ISOMETRY GROUPS OF NON-POSITIVELY CURVED SPACES: STRUCTURE THEORY

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ABSTRACT. We develop the structure theory of full isometry groups of locally compact non-positively curved metric spaces. Amongst the discussed themes are de Rham decompositions, normal subgroup structure and characterising properties of symmetric spaces and Bruhat–Tits buildings. Applications to discrete groups and further developments on non-positively curved lattices are exposed in a companion paper [CM08b].

1. INTRODUCTION

The study of non-positively curved metric spaces originates in the work of A. Wald [Wal36] and later of H. Busemann [Bus48]; A. D. Alexandrov [Ale57] introduced the notion of non-positive curvature now extensively used under the name of **CAT(0) spaces** popularised by the work of M. Gromov.

This theory offers a wide gateway to a form of generalised differential geometry, whose objects encompass Riemannian manifolds of non-positive sectional curvature as well as large families of singular spaces including Euclidean buildings and many other polyhedral complexes. It has found a broad range of applications to various fields, including semi-simple algebraic and arithmetic groups, and geometric group theory.

A recurrent theme in this area is the interplay between the geometry of a locally compact CAT(0) space X and the algebraic properties of a discrete group Γ acting properly on X by isometries. This interaction is expected to be especially rich and tight when the Γ-action is cocompact; the pair (X, Γ) is then called a **CAT(0) group**. The purpose of the present paper and its companion [CM08b] is to highlight the rôle of a third entity through which the interaction between X and Γ transits: namely the full isometry group $\mathrm{Is}(X)$ of X. The topology of uniform convergence on compacta makes $\mathrm{Is}(X)$ a locally compact second countable group which is thus canonically endowed with Haar measures. It therefore makes sense to consider lattices in $\mathrm{Is}(X)$, *i.e.* discrete subgroups of finite invariant covolume; we call such pairs (X, Γ) **CAT(0) lattices** (thus CAT(0) groups are precisely *uniform* CAT(0) lattices). This immediately suggests the following two-step programme:

(I) To develop the basic structure theory of the locally compact group $\mathrm{Is}(X)$ and deduce consequences on the overall geometry of the underlying proper CAT(0) space X. This is the main purpose of the present paper.

(II) To study CAT(0) lattices and thus in particular CAT(0) groups by building upon the structure results of the present paper, using new geometric density and superrigidity techniques. This is carried out in the subsequent paper [CM08b].

We now proceed to describe the main results of this first part in more detail. First, in § 1.A, we present results in the special case of **geodesically complete** CAT(0) spaces,

i.e. spaces in which every geodesic segment can be extended to a bi-infinite geodesic line — which need not be unique. Important examples of geodesically complete CAT(0) spaces are provided by Bruhat–Tits buildings and of course Hadamard manifolds, *e.g.* symmetric spaces. (Recall that a **Hadamard manifold** is a complete simply connected manifold of non-positive sectional curvature.)

The second and longer part of the Introduction, § 1.B, will present results valid for arbitrary locally compact CAT(0) spaces. In either case, the entire contents of the Introduction rely on more general, more detailed but probably also more cumbrous statements proved in the core of the text.

1.A. **Spaces with extensible geodesics.** The conclusions of several results become especially clear and perhaps more striking in the special case of geodesically complete CAT(0) spaces. Beyond Euclidean buildings and Hadamard manifolds, we recall that a complete CAT(0) space that is also a homology manifold has automatically extensible geodesics [BH99, II.5.12]. Note also that it is always possible artificially to make a CAT(0) space geodesically complete by gluing rays, though it is not always possible to preserve properness (consider a compact but total set in an infinite-dimensional Hilbert space).

Decomposing CAT(0) spaces into products of symmetric spaces and locally finite cell complexes. Prototypical examples of locally compact CAT(0) spaces are mainly provided by the following two sources.

— Hadamard manifolds, whose most prominent representatives are the Riemannian symmetric spaces of non-compact type. These spaces are **regular** in the sense that distinct geodesic segments intersect in at most one point. The full isometry group of such a space is a Lie group.

— Polyhedral complexes of piecewise constant non-positive curvature, such as trees or Euclidean buildings. These spaces are **singular** in the sense that geodesics do branch. The subgroup of the isometry group which preserves the cell structure is totally disconnected.

The following result seems to indicate that a CAT(0) space often splits as a product of spaces belonging to these two families.

