\nu} \prec \lambda_{G/G_z}$ for all $z \in X_1$.

4. Hyperbolic locally compact groups

A locally compact group G is called **hyperbolic** if it has a compact generating set with respect to which the word metric on G is Gromov-hyperbolic. By [CCMT15, Proposition 2.1], this is equivalent to requiring that G has a continuous, proper, cocompact, isometric action on a locally compact geodesic metric space X that is Gromov-hyperbolic. This space is automatically proper by the Hopf–Rinow theorem, since cocompactness ensures completeness.

In this section, we assume that G is locally compact hyperbolic and fix a space X as above.

We freely refer to Gromov's typology concerning isometries and isometric group actions on hyperbolic spaces, as recalled in e.g. [CCMT15, §3.1]. We recall that

the **displacement length** of an isometry g is $|g| = \inf\{d(gx, x) : x \in X\}$ and its asymptotic displacement length is $|g|_{\infty} = \lim_{n\to\infty} \frac{1}{n}d(g^nx, x)$, which does not depend on x. We have $|g|_{\infty} \leq |g|$ and $|g| \leq |g|_{\infty} + 16\delta$, where δ is a hyperbolicity constant for X. Moreover, $|g|_{\infty}$ is positive if and only if g is hyperbolic. For all this, see [CDP90, Chapter 10 §6].

We also use throughout that given $x \in X$ and $\xi \neq \xi' \in \partial X$, there exists a geodesic ray from x to ξ and a geodesic line from ξ' to ξ . See e.g. Proposition 4 in [GdlH90, Chap. 7] or [BH99, III.H.3.1] for the former and Proposition 6 in [GdlH90, Chap. 7] or [BH99, III.H.3.2] for the latter.

Whenever a constant can be chosen depending only on X, we shall call it an X-constant; most of the time it even depends only on the hyperbolicity constant of X.

4.1. The structure of relatively amenable subgroups. Our goal is to describe the algebraic structure of (relatively) amenable closed subgroups of the hyperbolic locally compact group G. Similar results for groups acting properly cocompact on proper CAT(0) spaces can be found in [CM13]. We first describe the geometric features of the action of a relatively amenable subgroup, and then derive algebraic information.

Proposition 4.1. For any closed subgroup $H \leq G$, the following assertions are equivalent.

- (i) H is amenable.
- (ii) H is relatively amenable in G.
- (iii) H is compact, or H fixes a point $\xi \in \partial X$, or H stabilizes a pair $\{\xi, \xi'\} \subseteq \partial X$.

Proof. That (i) implies (ii) is true in full generality.

Assume that H is relatively amenable. If the H-action on X is bounded, then H is compact since the G-action is proper and H is closed by hypothesis. If the H-action is horocyclic or focal (resp. lineal), then H fixes a point $\xi \in \partial X$ (resp. H stabilizes a pair $\{\xi, \xi'\} \subseteq \partial X$). Therefore, in order to show that (iii) holds, it suffices to prove that the H-action on X cannot be of general type. Suppose for a contradiction that it is. Then H contains a Schottky subgroup Λ (see [CCMT15, Lemma 3.3]). The action of a Schottky subgroup Λ on ∂X is minimal and strongly proximal, so that Λ , and hence also H, does not fix any probability measure on ∂X . This implies that H is not relatively amenable in G, thereby confirming that (ii) implies (iii).

That (iii) implies (i) is a consequence of a well-known result of S. Adams, cited as Lemma 3.10 in [CCMT15], ensuring that the stabilizer of each point $\xi \in \partial X$ in the full isometry group of X is amenable.

Our next goal is to describe the algebraic structure of the stabilizer G_{ξ} of a boundary point ξ , or the stabilizers $G_{\{\xi,\xi'\}}$ of a boundary pair. To that end, we shall use the **Busemann character** $\beta_{\xi} \colon G_{\xi} \to \mathbf{R}$ afforded by [CCMT15, Corollary 3.9]. It is a continuous homomorphism whose kernel is denoted by G_{ξ}^0 . Moreover, by [CCMT15, Lemma 3.8], the subgroup G_{ξ}^0 consists of those elements $g \in G_{\xi}$ that act as elliptic or parabolic isometries on X; in other words an element $g \in G_{\xi}$ satisfies $\beta_{\xi}(g) \neq 0$ if and only if it is a hyperbolic isometry of X.

Lemma 4.2. Let $\xi \neq \xi' \in \partial X$. Then $G^0_{\xi,\xi'} := G^0_{\xi} \cap G_{\xi'} = G_{\xi} \cap G^0_{\xi'}$ is a compact normal subgroup of $G_{\{\xi,\xi'\}}$. Moreover, the quotient group $G_{\{\xi,\xi'\}}/G^0_{\xi,\xi'}$ is trivial or isomorphic to $\mathbf{Z}, \mathbf{Z} \rtimes \{\pm 1\}, \mathbf{R}$ or $\mathbf{R} \rtimes \{\pm 1\}$.

Proof. Notice that $G_{\xi}^0 \cap G_{\xi'}$ consists of the elements of $G_{\xi} \cap G_{\xi'}$ that are not hyperbolic. The same description applies to $G_{\xi} \cap G_{\xi'}^0$. Hence $G_{\xi}^0 \cap G_{\xi'} = G_{\xi} \cap G_{\xi'}^0$. Denoting that subgroup by W, we may complete the proof by invoking the same argument as in the proof of [CCMT15, Proposition 5.6].

Following [CM13, §4.1], we say that a subgroup $A \leq G$ is **compactible** with limit K if $K \leq G$ is a compact subgroup such that for any neighbourhood U of K in G and every finite subset $F \subseteq A$, there is $\alpha \in \operatorname{Aut}(G)$ with $\alpha(F) \subseteq U$. If this is the case, we say that a sequence (g_n) in G is a **compacting sequence** for A if for all $a \in A$, the sequence $(g_n a g_n^{-1})$ is bounded and each of its accumulation point belongs to K.

Lemma 4.3. Let $\xi \in \partial X$. There exists a sequence (g_n) in G and a pair $\eta \neq \eta'$ in ∂X such that G^0_{ξ} is compactible with limit $G^0_{\eta,\eta'}$, and (g_n) is a compacting sequence for G^0_{ξ} .

Proof. Fix a geodesic ray $r: \mathbf{R}_+ \to X$ with endpoint ξ . Let $g_n \in G$ be such that the sequence $g_n(r(n))$ is bounded. Upon extracting, we may assume that $g_n r(\psi(n) + t)$ converges for every $t \in \mathbf{R}$, where $\psi: \mathbf{N} \to \mathbf{N}$ is a strictly increasing function. More precisely, note that for any fixed t, the point $r(\psi(n) + t)$ is defined when n is large enough and $g_n r(\psi(n) + t)$ remains in a bounded set. Finding a common subsequence for all $t \in \mathbf{R}$ can be achieved e.g. by applying a diagonal argument for $t \in \mathbf{Q}$ and then using that r is uniformly continuous. We can further assume that $g_n(\xi)$ converges to a point $\eta \in \partial X$ and that $g_n(r(0))$ converges to $\eta' \in \partial X$. There is now a geodesic line ℓ joining η' to η , namely $\ell(t) = \lim_n g_n r(\psi(n) + t)$. In particular we have $\eta \neq \eta'$. It now follows from the definition that for each $a \in G_{\xi}^0$, the sequence $(g_n a g_n^{-1})$ is bounded (because the sequence $n \mapsto d(a g_n^{-1}(\ell(0)), g_n^{-1}(\ell(0)))$ is bounded), and that each of its accumulation points belongs to $G_{\eta,\eta'}$.

We claim that each accumulation point of $(g_n a g_n^{-1})$ actually belongs to $G_{\eta,\eta'}^0$. To prove that claim, we shall use the existence of an X-constant K such that for each $g \in G_{\xi}^0$, we have $d(gr(n), r(n)) \leq K$ for all sufficiently large n (see [BMW12, Lemma 21]). Let now $t \in G_{\eta,\eta'}$ and a subsequence $(g_{\phi(n)}a g_{\phi(n)}^{-1})$ converging to t and such that $(g_{\phi(n)}(r(n)))$ converges to some point $x \in X$. For each integer N, the sequence $(g_{\phi(n)}a^N g_{\phi(n)}^{-1})$ converges to t^N . We have

$$d(t^{N}(x), x) = \lim_{n} d(g_{\phi(n)}a^{N}g_{\phi(n)}^{-1}(x), x)$$

=
$$\lim_{n} d(a^{N}g_{\phi(n)}^{-1}(x), g_{\phi(n)}^{-1}(x))$$

=
$$\lim_{n} d(a^{N}r(n), r(n))$$

< K.

This ensures that the isometry t is elliptic, thereby proving the claim.

Since $G^0_{\eta,\eta'}$ is compact by Lemma 4.2, we conclude that (g_n) is indeed a compacting sequence for G^0_{ξ} , and that G^0_{ξ} is compactible with limit $G^0_{\eta,\eta'}$.

Proposition 4.4. Assume that G is totally disconnected. Then, for each $\xi \in \partial X$, the group G^0_{ξ} is regionally elliptic.

Proof. In view of Lemma 4.3, this is a direct consequence of [CM13, Proposition 4.2]. \Box

We close this subsection by recording the following result from [CCMT15]. It will be used repeatedly to various questions on hyperbolic locally compact groups to the totally disconnected case. **Proposition 4.5.** Let G be a non-amenable hyperbolic locally compact group. Then G has a largest compact normal subgroup W and the quotient G/W is either a virtually connected rank one simple Lie group, or a totally disconnected group.

Proof. Since a non-amenable hyperbolic locally compact group is of general type by Proposition 4.1, the result follows from [CCMT15, Proposition 5.10]. \Box

Remark 4.6. In a σ -compact locally compact group G, every identity neighbourhood contains a compact normal subgroup N such that G/N is second countable, see [KK44]. In particular the quotient G/W appearing in Proposition 4.5 is second countable. Since W acts trivially on the Gromov boundary of G, it follows from Remark 2.9 that the set of continuity points of the stabilizer map $\partial G \to \mathbf{Sub}(G)$ is a dense G_{δ} , although the group G itself need not be second countable. We shall frequently use this fact without further notice.

4.2. Ballistic boundary points. The structure of hyperbolic locally compact groups with a cocompact amenable subgroup has been described in [CCMT15]. If the group is elementary, it is either compact, or 2-ended, in which case it is described by [CCMT15, Proposition 5.6]. If the group is non-elementary, it is described by Theorem A or D in [CCMT15], depending on whether the group is amenable or not. For a non-amenable locally compact hyperbolic group G, the existence of a cocompact amenable subgroup is equivalent to the fact that G acts 2-transitively on its Gromov boundary (see [CCMT15, Theorem 8.1]).

The following theorem establishes an additional powerful criterion, namely it suffices to verify $G_{\xi}^{0} \neq G_{\xi}$ for suitable points at infinity ξ . We shall call ξ a **ballistic** point when $G_{\xi}^{0} \neq G_{\xi}$ is satisfied.

Theorem 4.7. Let G be a non-amenable hyperbolic locally compact group and X be a proper geodesic metric space on which G acts continuously, properly and cocompactly by isometries.

The following conditions are equivalent.

- (i) G has a cocompact amenable subgroup.
- (ii) The G-action on the Gromov boundary ∂X is 2-transitive.
- (iii) For all $\xi \in \partial X$, we have $G_{\xi}^0 \neq G_{\xi}$.
- (iv) For some continuity point $\eta \in \partial X$ of the stabilizer map, we have $G_n^0 \neq G_n$.
- (v) There is a constant K such that, for every hyperbolic element $\gamma \in G$, there exists a hyperbolic element $\gamma' \in G$ with asymptotic displacement length $|\gamma'|_{\infty} \leq K$ such that γ and γ' share the same pair of fixed points in ∂X .

As indicated above, the equivalence between (i) and (ii) is taken from [CCMT15]. A special instance of the equivalence between (ii) and (iii) has been observed for specific families of groups of tree automorphisms by C. Ciobotaru in her PhD thesis [Cio14, Proposition 2.2.11].

Before embarking on the proof, we record the following classical criterion to identify hyperbolic isometries.