Theorem 1.1. *Let X be a proper geodesically complete* CAT(0) *space whose isometry group acts cocompactly without fixed point at infinity. Then X admits an* $\mathrm{Is}(X)$-*equivariant splitting*

$$X = M \times \mathbf{R}^n \times Y,$$

where M is a symmetric space of non-compact type and the isometry group $\mathrm{Is}(Y)$ *is totally disconnected and acts by semi-simple isometries on Y (each factor may be trivial).*

Furthermore, the space Y admits an $\mathrm{Is}(Y)$-*equivariant locally finite decomposition into convex cells, where the cell supporting a point $y \in Y$ is defined as the fixed point set of the isotropy group* $\mathrm{Is}(Y)_y$.

If X is regular, then $\mathrm{Is}(Y)$ is discrete. In other words, the space Y has branching geodesics as soon as $\mathrm{Is}(Y)$ is non-discrete. We refer to Theorem 1.6 and Addendum 1.8 below for a version of the above without the assumption of extensibility of geodesics.

We emphasize that the 'cells' provided by Theorem 1.1 need not be compact; in fact if $\mathrm{Is}(Y)$ acts freely on Y then the decomposition in question becomes trivial and consists of a single cell, namely the whole of Y. Conversely the cell decomposition is non-trivial provided $\mathrm{Is}(Y)$ does not act freely. The most obvious way for the $\mathrm{Is}(Y)$-action not to be free is if

Is(Y) is not discrete. A strong version of the latter condition is that *no open subgroup of* Is(Y) *fixes a point at infinity*; this holds notably for symmetric spaces and Bruhat–Tits buildings. A quite immediate consequence of this condition is that the above cells are then necessarily compact. We shall show that much additional structure can be derived from it (see Section 7.D below).

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Smoothness. The cell decomposition of the third factor in Theorem 1.1 is derived from the following *smoothness* result for isometric actions of totally disconnected groups. In this statement, **minimality** of the action means that there is no smaller invariant CAT(0) subspace; this notion will be further analysed and motived below.

Theorem 1.2. *Let X be a geodesically complete proper* CAT(0) *space X and $G < $ Is(G) a totally disconnected (closed) subgroup acting minimally.*
    Then the pointwise stabiliser in G of every bounded set is open.

This property, which is familiar from classical examples, *does in general fail* without geodesic completeness; we give such an example with cocompact isometry group in Remark 6.10 in [CM08b]. Smoothness is an important ingredient for the considerations of Section 7.D alluded to above, as well as for angle rigidity results regarding both the Alexandrov angle (Proposition 6.8) and the Tits angle (Proposition 7.15).

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A characterisation of symmetric spaces and Euclidean buildings. In symmetric spaces and Bruhat–Tits buildings, the stabilisers of points at infinity are exactly the parabolic subgroups; as such, they are cocompact. This cocompactness holds further for all Bass–Serre trees, namely bi-regular trees. Combining our results with work of B. Leeb [Lee00] and A. Lytchak [Lyt05], we establish a corresponding characterisation.

Theorem 1.3. *Let X be a geodesically complete proper* CAT(0) *space. Suppose that the stabiliser of every point at infinity acts cocompactly on X.*
    Then X is isometric to a product of symmetric spaces, Euclidean buildings and Bass–Serre trees.

The Euclidean buildings appearing in the preceding statement admit an automorphism group that is strongly transitive, *i.e.* acts transitively on pairs (c, A) where c is a chamber and A an apartment containing c. This property characterises the Bruhat–Tits buildings, except perhaps for some two-dimensional cases where this is a known open question.

The above characterisation is of a different nature and independent of the characterisations using lattices that will be presented in [CM08b]. A yet different phenomenon is the celebrated *rank rigidity* theorem for Hadamard manifolds established by Ballmann, Brin, Burns, Eberlein and Spatzier ([Bal85], [BS87], [BBE85], [BBS85]; see [Bal95, Theorem C]).

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Actions of simple algebraic groups. Both for the general theory and for the geometric superrigidity/arithmeticity statements of the companion paper [CM08b], it is important to understand how algebraic groups act on CAT(0) spaces.

Theorem 1.4. *Let k be a local field and $\mathbf{G}$ be an absolutely almost simple simply connected k-group. Let X be a CAT(0) space (not reduced to a point) on which $G = \mathbf{G}(k)$ acts continuously and cocompactly by isometries.*

Then X is isometric to X_{model}, the Riemannian symmetric space or Bruhat–Tits building associated with G.

A stronger and much more detailed statement is provided below as Theorem 7.4. In particular, a modification of the statement holds without extensibility of geodesics and the cocompactness assumption can be relaxed. However, we also show there by means of two examples that some assumptions remain necessary.

(As a point of terminology, we do not choose a particular scaling factor on X_{model}, so that the isometry of our statement could become a homothety for another model.)

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1.B. **General case.** When dealing with CAT(0) space in the highest possible level of generality, one has to face several technical difficulties caused by the flexibility of the CAT(0) condition. For example, given a CAT(0) space X, there are many ways to deform it in order to construct another space Y, non-isometric to X, but with the property that X and Y have isomorphic isometry groups or/and identical boundaries. Amongst the simplest constructions, one can form (possibly warped) products with compact CAT(0) spaces. Much wilder non-quasi-isometric examples can be constructed for instance by considering warped products with the very vast family of CAT(0) spaces having no isometries and a unique point at infinity. More prosaically, one can grow hair equivariantly along a discrete orbit to ruin the intrinsic nature of the space.

In order to address these issues, we introduce the following terminology.

Minimality.

> ...ὅσα τις ἂν εἴποι σφαίρας ἐγκώμια, αὐτὰ ταῦτα καὶ φαλάκρας ἐγκώμια διεξέρχεται.
>
> Συνέσιος Κυρεναίου, *Φαλάκρας ἐγκώμιον.*[1]

An isometric action of a group G on a CAT(0) space X is said to be **minimal** if there is no non-empty G-invariant closed convex subset $X' \subsetneq X$; the space X is itself called **minimal** if its full isometry group acts minimally. A CAT(0) space X is called **boundary-minimal** if it possesses no closed convex subset $Y \subsetneq X$ such that $\partial Y = \partial X$. Here is how these notions relate to one another.

Proposition 1.5. *Let X be a proper CAT(0) space.*

(i) *Assume ∂X finite-dimensional. If X is minimal, then it is boundary-minimal.*
(ii) *Assume $\mathrm{Is}(X)$ has full limit set. If X is boundary-minimal, then it is minimal.*
(iii) *If X is cocompact and geodesically complete, then it is both minimal and boundary-minimal.*

We emphasize that it is *not* true in general that a minimal CAT(0) space is geodesically complete, even if one assumes that the isometry group acts cocompactly and without global fixed point at infinity.

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[1]literally: *as much praise is due to the bald head as given to the spheres.* Synesius of Cyrene, *Φαλάκρας ἐγκώμιον* (known as *Calvitii encomium*), Chapter 8 (at 72A in the page numbering from Denis Pétau's 1633 edition). The *Encomium* was written around 402; we used the 1834 edition by J. G. Krabinger (Ch. G. Löflund, Stuttgart).

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Group decompositions. We now turn to structure results on the locally compact isometry group $\mathrm{Is}(X)$ of a proper $\mathrm{CAT}(0)$ space X in a broad generality; we shall mostly ask that no point at infinity be fixed simultaneously by all isometries of X. This non-degeneracy assumption will be shown to hold automatically in the presence of lattices (see Theorem 3.11 in [CM08b]).

The following result was the starting point of this work.

Theorem 1.6. *Let X be a proper $\mathrm{CAT}(0)$ space with finite-dimensional Tits boundary. Assume that $\mathrm{Is}(X)$ has no global fixed point in ∂X.*

Then there is a canonical closed convex $\mathrm{Is}(X)$-stable subset $X' \subseteq X$ such that $G = \mathrm{Is}(X')$ has a finite index open characteristic subgroup $G^* \lhd G$ which admits a canonical decomposition

(1.i) $G^* \cong S_1 \times \cdots \times S_p \times \left(\mathbf{R}^n \rtimes \mathbf{O}(n) \right) \times D_1 \times \cdots \times D_q \qquad (p, q, n \geq 0)$

where S_i are almost connected simple Lie groups with trivial centre and D_j are totally disconnected irreducible groups with trivial amenable radical. Any product decomposition of G^* is a regrouping of the factors in (1.i).

Moreover, all non-trivial normal, subnormal or ascending subgroups $N < D_j$ are still irreducible with trivial amenable radical and trivial centraliser in D_j. (These properties also hold for lattices in N and their normal, subnormal or ascending subgroups, see [CM08b].)

(A topological group is called **irreducible** if no finite index open subgroup splits nontrivially as a direct product of closed subgroups. The **amenable radical** of a locally compact group is the largest amenable normal subgroup; it is indeed a *radical* since the class of amenable locally compact groups is stable under group extensions.)