Lemma 4.8. Let g be an isometry of a δ -hyperbolic metric space and fix any a > 0. If there exists a point x with $d(g^{-1}x, gx) \ge d(x, gx) + 2\delta + a$, then g is hyperbolic with $|g|_{\infty} \ge a$.

Proof. The fact that g is hyperbolic is proved in [CDP90, Chapter 9, Lemma 2.2]. The proof given there proceeds by establishing $d(g^n x, x) \ge na$ for all positive integers n, which yields also the estimate on $|g|_{\infty}$.

There is a sort of converse upon replacing g by a power:

Lemma 4.9. If g is hyperbolic and a > 0, then for every x there is $n \ge 1$ with $d(g^{-n}x, g^nx) \ge d(x, g^nx) + 2\delta + a$.

Proof. Fix g, a and x and suppose there is no such n. Then, plugging the inequality

$$d(x, g^{2n}x) = d(g^{-n}x, g^nx) < d(x, g^nx) + 2\delta + a$$

in the definition of $|g|_{\infty}$ shows $|g|_{\infty} \leq \frac{1}{2}|g|_{\infty}$, hence $|g|_{\infty} = 0$ and g is not hyperbolic.

We also have the following behaviour with respect to limits in Isom(X).

Lemma 4.10. The set of hyperbolic elements is open in Isom(X).

Moreover, given a sequence (g_n) converging to $g \in \text{Isom}(X)$, we have $\limsup_n |g_n| \le |g|$ and $\limsup_n |g_n|_{\infty} \le |g|_{\infty} + 16\delta$.

Proof. Suppose first that (g_n) converges to a hyperbolic isometry g. By Lemma 4.9, we can choose x and n_0 such that $d(g^{-n_0}(x), g^{n_0}(x)) \ge d(x, g^{n_0}(x)) + 2\delta + 2$. It follows that for all n large enough we have $d(g_n^{-n_0}(x), g_n^{n_0}(x)) \ge d(x, g_n^{n_0}(x)) + 2\delta + 1$. Now Lemma 4.8 implies that $g_n^{n_0}$ is hyperbolic, and hence so is g_n .

Regardless of the type of isometries, the first estimate follows from the definition of the displacement length and the second from the estimates between $|\cdot|$ and $|\cdot|_{\infty}$ recalled earlier.

Within the stabilizer of a point at infinity, we have more continuity because the asymptotic displacement length can be read on the (continuous) Busemann character as follows.

Proposition 4.11. Let g be an isometry fixing $\xi \in \partial X$. Then $|g|_{\infty} = |\beta_{\xi}(g)|$.

Just like the three above lemmas, this proposition holds for any geodesic hyperbolic space X; however, we should recall that if one applies it beyond the *proper* case (to which locally compact groups reduce us here), then β_{ξ} is a priori only a continuous homogeneous quasimorphism rather than a homomorphism (cf. [CCMT15, §3]). This does not affect the proposition nor its proof, which relies on the following.

Lemma 4.12. Let g be an isometry fixing $\xi \in \partial X$ and let $x \in X$. If g is not parabolic, then the difference $n|\beta_{\xi}(g)| - d(g^n x, x)$ remains bounded over $n \ge 0$.

Proof of Lemma 4.12. Since $\beta_{\xi}(g)$ vanishes if and only if g is non-hyperbolic [CCMT15, Lemma 3.8], we can assume g hyperbolic. Upon replacing g with its inverse, we can further assume that ξ is the attracting point of g, or equivalently $\beta_{\xi}(g) > 0$ (see again [CCMT15, Lemma 3.8]). Upon enlarging the desired bound, we suffer no loss of generality when replacing g by a (fixed) power of itself before varying n. Therefore, using Lemma 4.9, we can assume $d(g^2x, x) \geq d(gx, x) + 2\delta + 1$. Then the orbit $(g^n x)$ is quasigeodesic, see e.g. the proof of Lemma 2.2 in [CDP90, Chap. 9]. By stability of quasigeodesics (Theorem 25(i) in [GdlH90, Chap. 5]), it follows that there is a constant D such that $(g^n x)_{n\geq 0}$ remains at distance at most D from a geodesic ray r. Note that $r(+\infty) = \xi$ since $g^n x \to \xi$, and upon increasing D we can assume r(0) = x. We choose $s_n \geq 0$ such that $d(r(s_n), g^n x) \leq D$ for all n.

According to Proposition 3.7 in [CCMT15], we can write $\beta_{\xi}(g) = \lim_{n \to \infty} \frac{1}{n}h(x, g^n x)$ where the function h(x, y) of x, y is any accumulation point (for the pointwise convergence) of d(x, r(s)) - d(y, r(s)) as $s \to \infty$. Note that d(x, r(s)) = s, while $|d(g^n x, r(s)) - |s - s_n||$ is bounded by D. Letting $s \to \infty$, we deduce that $|h(x, g^n x) - s_n|$ is bounded by D. Using one more time $d(r(s_n), g^n x) \leq D$, we conclude

$$\left|h(x, g^n x) - d(x, g^n x)\right| \le 2D.$$

It is also shown in Proposition 3.7 of [CCMT15] that the difference $|h(x, g^n x) - \beta_{\xi}(g^n)|$ is bounded independently of n; now the lemma follows since $\beta_{\xi}(g^n) = n\beta_{\xi}(g)$.

Proof of Proposition 4.11. We can assume that g is hyperbolic thanks to [CCMT15, Lemma 3.8]. The proposition then follows from Lemma 4.12 by dividing by n and letting $n \to \infty$.

As mentionned in the proof of Lemma 4.12, it is well-known that the orbits of hyperbolic isometries are quasigeodesic. We can now strengthen this conclusion as follows, removing any multiplicative error factor.

Corollary 4.13. Let g by an isometry of a geodesic hyperbolic space X and let $x \in X$. If g is not parabolic, then the difference

$$|g|_{\infty} \cdot |n-m| - d(g^n x, g^m x)$$

remains bounded over $n, m \in \mathbf{Z}$.

Proof. Since g is isometric, it suffices to show the statement for m = 0 and $n \ge 0$. We can again assume that g is hyperbolic; in particular it fixes some $\xi \in \partial X$. Now the statement follows by combining Lemma 4.12 with Proposition 4.11.

Lemma 4.14. Let G and X be as in Theorem 4.7 and assume that (iv) holds. There is a constant K such that for each $\xi \in \partial X$, there exists a hyperbolic isometry $t \in G_{\xi}$ with $|t|_{\infty} \leq K$.

Proof. Since $G_{\eta}^{0} \neq G_{\eta}$, there is $\gamma \in G_{\eta}$ hyperbolic. We can take any $K > |\gamma|_{\infty} + 16\delta$. Let indeed $\xi \in \partial X$ be arbitrary. By hypothesis, the group G is non-amenable, hence the G-action on X is of general type. This implies that the G-action on ∂X is minimal. Therefore, there exists (g_{n}) in G such that the sequence $(g_{n}\xi)$ converges to η . By the choice of η , we know that the sequence $(g_{n}G_{\xi}g_{n}^{-1})$ Chabauty converges to G_{η} . Therefore there exist $t_{n} \in G_{\xi}$ such that the sequence $(g_{n}t_{n}g_{n}^{-1})$ converges to γ . For all n large enough, it follows from Lemma 4.10 that $g_{n}t_{n}g_{n}^{-1}$ is hyperbolic, with asymptotic displacement length at most K. Since the asymptotic displacement length is invariant under conjugation, we deduce that t_{n} is hyperbolic with $|t_{n}|_{\infty} \leq K$.

We shall use the notion of **duality** due to Chen and Eberlein (compare [Bal95, III.1]). We say that a pair (ξ, ξ') in $\partial X \times \partial X$ is *G*-dual if there exist a sequence (γ_n) in *G* such that $\gamma_n(x_0) \to \xi$ and $\gamma_n^{-1}(x_0) \to \xi'$ for some (hence any) $x_0 \in X$. Since *G* is supposed to be non-amenable, its action on *X* is of general type by Proposition 4.1.

Lemma 4.15. If the G-action on X is of general type, then every pair $(\xi, \xi') \in \partial X \times \partial X$ is G-dual.

Proof. There exists at least some G-dual pair (ξ, ξ') , for instance the attracting and repelling points of some hyperbolic isometry. For this given $\xi' \in \partial X$, the non-empty collection of those $\xi \in \partial X$ such that (ξ, ξ') is G-dual is closed and G-invariant. Since the G-action on ∂X is minimal, we deduce that the pair (ξ, ξ') is G-dual for all $\xi \in \partial X$. By symmetry, it follows that every pair (ξ, ξ') is G-dual.

We now establish two strengthenings of Lemma 4.14.

Lemma 4.16. Let G and X be as in Theorem 4.7. If the assertion (iv) from the latter statement holds, then (v) holds as well.

Proof. Since $G_{\eta}^{0} \neq G_{\eta}$, there is $\gamma \in G_{\eta}$ hyperbolic. Let $\xi \in \partial X$ be the repelling fixed point of γ . By Lemma 4.14, the group G_{ξ} contains a hyperbolic element t with $|t|_{\infty} \leq K$. Notice that the sequence $(\gamma^{n}t\gamma^{-n})$ is bounded in G. Upon extracting, it converges to an element $\gamma' \in G_{\xi}$. Since the Busemann character $\beta_{\xi} \colon G_{\xi} \to \mathbf{R}$ is continuous and $\beta_{\xi}(\gamma^{n}t\gamma^{-n}) = \beta_{\xi}(t)$, we have $\beta_{\xi}(\gamma') = \beta_{\xi}(t)$. Thus Proposition 4.11 implies $|\gamma'|_{\infty} = |t|_{\infty}$, which shows both that γ' is hyperbolic (since this length is positive) and that $|\gamma'|_{\infty} \leq K$.

Finally, if ξ' denotes the attracting fixed point of γ and η the fixed point of t different from ξ , we have $\lim_n \gamma^n \eta = \xi'$, so that γ' fixes ξ' .

Proposition 4.17. Let G and X be as in Theorem 4.7 and suppose that the assertion (v) from the latter statement holds.

There is a constant L such that, for any pair (ξ, ξ') in $\partial X \times \partial X$ with $\xi \neq \xi'$, there is a hyperbolic isometry $\gamma \in G$ with $|\gamma|_{\infty} \leq L$ fixing ξ and ξ' .

The following general lemma will be needed in the proof and again later.

Lemma 4.18. Let X be a geodesic δ -hyperbolic space; there exist constants H, J depending only on δ with the following property.

Let $\ell \subseteq X$ be a geodesic line (the image of a geodesic $\mathbb{Z} \to X$) and choose a map $P: X \to \ell$ such that d(P(x), x) minimizes the distance from x to ℓ for any $x \in X$. If $x, y \in X$ satisfy $d(P(x), P(y)) \geq J$, then any choices of geodesic segments

$$[x, P(x)] \cup [P(x), P(y)] \cup [P(y), y]$$

remain within distance less than H from any geodesic segment [x, y].

Proof of Lemma 4.18. We recall that a (λ, C, J) -local quasigeodesic segment is a path satisfying the (λ, C) -quasigeodesic conditions for parameters less than Japart [GdlH90, Chap. 5 §1]. According to Theorem 21 in [GdlH90, Chap. 5], there are constants H and J depending only on δ , λ and C such that any (λ, C, J) -local quasigeodesic remains at distance less than H from any geodesic segment between its endpoints. Thus we can obtain the constants H, J of the lemma by applying this result to $\lambda = 1$ and C depending only on δ .

By hyperbolicity, the minimizing property of P(x) implies that any $z \in [P(x), P(y)]$ satisfies $d(x, z) \ge d(x, P(x)) + d(P(x), z) - C$ for some C depending only on δ ; indeed, this follows by comparing $[x, P(x)] \cup [P(x), P(y)]$ with a tripod in a tree, as can be done by Theorem 12 in [GdlH90, Chap. 2]. We deduce that $[x, P(x)] \cup [P(x), P(y)]$ is a (1, C)-quasigeodesic. The same holds for $[P(x), P(y)] \cup [P(y), y]$. Thus, as long as $d(P(x), P(y)) \ge J$, the concatenation $[x, P(x)] \cup [P(x), P(y)] \cup [P(y), y]$ is a (1, C, J)-local quasigeodesic as required. \Box

Proof of Proposition 4.17. We choose some point $x_0 \in X$. Since G is non-amenable, its action on X is of general type by Proposition 4.1. By Lemma 4.15, the pair (ξ, ξ') is G-dual, so that there exists (γ_n) in G with $\gamma_n x_0 \to \xi$ and $\gamma_n^{-1} x_0 \to \xi'$.