Remarks 1.7.
  (i) The finite-dimensionality assumption holds automatically when X has a cocompact group of isometries [Kle99, Theorem C]. It is also automatic for uniquely geodesic spaces, *e.g.* manifolds (Proposition 7.11).
  (ii) The conclusion fails in various ways if G fixes a point in ∂X.
  (iii) The quotient G/G^* is just a group of permutations of possibly isomorphic factors in the decomposition. In particular, $G = G^* \rtimes G/G^*$.
  (iv) The canonical continuous homomorphism $\mathrm{Is}(X) \to \mathrm{Is}(X') = G$ is proper, but its image sometimes has infinite covolume.

In Theorem 1.6, the condition that $\mathrm{Is}(X)$ has no global fixed point at infinity ensures the existence of a closed convex $\mathrm{Is}(X)$-invariant subset $Y \subseteq X$ on which $\mathrm{Is}(X)$ acts minimally (see Proposition 4.1). The set of these minimal convex subsets possesses a canonical element, which is precisely the space X' which appears in Theorem 1.6. Proposition 1.5 explains why the distinction between X and X' did not appear in Theorem 1.1.

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De Rham decompositions. It is known that product decompositions of isometry groups acting minimally and without global fixed point at infinity induce a splitting of the space (for cocompact Hadamard manifolds, this is the Lawson–Yau [LY72] and Gromoll–Wolf [GW71] theorem; in general and for more references, see [Mon06]). It is therefore natural that Theorem 1.6 is supplemented by a geometric statement.

Addendum 1.8. *In Theorem 1.6, there is a canonical isometric decomposition*

$$(1.\text{ii}) \qquad\qquad X' \cong\ X_1 \times \cdots \times X_p \times \mathbf{R}^n \times Y_1 \times \cdots \times Y_q$$

where G^* acts componentwise according to (1.i) and G/G^* permutes any isometric factors. All X_i and Y_j are irreducible and minimal.

As it turns out, a geometric decomposition is the first of two independent steps in the proof of Theorem 1.6. In fact, we begin with an analogue of the de Rham decomposition [dR52] whose proof uses (a modification of) arguments from the generalised de Rham theorem of Foertsch–Lytchak [FL06]. In purely geometrical terms, we have the following statement.

Theorem 1.9. *Let X be a proper boundary-minimal $\mathrm{CAT}(0)$ space with ∂X finite-dimensional. Then X admits a canonical maximal isometric splitting*

$$X \cong\ \mathbf{R}^n \times X_1 \times \cdots \times X_m \qquad (n, m \geq 0)$$

with each X_i irreducible and $\neq \mathbf{R}^0, \mathbf{R}^1$. Every isometry of X preserves this decomposition upon permuting possibly isometric factors X_i. Moreover, if X is minimal, so is every X_i.

To apply this theorem, it is desirable to know conditions ensuring boundary-minimality. In addition to the conditions provided by Proposition 1.5, we show that a canonical boundary-minimal subspace exists as soon as the boundary has circumradius $> \pi/2$ (Corollary 3.10).

In the second part of the proof of Theorem 1.6, we analyse the **irreducible** case where X admits no isometric splitting, resulting in Theorem 1.10 to which we shall now turn. Combining these two steps, we then prove the unique decomposition of the *groups*, using also the splitting theorem from [Mon06].

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Geometry of normal subgroups. In É. Cartan's correspondence between symmetric spaces and semi-simple Lie groups as well as in Bruhat–Tits theory, irreducible factors of the space correspond to *simple* groups. For general $\mathrm{CAT}(0)$ spaces and groups, simplicity fails of course very dramatically (free groups are perhaps the simplest, and yet most non-simple, $\mathrm{CAT}(0)$ groups). Nonetheless, we establish a geometric weakening of simplicity.

Theorem 1.10. *Let $X \neq \mathbf{R}$ be an irreducible proper $\mathrm{CAT}(0)$ space with finite-dimensional Tits boundary and $G < \mathrm{Is}(X)$ any subgroup whose action is minimal and does not have a global fixed point in ∂X.*

Then every non-trivial normal subgroup $N \triangleleft G$ still acts minimally and without fixed point in ∂X. Moreover, the amenable radical of N and the centraliser $\mathscr{Z}_{\mathrm{Is}(G)}(N)$ are both trivial; N does not split as a product.

This result can for instance be combined with the solution to Hilbert's fifth problem in order to understand the connected component of the isometry group.

Corollary 1.11. $\mathrm{Is}(X)$ *is either totally disconnected or an almost connected simple Lie group with trivial centre.*

The same holds for any closed subgroup acting minimally and without fixed point at infinity.

A more elementary application of Theorem 1.10 uses the fact that elements with a discrete conjugacy class have open centraliser.