By the definition of convergence to ξ and ξ' in terms of Gromov products, every geodesic segment joining $\gamma_n^{-1}x_0$ to $\gamma_n x_0$ passes through a ball around x_0 of radius R independent of n. Thus, $d(\gamma_n^{-1}x_0, \gamma_n x_0)$ is at least $2d(\gamma_n x_0, x_0) - 2R$.

By Lemma 4.8, this implies in particular that γ_n is hyperbolic for all sufficiently large n with $|\gamma_n|_{\infty} \to \infty$ since $d(\gamma_n x_0, x_0)$ goes to infinity. We denote by ξ_n^+ and $\xi_n^$ its attracting (resp. repelling) fixed point; that is, $\lim_{k\to\pm\infty} \gamma_n^k x = \xi_n^{\pm}$ for all x, and $\xi_n^+ \neq \xi_n^-$.

Consider the union A_n of all geodesic lines from ξ_n^- to ξ_n^+ ; note that γ_n preserves A_n . Any two such lines remain at distance less than an X-constant D from each another; for instance, this follows by applying the tree approximation theorem (Theorem 12(ii) in [GdlH90, Chap. 2]) to the union of the two geodesics.

We claim that the distance $d(x_0, A_n)$ is bounded independently of n.

Indeed, pick one of the lines ℓ defining A_n and let P, H and J be as in Lemma 4.18. Hyperbolicity implies that all choices for P are within some D' from each other, where D' depends only on the hyperbolicity constant. Since $|\gamma_n^2|_{\infty} = 2|\gamma_n|_{\infty} \to \infty$ and hence also $|\gamma_n^2| \to \infty$, we can assume that γ_n^2 moves every point by more than J + D + D'. Note that the point $\gamma_n^2 P(\gamma_n^{-1}x_0)$ of $\gamma_n^2 \ell$ minimizes the distance from any point $z \in \gamma_n^2 \ell$ to $\gamma_n x_0$ because $d(\gamma_n^2 P(\gamma_n^{-1}x_0), \gamma_n x_0) = d(P(\gamma_n^{-1}x_0), \gamma_n^{-1}x_0) \leq d(\gamma_n^{-2}z, \gamma_n^{-1}x_0)$ since $\gamma_n^{-2}z \in \ell$. Therefore, since ℓ and $\gamma_n^2 \ell$ remain within distance D, we see that $d(\gamma_n^2 P(\gamma_n^{-1}x_0), P(\gamma_n x_0)) \leq D + D'$ and hence $d(P(\gamma_n^{-1}x_0), P(\gamma_n x_0)) \geq J$. Thus Lemma 4.18 applied to

$$S = [\gamma_n^{-1}x_0, P(\gamma_n^{-1}x_0)] \cup [P(\gamma_n^{-1}x_0), P(\gamma_nx_0)] \cup [P(\gamma_nx_0), \gamma_nx_0]$$

shows that x_0 is at distance at most H + R of some $p \in S$. To reach our claim, it remains only to justify that p does not belong to $[\gamma_n^{-1}x_0, P(\gamma_n^{-1}x_0)]$ nor to $[P(\gamma_n x_0), \gamma_n x_0]$, and it suffices by symmetry to show the latter. The image of $[P(\gamma_n x_0), \gamma_n x_0]$ under P has diameter bounded by some X-constant E; this follows e.g. from Proposition 2.1 in of [CDP90, §10]). Therefore, using that P is non-expanding up to another additive X-constant E' (Corollary 2.2 loc. cit.), $p \in [P(\gamma_n x_0), \gamma_n x_0]$ would imply $d(P(x_0), P(\gamma_n x_0)) \leq E + E + H + R'$. Finally, comparing ℓ and $\gamma_n \ell$, we see that $d(P(\gamma_n x_0), \gamma_n P(x_0))$ is bounded by an X-constant and hence we contradict

$$d(P(x_0), \gamma_n P(x_0)) \ge |\gamma_n| \ge |\gamma_n|_{\infty} \to \infty,$$

thus establishing the claim.

Since the assertion (v) holds by hypothesis, there exists a hyperbolic element $\gamma'_n \in G$ with $|\gamma'_n|_{\infty} \leq K$ such that γ'_n and γ_n have the same attracting (resp. repelling) fixed point.

We claim that for any choice $b_n \in A_n$, the distance $d(\gamma'_n b_n, b_n)$ is bounded by K+C, where C is an X-constant. Since A_n is γ'_n -invariant and D-close to a geodesic, it suffices to prove this for some choice $b_n \in A_n$, upon increasing C. Let δ be the hyperbolicity constant of X. There is $y_n \in X$ such that $d(\gamma'_n y_n, y_n) \leq |\gamma'_n| + \delta$, which implies $d(\gamma'_n y_n, y_n) \leq |\gamma'_n|_{\infty} + 17\delta \leq K + 17\delta$. Define $b_n = P(y_n) \in \ell \subseteq A_n$, with ℓ and P as in the first claim. Noting once again that $\gamma'_n b_n$ is within an Xconstant of $P(\gamma' y_n)$ and that P is non-expanding up to an X-constant, we conclude that $d(\gamma'_n b_n, b_n)$ is bounded by $d(\gamma'_n y, y)$ plus an X-constant; the claim follows.

Since the first claim allows us to choose all b_n within a bounded set, the second claim implies that the sequence (γ'_n) is bounded in G, and subconverges to a hyperbolic isometry γ with $|\gamma| \leq L = K + C$, and hence also $|\gamma|_{\infty} \leq L$. It remains to show that γ fixes ξ and ξ' .

Since γ'_n fixes ξ_n^{\pm} , it suffices to show that ξ_n^- converges to ξ' and ξ_n^+ to ξ ; it is enough to show the latter. In terms of Gromov products, we need to show $\langle \xi_n^+, \xi \rangle_{x_0} \to \infty$. The hyperbolicity constant δ of X satisfies

$$\langle \xi_n^+, \xi \rangle_{x_0} + \delta \ge \min\left(\langle \xi_n^+, \gamma_n x_0 \rangle_{x_0}, \langle \gamma_n x_0, \xi \rangle_{x_0}\right)$$

(see e.g. [CDP90, 2§1]). Thus, since $\gamma_n x_0 \to \xi$, it suffices to show $\langle \xi_n^+, \gamma_n x_0 \rangle_{x_0} \to \infty$. This follows by representing ξ_n^+ by a ray within A_n because $d(\gamma_n x_0, A_n) = d(x_0, A_n)$ and this distance is bounded independently of n by the first claim, while $d(x_0, \gamma_n x_0)$ goes to infinity.

The following ingredient is again general.

Lemma 4.19. There is an X-constant R such that for every $\xi \in \partial X$ and every $x \in X$ there is a geodesic $\sigma \colon \mathbf{R} \to X$ with $\sigma(+\infty) = \xi$ and passing within distance R of x.

Proof. Let $\lambda: \mathbf{R} \to X$ be some geodesic. By cocompactness, we can arrange that $\lambda(0)$ is at distance less than D from x, where D is the codiameter of X. Let further $\tau: \mathbf{R}_+ \to X$ be geodesic ray pointing to ξ with $\tau(0) = \lambda(0)$. We obtain two maps $\sigma_{\pm}: \mathbf{R} \to X$ by letting $\sigma_{\pm}(t) = \tau(t)$ for $t \ge 0$ and $\sigma_{\pm}(t) = \lambda(\pm t)$ for t < 0. If X were a metric tree, one of σ_- or σ_+ would already be geodesic. For general X, the tree approximation theorem (Theorem 12(ii) in [GdlH90, Chap. 2]) gives an X-constant C such that one of σ_- or σ_+ is a (1, C)-quasi-geodesic. By Theorem 25(ii) in [GdlH90, Chap. 5], there is therefore a constant H depending only on X such that the corresponding σ_{\pm} is at distance less than H of a true geodesic σ . The statement follows with R = D + H.

We turn to the proof of Theorem 4.7.

Proof of Theorem 4.7. The equivalence between (i) and (ii) follows from [CCMT15, Theorem 8.1]. The latter result also implies that (ii) implies (iii), whereas the implication from (iii) to (iv) is obvious. We next observe that (iv) implies (v) by Lemma 4.16. It remains to prove that (v) implies (i).

Assuming (v), we shall prove that the stabilizer G_{ξ} acts cocompactly on X for every $\xi \in \partial X$. This indeed implies that (i) holds since G_{ξ} is amenable by Proposition 4.1.

Let thus $x_0, x_1 \in X$ be two arbitrary points.

Lemma 4.19 provides two geodesic lines σ_0, σ_1 with σ_i passing within R of x_i and with $\sigma_i(+\infty) = \xi$. This common endpoint implies that there are points y_i of σ_i with $d(y_0, y_1) \leq R'$ for some X-constant R'; this follows e.g. from the tree approximation theorem (Theorem 12(ii) in [GdlH90, Chap. 2]).

Proposition 4.17 implies the existence of a hyperbolic isometry γ_0 of asymptotic displacement length $\leq L$, having the endpoints of σ_0 as its pairs of fixed points in ∂X . The orbit $\{\gamma_0^n x_0\}$ remains at distance at most R + C of σ_0 for some Xconstant C; choose z_n on σ_0 with $d(\gamma_0^n x_0, z_n) \leq R + C$. As already observed in the proof of the second claim of that proposition, γ_0 will move any point of σ_0 by at most L + C' for some X-constant C'. Since $d(\gamma_0 z_n, z_n + 1)$ is bounded by an X-constant by the choice of z_n , it follows that the sequence z_n has gaps bounded by L + C'' for some X-constant C''. In conclusion, there is n_0 with $d(\gamma_0^{n_0} x_0, y_0) \leq$ $d(\gamma_0^{n_0} x_0, z_{n_0}) + d(z_{n_0}, y_0) \leq R + L + C + C''$.

We find likewise γ_1 and n_1 with $d(\gamma_1^{n_1}x_1, y_1) \leq R + L + C + C''$. In conclusion, $\gamma_1^{-n_1}\gamma_0^{n_0}$ is an element of G_{ξ} which maps x_0 to within R' + 2(R + L + C + C'') of x_1 . This witnesses that G_{ξ} acts cocompactly on X and concludes the proof of Theorem 4.7.

Combining Propositions 4.4 and Theorem 4.7, we now deduce the following.

Corollary 4.20. Let G be a non-amenable hyperbolic locally compact group. Let \mathcal{F} be the stabilizer URS for the G-action on its Gromov boundary.

If G does not have any cocompact amenable subgroup, then every $H \in \mathcal{F}$ is regionally elliptic.

Proof. By Theorem 4.7, for every continuity point $\eta \in \partial X$ of the stabilizer map, we have $G_n^0 = G_\eta$.

The corollary is unaffected by replacing G with its quotient by a compact normal subgroup, and such a quotient cannot be a virtually connected simple Lie group

of rank one, since in that case parabolic subgroups are cocompact and amenable. Therefore, by Proposition 4.5, we can assume that G is totally disconnected.

Let now $H \in \mathcal{F}$. Then $H \leq G_{\xi}$ for some $\xi \in \partial X$, and there is a sequence (g_n) in G such that $(g_n G_\eta g_n^{-1})$ Chabauty converges to H. In view of Proposition 4.4, it suffices to show $H \leq G_{\xi}^0$. Suppose for a contradition that this is not the case; then X contains a hyperbolic isometry t. By Chabauty convergence, we can choose $t_n \in G_\eta$ such that $(g_n t_n g_n^{-1})$ converges to t. We then deduce from Lemma 4.10 that t_n is hyperbolic for all large n. This is impossible since $G_\eta = G_\eta^0$.

4.3. Roughly similar word metrics. The goal of this section is to complete the proof of Theorem J. In view of Proposition 4.5, we shall focus on totally disconnected groups.