Corollary 1.12. *If G is non-discrete, N cannot be a finitely generated discrete subgroup.*

A feature of Theorem 1.10 is that is can be iterated and thus applies to subnormal subgroups. Recall that more generally a subgroup $H < G$ is **ascending** if there is a (possibly transfinite) chain of normal subgroups starting with H and abutting to G. Using limiting arguments, we bootstrap Theorem 1.10 and show:

Theorem 1.13. *Let $N < G$ be any non-trivial ascending subgroup. Then all conclusions of Theorem 1.10 hold for N.*

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A few cases of superrigidity. Combining the preceding general structure results with some of Margulis' theorems, we obtain the following superrigidity statement.

Theorem 1.14. *Let X be a proper $\mathrm{CAT}(0)$ space whose isometry group acts cocompactly and without global fixed point at infinity. Let $\Gamma = \mathrm{SL}_n(\mathbf{Z})$ with $n \geq 3$ and $G = \mathrm{SL}_n(\mathbf{R})$.*
For any isometric Γ-action on X there is a non-empty Γ-invariant closed convex subset $Y \subseteq X$ on which the Γ-action extends uniquely to a continuous isometric action of G.
(The corresponding statement applies to all those lattices in semi-simple Lie groups that have virtually bounded generation by unipotents.)

Observe that the above theorem has no assumptions whatsoever on the action; cocompactness is an assumption on the given $\mathrm{CAT}(0)$ space. It can happen that Γ fixes points in ∂X, but its action on Y is without fixed points at infinity and minimal (as we shall establish in the proof).

The assumption on bounded generation holds conjecturally for all non-uniform irreducible lattices in higher rank semi-simple Lie groups (but always fails in rank one). It is known to hold for arithmetic groups in split or quasi-split algebraic groups of a number field K of K-rank ≥ 2 by [Tav90], as well as in a few cases of isotropic but non-quasi-split groups [ER06]; see also [WM07].

More generally, Theorem 1.14 holds for (S-)arithmetic groups provided the arithmetic subgroup (given by integers at infinite places) satisfies the above bounded generation property. For instance, the SL_n example is as follows:

Theorem 1.15. *Let X be a proper $\mathrm{CAT}(0)$ space whose isometry group acts cocompactly and without global fixed point at infinity. Let m be an integer with distinct prime factors $p_1, \ldots, p_k$ and set*

$$\Gamma = \mathrm{SL}_n(\mathbf{Z}[\tfrac{1}{m}]), \quad G = \mathrm{SL}_n(\mathbf{R}) \times \mathrm{SL}_n(\mathbf{Q}_{p_1}) \times \cdots \times \mathrm{SL}_n(\mathbf{Q}_{p_k}),$$

where $n \geq 3$. Then for any isometric Γ-action on X there is a non-empty Γ-invariant closed convex subset $Y \subseteq X$ on which the Γ-action extends uniquely to a continuous isometric action of G.

A fixed point property for similar groups acting on low-dimensional $\mathrm{CAT}(0)$ cell complexes was established by B. Farb [Far08].

Some of our general results also allow us to improve on the generality of the $\mathrm{CAT}(0)$ superrigidity theorem for irreducible lattices in arbitrary products of locally compact groups proved in [Mon06]. For actions on proper $\mathrm{CAT}(0)$ spaces, the results of *loc. cit.* establish an unrestricted superrigidity *on the boundary* but require, in order to deduce superrigidity on the space itself, the assumption that the action be *reduced* (or alternatively "indecomposable").

We prove that, as soon as the boundary is finite-dimensional, any action without global fixed point at infinity is always reduced after suitably passing to subspaces and direct factors. It follows that the superrigidity theorem for arbitrary products holds in that generality, see Theorem 8.4 below.

Contents

2. Notation and preliminaries

A metric space is **proper** if every closed ball is compact.

We refer to Bridson and Haefliger [BH99] for background on CAT(0) spaces. We recall that the **comparison angle** $\overline{\angle}_p(x, y)$ determined by three points p, x, y in any metric space is defined purely in terms of the corresponding three distances by looking at the corresponding Euclidean triangle. In other words, it is defined by

$$d^2(x, y) = d^2(p, x) + d^2(p, y) - 2d(p, x)d(p, y)\cos\overline{\angle}_p(x, y).$$

The **Alexandrov angle** $\angle_p(x, y)$ in a CAT(0) space X is the non-increasing limit of the comparison angle near p along the geodesic segments $[p, x]$ and $[p, y]$, see [BH99, II.3.1]. In particular, $\angle_p(x, y) \leq \overline{\angle}_p(x, y)$. Likewise, geodesic rays from p determine the Alexandrov angle $\angle_p(\xi, \eta)$ for $\xi, \eta \in \partial X$. The **Tits angle** $\angle_{\mathrm{T}}(\xi, \eta)$ is defined as the supremum of $\angle_p(\xi, \eta)$ over all $p \in X$ and has several useful characterisations given in Proposition II.9.8 of [BH99].

Recall that to any point at infinity $\xi \in \partial X$ is associated the **Busemann function**

$$B_\xi : X \times X \to \mathbf{R} : (x, y) \mapsto B_{\xi,x}(y)$$

defined by $B_{\xi,x}(y) = \lim_{t\to\infty}(d(\varrho(t), y) - d(\varrho(t), x))$, where $\varrho : [0, \infty) \to X$ is any geodesic ray pointing towards ξ. The Busemann function does not depend on the choice of ϱ and satisfies the following:

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\begin{array}{rcl}
B_{\xi,x}(y) & = & -B_{\xi,y}(x) \\
B_{\xi,x}(z) & = & B_{\xi,x}(y) + B_{\xi,y}(z) \qquad \text{(the ``cocycle relation'')} \\
B_{\xi,x}(y) & \leq & d(x, y).
\end{array}
$$

Combining the definition of the Busemann function and of the comparison angle, we find that if r is the geodesic ray pointing towards ξ with $r(0) = x$, then for any $y \neq x$ we have

$$\lim_{t\to\infty}\cos\overline{\angle}_x(r(t), y) = -\frac{B_{\xi,x}(y)}{d(x, y)} \qquad \text{(the ``asymptotic angle formula'')}.$$

By abuse of language, one refers to **a** Busemann function when it is more convenient to consider the convex 1-Lipschitz function $b_\xi : X \to \mathbf{R}$ defined by $B_{\xi,x}$ for some (usually implicit) choice of base-point $x \in X$. We shall simply denote such a function by b_ξ in lower case; they all differ by a constant only in view of the cocycle relation.

The boundary at infinity ∂X is endowed with the cône topology [BH99, II.8.6] as well as with the (much finer) topology defined by the Tits angle. The former is often implicitly understood, but when referring to dimension or radius, the topology and distance defined by the Tits angle are considered (this is sometimes emphasised by referring to the "Tits boundary"). The later distance is not to be confused with the associated length metric called "Tits distance" in the literature; we will not need this concept (except in the discussions at the beginning of Section 7).

Recall that any complete CAT(0) space splits off a canonical maximal Hilbertian factor (Euclidean in the proper case studied here) and any isometry decomposes accordingly, see Theorem II.6.15(6) in [BH99].

Normalisers and centralisers in a group G are respectively denoted by $\mathscr{N}_G$ and $\mathscr{Z}_G$. When some group G acts on a set and x is a member of this set, the stabiliser of x in G is denoted by $\mathrm{Stab}_G(x)$ or by the shorthand G_x. For the notation regarding algebraic groups, we follow the standard notation as in [Mar91].

Finally, we present two remarks that we whall never use but that give some context to those statements where we assume X geodesically complete or ∂X finite-dimensional.

The first remark is that every geodesically complete locally compact CAT(0) space is proper. This is contained in S. Cohn-Vossen's extension of the Hopf-Rinow theorem [HR31] (Folgerung 4 §3 in [CV36], see also [Bal95, Theorem 2.4]) and complements nicely the statement [BH99, I.3.7].

The second fact is that if a proper CAT(0) space is finite-dimensional (in the sense of [Kle99]), then so is its Tits boundary (generalising for instance Proposition 7.11 below). The argument is given in [CL08, Proposition 2.1] and may be outlined as follows. For any sphere S in the space X, the "visual map" $\partial X \to S$ is Tits-continuous; if it were injective, the result would follow. However, it becomes injective after replacing S with the ultraproduct of spheres of unbounded radius by the very definition of the boundary; the ultraproduct construction preserves the bound on the dimension, finishing the proof.

3. Convex subsets of the Tits boundary

3.A. **Boundary subsets of small radius.** Given a metric space X and a subset $Z \subseteq X$, one defines the **circumradius** of Z in X as

$$\inf_{x \in X} \sup_{z \in Z} d(x, z).$$

A point x realising the infimum is called a **circumcentre** of Z in X. The **intrinsic circumradius** of Z is its circumradius in Z itself; one defines similarly an **intrinsic circumcentre**. It is called **canonical** if it is fixed by every isometry of X which stabilises Z. We shall make frequent use of the following construction of circumcentres, due to A. Balser and A. Lytchak [BL05, Proposition 1.4]:

Proposition 3.1. *Let X be a complete CAT(1) space and $Y \subseteq X$ be a finite-dimensional closed convex subset. If Y has intrinsic circumradius $\leq \pi/2$, then the set $C(Y)$ of intrinsic circumcentres of Y has a unique circumcentre, which is therefore a canonical (intrinsic) circumcentre of Y.* $\qquad\square$

Let now X be a proper CAT(0) space.