Let (X, d_X) and (Y, d_Y) be metric spaces. Recall that a map $\varphi \colon X \to Y$ is a **quasi-isometric embedding** if there exist constants L > 0 and $C \ge 0$ such that

$$\frac{1}{L}d_X(x,x') - C \le d_Y(\varphi(x),\varphi(x')) \le Ld_X(x,x') + C$$

We say that φ is a **roughly homothetic embedding** if there exist constants L > 0and $C \ge 0$ such that

$$Ld_X(x, x') - C \le d_Y(\varphi(x), \varphi(x')) \le Ld_X(x, x') + C.$$

Thus Corollary 4.13 states precisely that every orbit of a hyperbolic isometry of a hyperbolic geodesic metric space is a rough embedding of \mathbf{Z} into that space, where \mathbf{Z} is endowed with the Euclidean metric.

A quasi-isometric (resp. roughly homothetic) embedding $\varphi \colon X \to Y$ is called a **quasi-isometry** (resp. a **rough similarity**) if Y is contained in a bounded neighbourhood of $\varphi(X)$.

It is easy to see that any two word metrics on \mathbf{Z} given by finite generating sets are roughly similar (this can also be considered as a trivial case of Corollary 4.13). This contrasts with the free group F_2 , which has numerous pairs of finite generating sets giving rise to word metrics that are quasi-isometric, but not roughly similar.

Let now G be a compactly generated tdlc group and $U \leq G$ be a compact open subgroup. Given a compact generating set $\Sigma \subset G$, the graph with vertex set G/Uand edge set defined by $gU \sim hU \Leftrightarrow h^{-1}g \in U\Sigma U \cup U\Sigma^{-1}U$, is connected, locally finite, and preserved by the natural G-action. It is called the **Cayley–Abels graph** associated with the pair (U, Σ) .

Proposition 4.21. For a non-amenable hyperbolic tdlc group G, the following assertions are equivalent.

- (i) G has a cocompact amenable subgroup.
- (ii) Considering any Cayley–Abels graph \mathcal{G} for G and endowing G with the word metric associated to any compact generating set, each orbit map $G \to V\mathcal{G}$ is rough similarity.
- (iii) For some compact open subgroup $U \leq G$, the identity on G/U is a rough similary between the Cayley–Abels graphs associated with (U, Σ_1) and (U, Σ_2) , for any pair of compact generating sets Σ_1, Σ_2 .

Proof. (i) \Rightarrow (ii). Let $x \in V\mathcal{G}$. Since the orbit map $g \mapsto gx$ is *G*-equivariant, it suffices to show that there exist constants A, C > 0 such that

$$A|g| - C \le d_{\mathcal{G}}(x, gx) \le A|g| + C \quad (\forall g \in G)$$

where |g| is the length of g in the chosen word metric. We invoke [CCMT15, Theorem 8.1], which ensures that up to a compact kernel G is a tree automorphism group acting 2-transitively on the boundary of that tree. In particular, there is a compact subset $K \subseteq G$ and an element $a \in G$ acting as a hyperbolic isometry of that tree such that the Cartan-like decomposition $G = K\{a^n : n \in \mathbb{N}\}K$ holds. Given $g \in G$, we denote by n_g some choice of integer with $g \in Ka^{n_g}K$.

We recall that the Cayley graph associated to the generating set defining $|\cdot|$, although unlike \mathcal{G} it is far from locally finite, is a hyperbolic geodesic metric space (compare [CCMT15, §2]). Moreover, a is a hyperbolic isometry of this space because its displacement length does not vanish. Since Corollary 4.13 is valid in that generality, the difference $|a^n| - nA_0$ remains bounded independently of n, where A_0 is the asymptotic displacement length of a in that Cayley graph.

Since the word length is bounded over K, there is a constant C_0 with

$$|a^{n_g}| - C_0 \le |g| \le |a^{n_g}| + C_0$$

for all g, and hence

$$n_g A_0 - C_1 \le |g| \le n_g A_0 + C_1$$

for some C_1 . Similarly, since K has bounded orbits in \mathcal{G} , let C_2 be a bound for $d_{\mathcal{G}}(kx, x)$ over $k \in K$. Then, writing $g = k_1 a^{n_g} k_2$, the triangle inequality gives

$$d(x, a^{n_g}x) - 2C_2 \le d(x, gx) \le d(x, a^{n_g}x) + 2C_2$$

because $d(x, gx) = d(k_1^{-1}x, a^{n_g}k_2x)$. We apply again Corollary 4.13 but to the action of a on \mathcal{G} and deduce that the difference $d(x, a^{n_g}x) - n_gA_1$ remains bounded, where A_1 is the asymptotic displacement length of a in \mathcal{G} . Putting everything together, we obtain the required conclusion with $A = A_1/A_0$.

(ii) \Rightarrow (iii). Obvious.

(iii) \Rightarrow (i). Fix a compact generating set Σ and let \mathcal{G}_0 denote the Cayley–Abels graph for G with respect to (U, Σ) . The distance function on $V\mathcal{G}_0 = G/U$ given by that graph structure is denoted by d_0 .

For each geodesic line $\tau: \mathbf{Z} \to V\mathcal{G}_0$, we choose a map $P_{\tau}: V\mathcal{G}_0 \to \tau(\mathbf{Z})$ such that for all $v \in V\mathcal{G}_0 = G/U$, we have $d_0(v, P_{\tau}(v)) = \min\{d_0(v, \tau(n)) \mid n \in \mathbf{Z}\}$. Thus P_{τ} is a nearest-point-projection on the geodesic line τ . By [CDP90, Proposition 10.2.1], there exists an X-constant C such that

$$d_0(P_{\tau}(x), P_{\tau}(y)) \le \max\{C, C + d_0(x, y) - d_0(x, P_{\tau}(x)) - d_0(y, P_{\tau}(y))\}$$

for all vertices x, y.

Let $a \in G$ be a hyperbolic element. Denote by $\xi, \xi' \in \partial G$ its fixed points at infinity. Fix a geodesic line $\sigma: \mathbb{Z} \to V\mathcal{G}_0$ with endpoints (ξ, ξ') . We also consider the union A of all geodesic lines from ξ to ξ' . As in the proof of Proposition 4.17, any two such lines are at distance less than some X-constant D from one another.

We claim that for any geodesic line $\tau \colon \mathbf{Z} \to V\mathcal{G}_0$ and any L > 0, there exists $g_L \in G$ such that the diameter of $P_{\tau}(g_L(A))$ is larger than L.

Suppose for a contradiction that the claim fails for some geodesic line τ . Hence there exists $L_0 > 0$ such that, for any $g \in G$, the diameter of $P_{\tau}(g(A))$ is at most L_0 . Upon enlarging L_0 , we may assume that $L_0 \ge C + 1$.

Let now $L > L_0$ be an even integer and choose an element $h_L \in G$ be mapping $\sigma(0)$ to $\sigma(L)$. Set $\Sigma_L = \Sigma \cup \{h_L\}$, let \mathcal{G}_L be the Cayley–Abels graph for G with respect to (U, Σ_L) and d_L be the associated distance function on G/U. By construction we have $d_L(x, y) \leq d_0(x, y)$ for all $x, y \in G/U$ since every edge of \mathcal{G}_0 is also an edge of \mathcal{G}_L . Moreover, every edge of \mathcal{G}_L that is not in \mathcal{G}_0 is of the form $\{g\sigma(0), g\sigma(L)\}$ for some $g \in G$. Following the terminology introduced in [BF09], every G-translate of the geodesic segment $\sigma|_{[0,L]}$ in \mathcal{G}_0 is called an **expressway**.

By the hypothesis (iii), there exist constants A_L, C_L such that

$$A_L d_L(x, y) - C_L \le d_0(x, y) \le A_L d_L(x, y) + C_L$$

Apply that inequality with $x = \sigma(0)$ and $y = a^n(x)$, divide by n and let n tend to infinity. We deduce that $A_L = |a|_{\infty,0}/|a|_{\infty,L}$, where $|a|_{\infty,0}$ and $|a|_{\infty,L}$ denote the asymptotic displacement length of a with respect to the metrics d_0 and d_L . Observe that any point of the $\langle a \rangle$ -orbit of $\sigma(0)$ remains D-close to $\sigma(\mathbf{Z})$ in the metric d_0 . Since $d_L \leq d_0$, we deduce from the definition of the metric d_L that the asymptotic displacement length $|a|_{\infty,L}$ tends to 0 as L tends to infinity. We may therefore assume, upon taking L large enough, that $A_L > 2L_0$.

Let now $N > C_L(1 + \frac{A_L}{L_0})$ be an integer and consider a \mathcal{G}_L -geodesic segment

$$(v_0 = \tau(0), v_1, \dots, v_M = \tau(N))$$

joining $\tau(0)$ to $\tau(N)$. We have $M \leq N$ since $d_L(x, y) \leq d_0(x, y)$ for all x, y. For each pair of vertices v_i, v_{i+1} , either $\{v_i, v_{i+1}\}$ is an edge of \mathcal{G}_0 , or there exists an expressway with endpoints $\{v_i, v_{i+1}\}$. By concatenating those edges and expressways, we obtain a continuous path $[v_0, v_1] \cup [v_1, v_2] \cup \ldots$ in the graph \mathcal{G}_0 . By the definition of L_0 , we know that for each i such that $[v_i, v_{i+1}]$ is an expressway, we have $d(P_{\tau}(v_i), P_{\tau}(v_{i+1})) \leq L_0$. If $[v_i, v_{i+1}]$ is an edge, then by the defining property of the constant C, we have $d(P_{\tau}(v_i), P_{\tau}(v_{i+1})) \leq 1 + C \leq L_0$.

Since $(v_0 = P_{\tau}(v_0), P_{\tau}(v_1), \dots, P_{\tau}(v_M) = v_M)$ defines a path from v_0 to v_M , we deduce from the triangle inequality that

$$N = d_0(\tau(0), \tau(N)) = d_0(v_0, v_M) \le \sum_{i=1}^M d_0(P_\tau(v_{i-1}), P_\tau(v_i)) \le L_0 M.$$

On the other hand, by the hypothesis (iii), we have

$$A_L M - C_L = A_L d_L(v_0, v_M) - C_L \le d_0(\tau(0), \tau(N)) = N.$$

Since $A_L > 2L_0$, we obtain

$$2L_0M - C_L < L_0M,$$

so that $M < \frac{C_L}{L_0}$. The hypothesis (iii) also implies that

$$N = d_0(\tau(0), \tau(N)) \le A_L d_L(v_0, v_M) + C_L = A_L M + C_L,$$

so that

$$N < \frac{A_L C_L}{L_0} + C_L.$$

This contradicts the choice of N, thereby establishing the claim.

Let now $b \in G$ be a hyperbolic element with attracting and repelling fixed points $\eta_+, \eta_- \in \partial G$. Fix a geodesic line $\tau \colon \mathbf{Z} \to V\mathcal{G}_0$ with $\eta_+ = \tau(\infty)$ and $\eta_- = \tau(-\infty)$. For each even integer L > 0, the claim ensures the existence of an element $g_L \in G$ be such that $P_{\tau}(g_L(A))$ has diameter strictly greater than 2C + 2D + 2L. Let $x, y \in A$ with $d_0(P_{\tau}(g_L(x)), P_{\tau}(g_L(y))) > 2C + 2D + 2L$. Pick $x', y' \in \sigma(\mathbf{Z})$ with $d_0(x, x') \leq D$ and $d_0(y, y') \leq D$. In particular we have $d_0(P_{\tau}(g_L(x')), P_{\tau}(g_L(y'))) > 2L$. We now apply Lemma 4.18 to obtain constants H, J and henceforth consider only L large enough to have $L \geq 2 \max\{C, H, J\}$. Thus, the piecewise geodesic segment

$$[g_L(x'), P_\tau(g_L(x'))] \cup [P_\tau(g_L(x')), P_\tau(g_L(y'))] \cup [P_\tau(g_L(y')), g_L(y')]$$

lies in an *H*-neighbourhood of the subsegment of the geodesic line $g_L(\sigma)$ joining $g_L(x')$ to $g_L(y')$. Let n_L be an integer such that $d_0(P_\tau(g_L(x')), g_L(\sigma(n_L))) \leq H$, and $n'_L \geq 0$ be an integer such that $d_0(P_\tau(g_L(y')), g_L(\sigma(n_L + n'_L))) \leq H$. By the definition of n_L and n'_L , we have $d_0(g_L(\sigma(n_L)), P_\tau(g_L(\sigma(n_L)))) \leq H$ and $d_0(g_L(\sigma(n_L + n'_L)), P_\tau(g_L(\sigma(n_L + n'_L)))) \leq H$. Therefore,

$$d_0(g_L(\sigma(n_L)), g_L(\sigma(n_L + n'_L))) \ge d_0(P_\tau(g_L(x')), P_\tau(g_L(y'))) - 2H > 2L - 2H > L.$$

This ensures that $n'_L > L$. In particular, the Hausdorff distance between the geodesic segments $g_L \circ \sigma|_{[n_L,n_L+L]}$ and $[P_{\tau}(g_L(x')), P_{\tau}(g_L(y'))]$ is bounded by an X-constant. Therefore it lies in a bounded neighbourhood of $\tau(\mathbf{Z})$, where the bound is independent of L.

Using Corollary 4.13, we may find suitable integers s, t such that the element $h_L = b^s g_L a^t$ maps the pair $\{\sigma(-L/2), \sigma(L/2)\}$ at uniformly bounded distance from $\{\tau(-L/2), \tau(L/2)\}$. In particular $d_0(h_L(\sigma(0)), \tau(0))$ is bounded independently of L. Therefore the sequence (h_L) is bounded in G, and thus subconverges to an element $h \in G$. By construction h maps the pair $\{\xi, \xi'\}$ to the pair $\{\eta_+, \eta_-\}$.

Notice that hah^{-1} is a hyperbolic element with the same asymptotic displacement length as a, and with the same pair of fixed points in ∂X as b. Fixing a and letting b vary over the collection of all hyperbolic isometries in G, we deduce that the assertion (v) from Theorem 4.7 is satisfied, which therefore yields the required conclusion.

Proof of Theorem J. The equivalences between the assertions (i)–(v) are established by Theorem 4.7. Their equivalence with (vi) follows from Proposition 4.21 in case G is totally disconnected.

In general, since G is non-amenable it is of general type, we see from [CCMT15, Proposition 5.10] that, after dividing out a compact normal subgroup, it is either a virtually connected rank one simple Lie group, or a totally disconnected group. Since every connected Lie group has a cocompact solvable subgroup, it suffices to show that a virtually connected rank one simple Lie group satisfies (vi). This follows from the KAK-decomposition as in the first part of the proof of Proposition 4.21; we omit the details.

5. Boundary representations of hyperbolic groups and the type I property

The goal of this section is to complete the proofs of the remaining results from the introduction.

5.1. Quasi-regular representations defined by cocompact subgroups. In the special case where P is unimodular, the following result can be extracted from the proof of [DE14, Theorem 9.2.2].

Proposition 5.1. Let G be a tdlc group and P be a closed cocompact subgroup. Then every element of $C^*_{\lambda_{G/P}}(G)$ is a compact operator. In particular, it splits as a direct sum of irreducible CCR representations.

Proof. Let U be a compact open subgroup of G and ν be a quasi-invariant probability measure on G/P. By Lemmas 2.2 and 2.1, we may replace ν by $\mu * \nu$, where μ is a U-invariant probability measure on G, so as to ensure that ν is U-invariant. For any open subgroup $V \leq U$, the dimension of the space of V-invariant vectors in $L^2(G/P, \nu)$ equals the cardinality $|V \setminus G/P|$ of the set of double cosets modulo (V, P). Since P is cocompact and V is open, that number is finite. It follows that the orthogonal projection p_V on the space of V-invariant vectors, is a compact operator. Observe moreover that $p_V = \frac{1}{\mu_G(V)} \lambda_{G/P}(\mathbf{1}_V)$, where μ_G denotes a left Haar measure on G and $\mathbf{1}_V$ is the characteristic function of V. Hence the operators (p_V) , where V runs over the compact open subgroups of U, form an approximate unit in $C^*_{\lambda_{G/P}}(G)$, consisting entirely of compact operators. This implies that every element of $C^*_{\lambda_{G/P}}(G)$ is a compact operator. The required conclusions then follow from [Arv76, Theorem 1.4.4].

In [Rau16, Theorem A] Raum proved that a locally compact second countable unimodular C^* -simple group G does not admit any cocompact amenable closed subgroups. Using the above proposition we give a simple proof of this fact for general locally compact groups G, dropping the assumptions of second countability and unimodularity.

We need the following fact which is certainly known to the experts. We include a proof for the sake of completeness.

Recall that a C^* -algebra is called **elementary** if it is isomorphic to the C^* -algebra of compact operators on some Hilbert space.

Lemma 5.2. If G is a non-trivial locally compact group, then the C^* -algebra $C^*_r(G)$ is non-elementary.

Proof. Suppose $C_r^*(G)$ is elementary. Then it follows from [Run08, Proposition 4.3] that $L^1(G)$ has a unique C^* -norm. In particular, we have $C^*(G) = C_r^*(G)$, which implies $C_r^*(G)$ admits a character. Since the C^* -algebra of compact operators is simple, it follows G is trivial.

Theorem 5.3. Let G be a non-trivial locally compact group containing a cocompact amenable closed subgroup. Then $C_r^*(G)$ is not simple.

Proof. Assume G contains a cocompact amenable closed subgroup P, and $C_r^*(G)$ is simple. By [Rau19, Theorem 6.1], G is totally disconnected. Since P is amenable, λ_G weakly contains $\lambda_{G/P}$, therefore they are weakly equivalent by C*-simplicity. Then Proposition 5.1 implies $C_r^*(G)$ is CCR, and since it is also simple, it follows it is elementary. Hence, G is trivial by Lemma 5.2.

5.2. Weak equivalence of boundary representations.

Proof of Theorem E. Let us first observe that the required assertion holds in the special case where G has a cocompact amenable subgroup P. Then, by Theorem 4.7, the G-action on ∂G is transitive, so that there is a unique G-invariant measure class on ∂G , see [BdlHV08, Theorem B.1.4] and [CCMT15, Lemma 8.1]. Hence, by Lemma 2.1, any two boundary representations are unitarily equivalent. In particular, they are weakly equivalent.

We next invoke Proposition 4.5 and denote by W the compact normal subgroup afforded in that way. Since W acts trivially on the Gromov boundary ∂G , and is thus contained in the kernel of every boundary unitary representation, there is no loss of generality in assuming that W is trivial. Hence, either G is a virtually connected rank one simple Lie group, or G is totally disconnected.

In the former case, the group G has a cocompact amenable subgroup, and the required conclusion follows by the first paragraph. We assume henceforth that G is totally disconnected and that G does not have any cocompact amenable subgroup. Therefore, Corollary 4.20 ensures that some point $\xi \in \partial G$ has a regionally elliptic stabilizer. Since the G-action on ∂G is topologically amenable (see [Ada96], [Kai04]), all points have an amenable stabilizer, and the G-action on $(\partial G, \nu)$ is Zimmeramenable with respect to quasi-invariant probability measure ν (see Proposition 3.3.5 in [ADR00]). Therefore the hypotheses of Corollary H are fulfilled. The required conclusion follows. $\hfill \Box$

5.3. **Type I hyperbolic groups.** It remains to prove Theorems B and K. As announced in the introduction, we shall rely on Garncarek's work [Gar14], which concerns the boundary representations of discrete hyperbolic groups associated with Patterson–Sullivan measures.

For our purposes, it is sufficient to consider those measures in the setting of hyperbolic locally finite graphs with a vertex transitive automorphism group. Given such a graph X, the Patterson–Sullivan construction (developed by M. Coornaert [Coo93] in the general hyperbolic setting) yields a canonical measure class on the Gromov boundary ∂X , which is invariant under the full automorphism group Aut(X). The probability measures in that class can be defined by Hausdorff measures associated with visual metrics or, alternatively, as weak*-limits of normalized counting measures on balls around a fixed vertex of the graph X. The Koopman representation defined by any probability measure in the Patterson–Sullivan class is called the **PSrepresentation** associated with X. We denote it by κ_X . It is well defined up to equivalence (see Lemma 2.1). Its domain is the full automorphism group Aut(X).

In order to clarify how Garncarek's work comes into play, we introduce the following two conditions on a hyperbolic tdlc groups G and compact open subgroup $U \leq G$:

- (G1): For any Cayley–Abels graph X on G/U, the PS-representation κ_X is an irreducible representation of G.
- (G2): For any two Cayley–Abels graphs X_1, X_2 on G/U, if the PS-representations $\kappa_{X_1}, \kappa_{X_2}$ are unitarily equivalent, then the identity on G/U, viewed as a map from VX_1 to VX_2 , is a rough similarity.

The main results of [Gar14] state notably that every non-elementary *discrete* hyperbolic group satisfies (G1) and (G2). Applying this to a uniform lattice in the non-discrete case, we obtain the following.

Theorem 5.4 (Garncarek). Let G be a non-amenable hyperbolic tdlc group. If G has a cocompact lattice, then for every compact open subgroup $U \leq G$, the conditions (G1) and (G2) are satisfied.

Proof. Let $U \leq G$ be any compact open subgroup. Let also $\Gamma \leq G$ be a cocompact lattice. For any Cayley–Abels graph X on G/U, the restriction of the PS-representation κ_X to Γ is irreducible by [Gar14, Theorem 6.2]. In particular, it is irreducible as a representation of G, so (G1) holds.

Given two Cayley–Abels graphs X_1, X_2 on G/U, if the PS-representations $\kappa_{X_1}, \kappa_{X_2}$ are unitarily equivalent, then their restrictions to Γ are equivalent. It then follows from (the proof of) [Gar14, Theorem 7.4] that the identity on Γ defines a rough similarity of (Γ, d_1) to (Γ, d_2) , where d_i is the pseudo-metric on Γ induced by the orbit map $\Gamma \to VX_i = G/U : \gamma \mapsto \gamma U$. Let V'_i be the image of that map.

Since Γ is cocompact in G, we have $G = \Gamma K$ for some compact subset K of G. Since K is covered by finitely many left U-cosets, it follows that Γ acts with finitely many orbits on VX_i . In particular, by choosing for each $v \in VX_i$ a vertex $p_i(v) \in V'_i$ that is at minimal distance from v, we obtain a map $p_i \colon VX_i \to V'_i$ such that the distance from v to $p_i(v)$ is uniformly bounded.

By the very definition, the orbit map $\Gamma \to VX_i$ defines a rough similarity of (Γ, d_i) to V'_i . It follows that the identity on $\Gamma U/U$, viewed as a map $\varphi' \colon V'_1 \to V'_2$, is a rough similarity. By the triangle inequality, it follows that the map $\varphi \colon VX_1 \to VX_2 \colon v \mapsto \varphi'(p_1(v))$ is a rough similarity. Let $\psi \colon VX_1 \to VX_2$ be the map defined

by the identity on G/U. By construction the maps φ and ψ coincide on V'_1 . Since ψ is a quasi-isometry, it follows that the distance from $\psi(v)$ to $\psi(p_1(v))$ is uniformly bounded. Therefore, the maps φ and ψ are within bounded distance of each other. Since φ is a rough similarity, it follows that ψ has also this property. Thus (G2) holds.

To complete the proofs of the results announced in the introduction, it remains to establish the following.

Theorem 5.5. Let G be a non-amenable hyperbolic tdlc group and $U \leq G$ be a compact open subgroup satisfying (G1) and (G2). If for some boundary representation κ of G, the C*-algebra $C_{\kappa}^*(G)$ contains a non-zero CCR two-sided ideal, then G has a cocompact amenable subgroup.

Proof. Set $A = C_{\kappa}^{*}(G)$ and let I be a non-zero CCR two-sided ideal in A. By Theorem E, any two boundary representations of G are weakly equivalent. Thus, it follows that for every boundary representation π of G, we have $C_{\pi}^{*}(G) \cong A$.

Let now X be any Cayley–Abels graph on G/U. By (G1), the representation κ_X is irreducible. Since $C^*_{\kappa_X}(G) \cong A$, we deduce that $C^*_{\kappa_X}(G)$ contains a non-zero CCR two-sided ideal, which acts irreducibly on \mathcal{H}_{κ_X} by [Arv76, Theorem 1.3.4]. Since that ideal is CCR, it entirely consists of compact operators. Since this is valied for any Cayley–Abels graph X, we deduce from [Dix96, Corollary 4.1.10] that, for any two Cayley–Abels graphs X_1, X_2 on G/U, the representations κ_{X_1} and κ_{X_2} are unitary equivalent. Since the pair (G, U) satisfies (G2), we infer that the identity on G/U, viewed as a map from VX_1 to VX_2 , is a rough similarity. Proposition 4.21 now ensures that G has a cocompact amenable subgroup.