Proposition 3.2. *Let $X_0 \supset X_1 \supset \ldots$ be a nested sequence of non-empty closed convex subsets of X such that $\bigcap_n X_n$ is empty. Then the intersection $\bigcap_n \partial X_n$ is a non-empty closed convex subset of ∂X of intrinsic circumradius at most $\pi/2$.*

In particular, if the Tits boundary is finite-dimensional, then $\bigcap_n \partial X_n$ has a canonical intrinsic circumcentre.

Proof. Pick any $x \in X$ and let x_n be its projection to X_n. The assumption $\bigcap_n X_n = \varnothing$ implies that x_n goes to infinity. Upon extracting, we can assume that it converges to some point $\xi \in \partial X$; observe that $\xi \in \bigcap_n \partial X_n$. We claim that any $\eta \in \bigcap_n \partial X_n$ satisfies $\angle_{\mathrm{T}}(\xi, \eta) \leq \pi/2$. The proposition then follows because (i) the boundary of any closed convex set is closed and π-convex [BH99, II.9.13] and (ii) each ∂X_n is non-empty since otherwise X_n would be bounded, contradicting $\bigcap_n X_n = \varnothing$. When ∂X has finite dimension, there is a canonical intrinsic circumcentre by Proposition 3.1.

For the claim, observe that there exists a sequence of points $y_n \in X_n$ converging to η. It suffices to prove that the comparison angle $\overline{\angle}_x(x_n, y_n)$ is bounded by $\pi/2$ for all n, see [BH99, II.9.16]. This follows from

$$\overline{\angle}_{x_n}(x, y_n) \geq \angle_{x_n}(x, y_n) \geq \pi/2,$$

where the second inequality holds by the properties of the projection on a convex set [BH99, II.2.4(3)].   □

The combination of the preceding two propositions has the following consequence, which improves the results established by Fujiwara, Nagano and Shioya (Theorems 1.1 and 1.3 in [FNS06]).

Corollary 3.3. *Let g be a parabolic isometry of X. The following assertions hold:*
(i) *The fixed point set of g in ∂X has intrinsic circumradius at most $\pi/2$.*
(ii) *If ∂X finite-dimensional, then the centraliser $\mathscr{Z}_{\mathrm{Is}(X)}(g)$ has a canonical global fixed point in ∂X.*
(iii) *For any subgroup $H < \mathrm{Is}(X)$ containing g, the (possibly empty) fixed point set of H in ∂X has circumradius at most $\pi/2$.*   □

Here is another immediate consequence.

Corollary 3.4. *Let G be a topological group with a continuous action by isometries on X without global fixed point. Suppose that G is the union of an increasing sequence of compact subgroups and that ∂X is finite-dimensional. Then there is a canonical G-fixed point in ∂X, fixed by all isometries normalising G.*

Proof. Consider the sequence of fixed point sets X^{K_n} of the compact subgroups K_n. Its intersection is empty by assumption and thus Proposition 3.2 applies.   □

Finally, we record the following elementary fact, which may also be deduced by means of Proposition 3.2 (for manifolds, see Lemma 4.10(iv) in [BGS85]).

Lemma 3.5. *Let $\xi \in \partial X$. Given any closed horoball B centred at ξ, the boundary ∂B coincides with the ball of Tits radius $\pi/2$ centred at ξ in ∂X.*

Proof. Any two horoballs centred at the same point at infinity lie at bounded Hausdorff distance from one another. Therefore, they have the same boundary at infinity. In particular, the boundary ∂B of the given horoball coincides with the intersection of the boundaries of all horoballs centred at ξ. By Proposition 3.2, this is of circumradius at most $\pi/2$; in fact the proof of that proposition shows precisely that the set is contained in the ball of radius at most $\pi/2$ around ξ.