Proof of Theorem K. The implications (iv) \Rightarrow (iii) \Rightarrow (i) \Rightarrow (i) are obvious. If (v) holds, we see as in the proof of Theorem E that any two boundary representations are unitarily equivalent (and not only weakly equivalent). Moreover, by [CCMT15, Theorem 8.1], we may assume that G is either a virtually connected rank one simple Lie group, or a closed subgroup of the automorphism group of a non-elementary locally finite tree, acting without inversion, with exactly 2 orbits of vertices, and acting 2-transitively on the set of ends. In the former case, the assertion (iii) follows from known results on unitary representations theory of simple Lie groups. Indeed κ is the representation unitarily induced from the trivial representation of a parabolic subgroup and therefore it is admissible, see Proposition 8.4 in [Kna86]. For the fact that an admissible representation maps every element of the group C*-algebra to a compact operator, we refer e.g. to the proof of (i) and (ii) in Lemma 15.5.1 in [Dix96] or to the proof of Proposition 6.E.11 in [BdlH20]. In the totally disconnected case, the assertion (iv) follows from Proposition 5.1.

Assume finally that (i) holds; we shall show (v). We may assume that G is non-amenable. By Proposition 4.5 we may further assume that G is totally disconnected, since every connected Lie group has a cocompact solvable subgroup. The required conclusion then follows from Theorems 5.4 and 5.5.

Proof of Theorem B. If G is type I, then since G is σ -compact, every unitary representation of G is GCR (see [BdlH20, Theorem 8.F.3]). If G is amenable, there is nothing to prove; otherwise, the theorem follows from the implication (ii) \Rightarrow (v) in Theorem K.

Remark 5.6. We strongly believe that Theorems B and K hold without the requirement that G contains a uniform lattice. This would follow from Theorem 5.5 by the same reasoning as above, provided one shows that for every non-amenable hyperbolic tdlc group G, there is a compact open subgroup U such that (G1) and (G2) are

satisfied. Whether this non-discrete generalization of Garncarek's results [Gar14] holds is a highly interesting question in its own. This question can be envisaged in the context of a wide-ranging conjecture of Bader and Muchnik [BM11]. That conjecture predicts that the Koopman representation associated with the Poisson boundary of the random walk defined by a spread-out probability measure μ on an arbitrary locally compact is irreducible. (Recent results by Björklund–Hartman–Oppelmayer [BHO20] suggest that the measure μ should be assumed to be symmetric in the Bader–Muchnik conjecture.)

Proof of Corollary C. If G is compact or 2-ended, then (iii) holds by [CCMT15, Proposition 5.6]. We assume henceforth that this is not the case, so G is nonelementary. The existence of a lattice implies that G is unimodular, hence G is nonamenable by [CCMT15, Theorem 7.3]. It then follows from [CCMT15, Theorem D] that (i) or (ii) holds.

Proof of Corollary D. If G were compactly generated, then it would act cocompactly on T by [CDM11, Lemma 2.4], hence it would be Gromov hyperbolic. If moreover G were unimodular, then it would contain a uniform lattice by the main result of [BK90]. Hence, in that situation, we may invoke Corollary C. Although the tree given by Corollary C need not be T itself, its boundary is canonically identified with ∂T as a G-space by cocompactness of the G-action on T. The required conclusion follows.

We now present two different approaches to finish the proof. The first one consists in invoking [HR19, Theorem A], which ensures that, after discarding vertices of degree 2 of T, the action of every vertex stabilizer in G is 2-transitive on the neighbouring edges. It is easy to deduce that G acts edge-transitively on T, that G is compactly generated, and that G is unimodular (because it is generated by compact subgroups). The hypotheses required by the argument of the first paragraph above are fulfilled, and we are done.

We may alternatively avoid invoking the full strength of [HR19, Theorem A], as follows. Let G^+ be the open subgroup of index at most 2 preserving the canonical bipartition of T. Then G^+ acts without inversion. If the quotient graph $G^+ \setminus T$ is not a tree, then G cannot be a type I group by [HR19, Proposition 4.1 and 4.2]. Since G, hence also G^+ , is of type I (see [Kal73, Proposition 2.4]), we infer that G^+ is the fundamental group of a tree of profinite groups. In particular it is generated by compact subgroups, hence it is unimodular. Let us write G^+ as an ascending union of compactly generated open subgroups $G^+ = \bigcup_n H_n$. Without loss of generality we assume that H_0 is not compact. For each n let $T_n \subset T$ be a minimal H_n -invariant subtree of T. By [CDM11, Lemma 2.4], the group H_n acts cocompactly on T_n , so that H_n is hyperbolic. As an open subgroup of G^+ , the group H_n is unimodular. Applying Corollary C to H_n , we invoke that the H_n -action is 2-transitive on the ends of T_n . Let now $v \in V(T_0)$ be a vertex and set $K = G_v^+$. For n large enough we have $K \leq H_n$. Since T_0 is contained in T_n , we have $H_n = K \langle a_n \rangle K$ for a suitable hyperbolic element $a_n \in H_n$. It follows that K is a maximal subgroup of H_n . Similarly K is maximal in H_m for all $m \ge n$. Since $K \le H_n \le H_m$, we infer that $H_n = H_m$ for all $m \ge n$. Hence $G^+ = H_n$, which implies that G^+ is compactly generated. This confirms again that the hypotheses required by the argument of the first paragraph above are fulfilled.

Remark 5.7. We note that in proving Theorem B we only used that boundary representations of G are GCR. This is formally weaker than type I property, which is equivalent to all representations of G being GCR. In fact, for every almost invariant measure ν on ∂G , the G-space ($\partial G, \nu$) is amenable by [Ada96, Theorem 6.8],

so that every boundary representation κ_{ν} of G is weakly contained in the regular representation λ_G of G by [AD03, Corollary 3.2.2]. Hence, it follows that if λ_G is GCR, then G has a cocompact amenable subgroup.

In non-discrete setting, the GCR property for the group and its regular representation are not equivalent, see [Mac61]. Examples of locally compact groups whose regular representation is GCR include all linear algebraic groups over a non-archimedean local field k of characteristic zero [GK79].

We finish this section by mentioning an example of an amenable hyperbolic group that is not type I.

Proposition 5.8. Let F be a non-abelian finite simple group. Then the semirestricted wreath product

$$G = (\bigoplus_{\mathbf{Z}_{<0}} F) \oplus (\prod_{\mathbf{Z}_{\geq 0}} F) \rtimes \mathbf{Z}$$

is an amenable hyperbolic locally compact group that is not type I.

Proof. That G is amenable and hyperbolic follows from [CCMT15, Theorem A] (the amenability is obvious since G is {regionally elliptic}-by-abelian). The regionally elliptic subgroup $G^0 = (\bigoplus_{\mathbf{Z}_{<0}} F) \oplus (\prod_{\mathbf{Z}_{\geq 0}} F)$ has the compact group $K = \prod_{\mathbf{Z}_{\geq 0}} F$ as an open normal subgroup. The quotient G^0/K is the discrete group $\bigoplus_{\mathbf{Z}_{<0}} F$. The latter is not virtually abelian since F is non-abelian simple. Hence it is not type I by Thoma's theorem [Tho68]. Therefore G^0 is not type I, hence G is not type I either, since the type I property is inherited by open subgroups, see [Kal73, Proposition 2.4].

6. Conjectures and relation to C^* -simplicity

In this section, we wish to relate Conjecture A with another conjecture, pertaining to C^* -simplicity. Important recent results from [KK17] ensure that if a discrete group G has a (topologically) free action on its Furstenberg boundary, then its reduced C^* -algebra is simple. It is currently unknown whether the same result holds for non-discrete locally compact groups:

Conjecture 6.1. Let G be a second countable locally compact group. If some point of the Furstenberg boundary of G has a trivial stabilizer, then $C_r^*(G)$ is simple.

A few comments relevant to Conjecture 6.1 may be found in the introduction of [CBB21].

Known results in the discrete case make it natural to strengthen that conjecture as follows. Let \mathcal{A} be the stabilizer URS associated with the *G*-action on its Furstenberg boundary. It is not difficult to see that \mathcal{A} is the unique largest URS of *G* consisting of relatively amenable subgroups. Moreover, that URS is reduced to the trivial subgroup if and only if some point of the Furstenberg boundary of *G* has a trivial stabilizer. This is in turn equivalent to the condition that the Chabauty closure of the conjugacy class of each relatively amenable closed subgroup of *G* contains the trivial subgroup.

Conjecture 6.2. Let G be a second countable locally compact group and $\partial_F G$ be its Furstenberg boundary. For any continuity point $z \in \partial_F G$ of the stabilizer map $\partial_F G \to \mathbf{Sub}(G)$, the C^{*}-algebra $C^*_{\lambda_{G/Gz}}(G)$ is simple.

In other words, using the notation of Section 2.2, this conjecture predicts that $C^*(\mathcal{A})$ is simple, where \mathcal{A} is the stabilizer URS for the *G*-action on its Furstenberg boundary.

Conjecture 6.2 is known to hold if G is discrete, see [Kaw17, Corollary 8.5] and [KS20, Theorem 5.8]. Moreover, that conjecture is formally stronger than Conjecture 6.1: indeed, the latter is the special case of the former when G_z is trivial (equivalently $\mathcal{A} = \{\langle e \rangle\}$). In the setting of hyperbolic groups, Conjecture 6.2 would imply that for any boundary representation π of a non-amenable hyperbolic group G, the C^* -algebra $C^*_{\pi}(G)$ is simple, while Theorem E ensures that this algebra does not depend on the choice of π .

The following result highlights a link between Conjecture 6.2 and Conjecture A.

Proposition 6.3. Let G be a second countable tdlc group satisfying the conclusion of Conjecture 6.2. If G is a type I group, then it has a cocompact amenable subgroup.

The proof will be presented at the end of this section. The argument relies on the following two propositions of independent interest.

Proposition 6.4. For every locally compact group G, the following assertions are equivalent.

- (i) G has a cocompact amenable subgroup.
- (ii) G acts transitively on its Furstenberg boundary.

Proof. Suppose that the topological group G has a homogeneous Furstenberg boundary $\partial_F G$ and write $\partial_F G \cong G/P$ for a subgroup P < G which is necessarily cocompact. The universal property of $\partial_F G$ shows that P is relatively amenable in G, but by cocompactness it is actually amenable. This latter point is proved e.g. in the Proposition in [Mon20].

Conversely, suppose that G contains a cocompact amenable subgroup P. Since P must fix a point in the space ΔG of probability measures on $\partial_F G$, it follows that G has a compact orbit in ΔG . By the converse to Krein–Milman's theorem [DS58, V.8.5], this orbit contains all extremal points of ΔG and hence $\partial_F G$ is a single orbit (see [Gla76] for details).

Proposition 6.5. Let G be a σ -compact locally compact group, let Z be a Gboundary, let ν be a G-quasi-invariant Radon probability measure on Z such that $L^2(Z,\nu)$ is separable. Let κ denote the associated Koopman representation, and let $U \leq G$ be a compact open subgroup that fixes the measure ν . If the image of the convolution algebra $C_c(U \setminus G/U)$ under κ is finite-dimensional, then the G-action on Z is transitive.

We need the following elementary fact.

Lemma 6.6. Let G be a locally compact group with left Haar measure μ , let $U \leq G$ be a compact open subgroup and π be a unitary representation of G. For each $g \in G$, the characteristic function $\mathbf{1}_{UgU}$ belongs to $C_c(G)$ and we have

$$\pi(\mathbf{1}_{UqU}) = \mu(UgU)p_U\pi(g)p_U,$$

where p_U is the orthogonal projection on the subspace of U-invariant vectors in \mathcal{H}_{π} .

Proof. The identity is unaffected by renormalizing μ , so we can assume $\mu(U) = 1$. The integral expression $p_U = \pi(\mathbf{1}_U)$ for p_U is well-known and easy to see. It follows that the right hand side of the identity to establish is $\mu(UgU)\pi(\mathbf{1}_U * \mathbf{1}_{gU})$ and it suffices to check the convolution equation $\mathbf{1}_{UgU} = \mu(UgU) \cdot \mathbf{1}_U * \mathbf{1}_{gU}$. By definition, $(\mathbf{1}_U * \mathbf{1}_{gU})(z) = \int_U \mathbf{1}_{gU}(y^{-1}z) d\mu(y) = \mu(U \cap zUg^{-1})$. Since U is compact open, it is partitioned into $m \geq 1$ cosets $u_i(U \cap gUg^{-1})$ of measure 1/m, with $u_i \in U$. Writing $U = \bigsqcup_{i=1}^m U \cap u_i gUg^{-1}$, it follows that $\mu(U \cap zUg^{-1})$ is 1/m or 0 according to whether $z \in UgU$ or not. This implies It remains to check $m = \mu(UgU)$, which can be derived from integrating both sides of (*).

Proof of Proposition 6.5. Suppose that $\kappa(C_c(U\backslash G/U))$ is finite-dimensional. Then there exist elements $g_0, g_1, \ldots, g_k \in G$ such that $\kappa(C_c(U\backslash G/U))$ is spanned by the images of the characteristic functions $\mathbf{1}_{Ug_0U}, \ldots, \mathbf{1}_{Ug_kU}$. Suppose for a contradiction that there exist points $x, y \in Z$ in distinct *G*-orbits. Since *U* is compact, it follows that there exists a compact neighbourhood α of *x* such that $y \notin \bigcup_{i=0}^k Ug_iU\alpha$. Since the set $\bigcup_{i=0}^k Ug_iU\alpha$ is closed, there exists also a neighbourhood β of *y* with $\beta \cap$ $(\bigcup_{i=0}^k Ug_iU\alpha) = \emptyset$. In particular $\beta \cap Ug_iU\alpha = \emptyset$ for all *i*, so that

$$\langle \kappa(\mathbf{1}_{Uq_iU})\mathbf{1}_{\alpha},\mathbf{1}_{\beta}\rangle=0$$

for all *i*, in the space $L^2(Z, \nu)$.

Choose neighbourhoods $\alpha' \subsetneq \alpha$ of x and $\beta' \subsetneq \beta$ of y such that $U' := \{g \in U \mid g(\alpha') \subseteq \alpha, g(\beta') \subseteq \beta\}$ is an identity neighbourhood in G. Upon replacing U' by $U' \cap (U')^{-1}$ we shall assume that U' is symmetric.

Since Z is a G-boundary, there exists $t \in G$ such that $\nu(t\alpha') > 1 - \nu(\beta')$. By the definition of g_0, \ldots, g_k , there exist $\lambda_0, \ldots, \lambda_k \in \mathbf{C}$ such that $\kappa(\mathbf{1}_{UtU}) = \sum_{i=0}^k \lambda_i \kappa(\mathbf{1}_{Ug_iU})$. By the above, it follows that $\langle \kappa(\mathbf{1}_{UtU})\mathbf{1}_{\alpha}, \mathbf{1}_{\beta} \rangle = 0$. Using Lemma 6.6, we deduce that

$$0 = \langle p_U \kappa(t) p_U \mathbf{1}_{\alpha}, \mathbf{1}_{\beta} \rangle$$

= $\langle \kappa(t) p_U \mathbf{1}_{\alpha}, p_U \mathbf{1}_{\beta} \rangle$
= $\int_Z (p_U \mathbf{1}_{\alpha}) (t^{-1} z) (p_U \mathbf{1}_{\beta}) (z) \sqrt{\frac{dt\nu}{d\nu}} (z) d\nu(z)$

Therefore the non-negative map $z \mapsto p_U \mathbf{1}_{\alpha}(t^{-1}z)p_U \mathbf{1}_{\beta}(z)\sqrt{\frac{dt\nu}{d\nu}(z)}$ vanishes ν -almost everywhere. Since $t\nu$ is in the same measure class as ν , the Radon-Nikodym derivative $dt\nu/d\nu$ is non-zero ν -a.e. Therefore, the map $z \mapsto p_U \mathbf{1}_{\alpha}(t^{-1}z)p_U \mathbf{1}_{\beta}(z)$ also vanishes ν -a.e. Moreover, by Lemma 6.6 we have

$$p_U \mathbf{1}_{\beta}(z) = \frac{1}{\mu(U)} \int_U \mathbf{1}_{\beta}(u^{-1}z) d\mu(u)$$

$$\geq \frac{1}{\mu(U)} \int_{U'} \mathbf{1}_{\beta}(u^{-1}z) d\mu(u)$$

$$\geq \frac{1}{\mu(U)} \int_{U'} \mathbf{1}_{\beta'}(z) d\mu(u)$$

$$= \frac{\mu(U')}{\mu(U)} \mathbf{1}_{\beta'}(z).$$

Similarly, we have $p_U \mathbf{1}_{\alpha}(z) \geq \frac{\mu(U')}{\mu(U)} \mathbf{1}_{\alpha'}(z)$. It follows that

$$0 = \int_{Z} (p_U \mathbf{1}_{\alpha})(t^{-1}z)(p_U \mathbf{1}_{\beta})(z)d\nu(z)$$

$$\geq \frac{\mu(U')^2}{\mu(U)^2} \int_{Z} \mathbf{1}_{\alpha'}(t^{-1}z)\mathbf{1}_{\beta'}(z)d\nu(z)$$

$$= \frac{\mu(U')^2}{\mu(U)^2}\nu(t\alpha' \cap \beta').$$

Therefore $\nu(t\alpha' \cap \beta') = 0$ so that $\nu(t\alpha') \le 1 - \nu(\beta')$. This contradicts the definition of t.

Proof of Proposition 6.3. Let G be a second countable tdlc group of type I and Z be its Furstenberg boundary. Let $z \in Z$ be a continuity point of the stabilizer map $Z \to \mathbf{Sub}(G)$, so that G_z belongs to the URS \mathcal{A} . Let ν_0 be a G-quasi-invariant Radon probability measure on G/G_z , and ν denote the push-forward of ν_0 along the orbit map $G/G_z \to Z$. Then $L^2(Z,\nu)$ is separable since $L^2(G/G_z,\nu_0)$ is, and by construction the Koopman representation κ on $L^2(Z,\nu)$ is unitarily equivalent to the quasi-regular representation λ_{G/G_z} . Let $U \leq G$ be a compact open subgroup. By Lemma 2.2, there exists a U-invariant probability measure ν' in the measure-class of ν . In view of Lemma 2.1, we may replace ν by ν' and assume henceforth that ν is U-invariant.

By construction, we have $C_{\kappa}^*(G) \cong C_{\lambda_{G/G_z}}^*(G) = C^*(\mathcal{A})$. It follows by hypothesis that $C_{\kappa}^*(G)$ is simple. Since moreover G is a type I group by assumption, we infer that κ is CCR. Let π be an irreducible representation of $C_{\kappa}^*(G)$ (considered also as a unitary representation of G). Then since the operator $\pi(\mathbf{1}_U)$ is compact, $\pi(C_c(U \setminus G/U))$ is finite-dimensional. Therefore $\kappa(C_c(U \setminus G/U))$ is finite-dimensional since $C_{\pi}^*(G) = C_{\kappa}^*(G)$. This ensures that all the hypotheses of Proposition 6.5 are satisfied. Hence G acts transitively on Z, so that G has a cocompact amenable subgroup by Proposition 6.4.

Remark 6.7. Proposition 6.3 does not require any hypothesis of irreducibility of boundary representations. Hence, for a hyperbolic group G satisfying the conclusion of Conjecture 6.2, it provides a conceptually different approach to the existence of a cocompact amenable subgroup, in comparison with the proof Theorem B.

We conclude this discussion by showing that, in order to establish Conjecture A, it suffices to prove it in the special case of unimodular tdlc groups. We first record the following observation.

Proposition 6.8. Let $Q_0 \triangleleft Q$ be an open normal subgroup of a locally compact group Q and assume that Q/Q_0 is amenable. If Q_0 admits a cocompact amenable subgroup, then so does Q.

Proof. According to (the proof of) Proposition 6.4, the Furstenberg boundary of Q_0 is of the form Q_0/P_0 for a cocompact amenable subgroup $P_0 < Q_0$. Moreover, the action of Q_0 on its Furstenberg boundary can be extended to a Q-action by homeomorphisms, see Proposition II.4.3 and page 32 in [Gla76]. This Q-action is continuous because Q_0 is open in Q. It is moreover transitive since already Q_0 is transitive. Therefore, Q admits a cocompact subgroup P < Q with $P \cap Q_0 = P_0$. This subgroup is amenable since it is an extension of P_0 by a subgroup of the amenable discrete group Q/Q_0 .

Proposition 6.9. If every (second countable) unimodular tdlc group satisfies the conclusion of Conjecture A, then every (second countable) locally compact group does as well.

Proof. Let G be an arbitrary (second countable) locally compact group of type I. Let $R \lhd G$ be its amenable radical, that is, the largest normal (topologically) amenable subgroup, which is automatically closed and unique. It follows from structure theory and Lie theory that G/R admits a finite index open subgroup H < G/R which splits as a direct product $H = S \times Q$, where S is a connected semi-simple Lie group and Q is totally disconnected. This is proved as Theorem 3.3.3 in [BM02] (see also [Mon01, 11.3.4]). Finally, let Q_0 be the kernel of the modular function of Q.

Recall first that Q_0 is unimodular: it is actually the maximal unimodular closed normal subgroup of Q, see [Bou63b, VII §2 No 7]. Moreover Q_0 is second countable if G is so. Observe next that Q_0 is open in Q because it contains every compact subgroup of Q and there are open compact subgroups by van Dantzig's theorem. Finally, we claim that Q_0 is of type I. Indeed, the type I condition passes to quotients by definition, and is also passes to open subgroups, see e.g. Proposition 2.4 in [Kal73] (this reference assumes second countability but it is not used in 2.4). Applying each of these hereditary properties twice, we can pass from G to Q_0 .

By hypothesis, the group Q_0 satisfies the conclusion of Conjecture A, hence it contains a cocompact amenable subgroup P_0 . In view of Proposition 6.8, we now know that Q admits a cocompact amenable subgroup $P_Q < Q$. On the other hand, S also admits a cocompact amenable subgroup $P_S < S$, namely a minimal parabolic subgroup. The amenable group $P_Q \times P_S$ is still cocompact in G/R since $S \times Q$ has finite index in the latter. It follows that the pre-image P < G of $P_Q \times P_S$ in G is cocompact in G. On the other hand, P is amenable since it is an extension of R by $P_Q \times P_S$.

Remark 6.10. Combining Propositions 6.3 and 6.9, we see that if Conjecture 6.2 holds for all second countable unimodular tdlc groups, then Conjecture A is true.

Acknowledgements. The first named author is grateful to Adrien Le Boudec for stimulating discussions around Conjecture 6.1. We thank Uri Bader, Bachir Bekka, Pierre de la Harpe, Cyril Houdayer, Basile Morando and Sven Raum for their comments on a preliminary version of this paper.

References

- [AD03] Claire Anantharaman-Delaroche, On spectral characterizations of amenability, Israel J. Math. 137 (2003), 1–33. 5, 12, 29
- [Ada94] S. Adams, Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups, Topology 33 (1994), no. 4, 765–783. 3
- [Ada96] Scot Adams, Reduction of cocycles with hyperbolic targets, Ergodic Theory Dynam. Systems 16 (1996), no. 6, 1111–1145. 5, 12, 25, 28
- [ADR00] C. Anantharaman-Delaroche and J. Renault, Amenable groupoids, Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique], vol. 36, L'Enseignement Mathématique, Geneva, 2000, With a foreword by Georges Skandalis and Appendix B by E. Germain. 12, 26
- [AM66] Louis Auslander and Calvin C. Moore, Unitary representations of solvable Lie groups, Mem. Amer. Math. Soc. 62 (1966), 199. 1
- [Arv76] William Arveson, An invitation to C*-algebras, Graduate Texts in Mathematics, No. 39, Springer-Verlag, New York-Heidelberg, 1976. 25, 27
- [Bal95] Werner Ballmann, Lectures on spaces of nonpositive curvature, DMV Seminar, vol. 25, Birkhäuser Verlag, Basel, 1995, With an appendix by Misha Brin. 17
- [BdlH20] Bachir Bekka and Pierre de la Harpe, Unitary representations of groups, duals, and characters, Mathematical Surveys and Monographs, vol. 250, American Mathematical Society, Providence, RI, 2020. 1, 27
- [BdlHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette, Kazhdan's property (T), New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008. 3, 7, 9, 25
- [BF09] Mladen Bestvina and Koji Fujiwara, A characterization of higher rank symmetric spaces via bounded cohomology, Geom. Funct. Anal. **19** (2009), no. 1, 11–40. 22
- [BH99] Martin R. Bridson and André Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. 13
- [BHC62] Armand Borel and Harish-Chandra, Arithmetic subgroups of algebraic groups, Ann. of Math. (2) 75 (1962), 485–535. 2
- [BHO20] Michael Björklund, Yair Hartman, and Hanna Oppelmayer, Random walks on dense subgroups of locally compact groups, 2020, Preprint, arXiv:2006.15705. 28
- [BK90] Hyman Bass and Ravi Kulkarni, Uniform tree lattices, J. Amer. Math. Soc. 3 (1990), no. 4, 843–902. 2, 28

- [BKK017] Emmanuel Breuillard, Mehrdad Kalantar, Matthew Kennedy, and Narutaka Ozawa, C*-simplicity and the unique trace property for discrete groups, Publ. Math. Inst. Hautes Études Sci. 126 (2017), 35–71. 3
- [Bla63] Robert J. Blattner, *Positive definite measures*, Proc. Amer. Math. Soc. **14** (1963), 423–428. 8
- [BM02] Marc Burger and Nicolas Monod, *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. **12** (2002), no. 2, 219–280. 32
- [BM11] Uri Bader and Roman Muchnik, *Boundary unitary representations—irreducibility and rigidity*, J. Mod. Dyn. **5** (2011), no. 1, 49–69. 4, 28
- [BMW12] Udo Baumgartner, Rögnvaldur G. Möller, and George A. Willis, Hyperbolic groups have flat-rank at most 1, Israel J. Math. 190 (2012), 365–388. 14
- [Bou63a] N. Bourbaki, Éléments de mathématique. Fascicule XXIX. Livre VI: Intégration. Chapitre 7: Mesure de Haar. Chapitre 8: Convolution et représentations, Actualités Scientifiques et Industrielles, No. 1306, Hermann, Paris, 1963. 9, 11
- [Bou63b] _____, Intégration. Chapitre 7 et 8, Actualités Scientifiques et Industrielles, No. 1306, Hermann, Paris, 1963. 32
- [Bou65] _____, Éléments de mathématique. Fasc. XIII. Livre VI: Intégration. Chapitres 1, 2, 3 et 4: Inégalités de convexité, Espaces de Riesz, Mesures sur les espaces localement compacts, Prolongement d'une mesure, Espaces L^p, Actualités Scientifiques et Industrielles, No. 1175, Hermann, Paris, 1965, Deuxième édition revue et augmentée. 8, 9
- [CBB21] Pierre-Emmanuel Caprace, Adrien Le Boudec, and Nicolás Matte Bon, Piecewise strongly proximal actions, free boundaries and the neretin groups, 2021, Preprint, arXiv:2107.07765. 29
- [CCMT15] Pierre-Emmanuel Caprace, Yves Cornulier, Nicolas Monod, and Romain Tessera, Amenable hyperbolic groups, J. Eur. Math. Soc. (JEMS) 17 (2015), no. 11, 2903–2947. 1, 2, 5, 6, 12, 13, 14, 15, 16, 17, 20, 21, 22, 24, 25, 27, 28, 29
- [CDM11] Pierre-Emmanuel Caprace and Tom De Medts, Simple locally compact groups acting on trees and their germs of automorphisms, Transform. Groups 16 (2011), no. 2, 375–411. 3, 28
- [CDP90] M. Coornaert, T. Delzant, and A. Papadopoulos, Géométrie et théorie des groupes, Lecture Notes in Mathematics, vol. 1441, Springer-Verlag, Berlin, 1990, Les groupes hyperboliques de Gromov. [Gromov hyperbolic groups], With an English summary. 13, 15, 16, 19, 22
- [Cio14] Corina Ciobotaru, Analytic aspects of locally compact groups acting on buildings, Ph.D. thesis, UCLouvain, 2014. 6, 15
- [CM09] Pierre-Emmanuel Caprace and Nicolas Monod, *Isometry groups of non-positively curved spaces: structure theory*, J Topology **2** (2009), no. 4, 661–700. 1
- [CM13] _____, Fixed points and amenability in non-positive curvature, Math. Ann. **356** (2013), no. 4, 1303–1337. 1, 13, 14
- [CM15] _____, An indiscrete Bieberbach theorem: from amenable CAT(0) groups to Tits buildings, J. Éc. polytech. Math. 2 (2015), 333–383. 1
- [Coo93] Michel Coornaert, Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov, Pacific J. Math. 159 (1993), no. 2, 241–270. 26
- [DE14] Anton Deitmar and Siegfried Echterhoff, *Principles of harmonic analysis*, second ed., Universitext, Springer, Cham, 2014. 24
- [DG17] Artem Dudko and Rostislav Grigorchuk, On spectra of Koopman, groupoid and quasiregular representations, J. Mod. Dyn. 11 (2017), 99–123. 5
- [Dix96] Jacques Dixmier, Les C*-algèbres et leurs représentations, Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics], Éditions Jacques Gabay, Paris, 1996, Reprint of the second (1969) edition. 6, 8, 27
- [dlH88] Pierre de la Harpe, Groupes hyperboliques, algèbres d'opérateurs et un théorème de Jolissaint, C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), no. 14, 771–774. 3
- [dlH07] _____, On simplicity of reduced C^* -algebras of groups, Bull. Lond. Math. Soc. **39** (2007), no. 1, 1–26. 3
- [DS58] Nelson Dunford and Jacob Theodore Schwartz, Linear Operators. I. General Theory, With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7, Interscience Publishers Inc., New York, 1958. 30

- [Ele18] Gábor Elek, Uniformly recurrent subgroups and simple C*-algebras, J. Funct. Anal. 274 (2018), no. 6, 1657–1689. 10
- [Eym72] Pierre Eymard, Moyennes invariantes et représentations unitaires, Lecture Notes in Mathematics, Vol. 300, Springer-Verlag, Berlin-New York, 1972. MR 0447969 12
- [Fel64] J. M. G. Fell, Weak containment and induced representations of groups. II, Trans. Amer. Math. Soc. 110 (1964), 424–447. 9
- [Gar14] Lukacz Garncarek, Boundary representations of hyperbolic groups, preprint arXiv:1404.0903, 2014. 4, 26, 28
- [GdlH90] Étienne Ghys and Pierre de la Harpe (eds.), Sur les groupes hyperboliques d'après Mikhael Gromov, Birkhäuser Verlag, Basel, 1990, Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988. 13, 16, 18, 19, 20
- [GK79] Elliot Charles Gootman and Robert Richard Kallman, The left regular representation of a p-adic algebraic group is type I, Studies in algebra and number theory, Adv. in Math. Suppl. Stud., vol. 6, Academic Press, New York-London, 1979, pp. 273–284. 29
- [Gla76] Shmuel Glasner, Proximal flows, Lecture Notes in Mathematics, Vol. 517, Springer-Verlag, Berlin-New York, 1976. 30, 32
- [Gli61] James Glimm, Type I C^{*}-algebras, Ann. of Math. (2) **73** (1961), 572–612. 3
- [GW15] Eli Glasner and Benjamin Weiss, Uniformly recurrent subgroups, Recent trends in ergodic theory and dynamical systems, Contemp. Math., vol. 631, Amer. Math. Soc., Providence, RI, 2015, pp. 63–75. 10
- [HR19] Cyril Houdayer and Sven Raum, Locally compact groups acting on trees, the type I conjecture and non-amenable von Neumann algebras, Comment. Math. Helv. 94 (2019), no. 1, 185–219. 3, 28
- [Kai04] Vadim A. Kaimanovich, Boundary amenability of hyperbolic spaces, Discrete geometric analysis, Contemp. Math., vol. 347, Amer. Math. Soc., Providence, RI, 2004, pp. 83– 111. 5, 12, 25
- [Kal73] Robert Richard Kallman, Certain topological groups are type I. II, Advances in Math. 10 (1973), 221–255. 28, 29, 33
- [Kaw17] Takuya Kawabe, Uniformly recurrent subgroups and the ideal structure of reduced crossed products, 2017, Preprint, arXiv:1701.03413. 10, 30
- [KK44] Shizuo Kakutani and Kunihiko Kodaira, Uber das Haarsche Mass in der lokal bikompakten Gruppe, Proc. Imp. Acad. Tokyo 20 (1944), 444–450. 15
- [KK17] Mehrdad Kalantar and Matthew Kennedy, Boundaries of reduced C*-algebras of discrete groups, J. Reine Angew. Math. 727 (2017), 247–267. 29
- [Kna86] Anthony William Knapp, Representation theory of semisimple groups, Princeton Mathematical Series, vol. 36, Princeton University Press, Princeton, NJ, 1986, An overview based on examples. 27
- [KS20] Mehrdad Kalantar and Eduardo Scarparo, Boundary maps, germs and quasi-regular representations, 2020, Preprint, arXiv:2010.02536. 30
- [Kuh94] M. Gabriella Kuhn, Amenable actions and weak containment of certain representations of discrete groups, Proc. Amer. Math. Soc. 122 (1994), no. 3, 751–757. 3
- [Kur28] Casimir Kuratowski, Sur les décompositions semi-continues d'espaces métriques compacts, Fundamenta Mathematicae 11 (1928), no. 1, 169–185 (fre). 10
- [Mac61] George Whitelaw Mackey, Induced representations and normal subgroups, Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1961, pp. 319–326. 29
- [Mon01] Nicolas Monod, Continuous bounded cohomology of locally compact groups, Lecture Notes in Mathematics 1758, Springer, Berlin, 2001. 32
- [Mon20] _____, Gelfand pairs admit an Iwasawa decomposition, Math. Ann. **378** (2020), no. 1-2, 605–611. 2, 30
- [MvN36] Francis Joseph Murray and Johann von Neumann, On rings of operators, Ann. of Math.
 (2) 37 (1936), no. 1, 116–229. 1
- [Neb99] Claudio Nebbia, Groups of isometries of a tree and the CCR property, Rocky Mountain J. Math. 29 (1999), no. 1, 311–316. 2, 6
- [Rau16] Sven Raum, Cocompact amenable closed subgroups: weakly inequivalent representations in the left-regular representation, Int. Math. Res. Not. IMRN (2016), no. 24, 7671–7685. 25
- [Rau19] _____, C*-simplicity of locally compact Powers groups, J. Reine Angew. Math. 748 (2019), 173–205. 25

36	PE. CAPRACE, M. KALANTAR, AND N. MONOD
[Run08]	Volker Runde, Characterizations of compact and discrete quantum groups through sec- ond duals J. Operator Theory 60 (2008) no. 2, 415–428, 25
[SvN50]	I. E. Segal and John von Neumann, A theorem on unitary representations of semisimple Lie groups, Ann. of Math. (2) 52 (1950), 509–517. 7
[Tho64]	Elmar Thoma, Über unitäre Darstellungen abzählbarer, diskreter Gruppen, Math. Ann. 153 (1964), 111–138. 1
[Tho68]	, Eine Charakterisierung diskreter Gruppen vom Typ I, Invent. Math. 6 (1968), 190–196. 1, 29
[Var63]	V. S. Varadarajan, <i>Groups of automorphisms of Borel spaces</i> , Trans. Amer. Math. Soc. 109 (1963), 191–220. 10
UCLO	UVAIN, IRMP, CHEMIN DU CYCLOTRON 2, BTE L7.01.02, 1348 LOUVAIN-LA-NEUVE,

Belgique

 $Email \ address: \verb"pe.caprace@uclouvain.be"$

UNIVERSITY OF HOUSTON, USA Email address: mkalantar@uh.edu

EPFL, SWITZERLAND $Email \ address: \verb"micolas.monod@epfl.ch"$

36