Conversely, let $\eta \in \partial X$ be a point which does not belong to ∂B. We claim that $\angle_{\mathrm{T}}(\xi, \eta) \geq \pi/2$. This shows that every point of ∂X at Tits distance less than $\pi/2$ from ξ belongs to ∂B. Since the latter is closed, it follows that ∂B contains the closed ball of Tits radius $\pi/2$

We turn to the claim. Let b_ξ be a Busemann function centred at ξ. Since every geodesic ray pointing towards η escapes every horoball centred at ξ, there exists a ray $\varrho : [0, \infty) \to X$ pointing to η such that $b_\xi(\varrho(0)) = 0$ and $b_\xi(\varrho(t)) > 0$ for all $t > 0$ (actually, this increases to infinity by convexity). Let $c : [0, \infty) \to X$ be the geodesic ray emanating from $\varrho(0)$ and pointing to ξ. We have $\angle_{\mathrm{T}}(\xi, \eta) = \lim_{t,s \to \infty} \overline{\angle}_{\varrho(0)}(\varrho(t), c(s))$, see [BH99, II.9.8]. Therefore the claim follows from the asymptotic angle formula (Section 2) by taking $y = c(s)$ with s large enough.   □

3.B. Subspaces with boundary of large radius. As before, let X be a proper CAT(0) space. The following result improves Proposition 2.2 in [Lee00]:

Proposition 3.6. *Let $Y \subseteq X$ be a closed convex subset such that ∂Y has intrinsic circumradius $> \pi/2$. Then there exists a closed convex subset $Z \subseteq X$ with $\partial Z = \partial Y$ which is minimal for these properties. Moreover, the union Z_0 of all such minimal subspaces is closed, convex and splits as a product $Z_0 \cong Z \times Z'$.*

Proof. If no minimal such Z existed, there would be a chain of such subsets with empty intersection. The distance to a base-point must then go to infinity and thus the chain contains a countable sequence to which we apply Proposition 3.2, contradicting the assumption on the circumradius.

Let Z' denote the set of all such minimal sets and $Z_0 = \bigcup Z'$ be its union. As in [Lee00, p. 10] one observes that for any $Z_1, Z_2 \in Z'$, the distance $z \mapsto d(z, Z_2)$ is constant on Z_1 and that the nearest point projection p_{Z_2} restricted to Z_1 defines an isometry $Z_1 \to Z_2$. By the Sandwich Lemma [BH99, II.2.12], this implies that Z_0 is convex and that the map $Z' \times Z' \to \mathbf{R}_+ : (Z_1, Z_2) \mapsto d(Z_1, Z_2)$ is a geodesic metric on Z'. As in [Mon06, Section 4.3], this yields a bijection $\alpha : Z_0 \to Z \times Z' : x \mapsto (p_Z(x), Z_x)$, where Z_x is the unique element of Z' containing x. The product of metric spaces $Z \times Z'$ is given the product metric. In order to establish that α is an isometry, it remains as in [Mon06, Proposition 38], to trivialise "holonomy"; it the current setting, this is achieved by Lemma 3.7, which thus concludes the proof of Proposition 3.6. (Notice that Z_0 is indeed closed since otherwise we could extend α^{-1} to the completion of $Z \times Z'$.) $\qquad\square$

Lemma 3.7. *For all $Z_1, Z_2, Z_3 \in Z'$, we have $p_{Z_1} \circ p_{Z_3} \circ p_{Z_2}|_{Z_1} = \mathrm{Id}_{Z_1}$.*

Proof of Lemma 3.7. Let $\vartheta : Z_1 \to Z_1$ be the isometry defined by $p_{Z_1} \circ p_{Z_3} \circ p_{Z_2}|_{Z_1}$ and let f be its displacement function. Then $f : Z_1 \to \mathbf{R}$ is a non-negative convex function which is bounded above by $d(Z_1, Z_2) + d(Z_2, Z_3) + d(Z_3, Z_1)$. In particular, the restriction of f to any geodesic ray in Z_1 is non-increasing. Therefore, a sublevel set of f is a closed convex subset Z of Z_1 with full boundary, namely $\partial Z = \partial Z_1$. By definition, the subspace Z_1 is minimal with respect to the property that $\partial Z_1 = \partial Y$ and hence we deduce $Z = Z_1$. It follows that the convex function f is constant. In other words, the isometry ϑ is a Clifford translation. If it is not trivial, then Z_1 would contain a ϑ-stable geodesic line on which ϑ acts by translation. But by [BH99, Lemma II.2.15], the restriction of ϑ to any geodesic line is the identity. Therefore ϑ is trivial, as desired. $\qquad\square$

Let Γ be a group acting on X by isometries. Following [Mon06, Definition 5], we say that the Γ-action is **reduced** if there is no unbounded closed convex subset $Y \subsetneq X$ such that $g.Y$ is at finite Hausdorff distance from Y for all $g \in \Gamma$.

Corollary 3.8. *Let X be a proper irreducible $\mathrm{CAT}(0)$ space with finite-dimensional Tits boundary, and $\Gamma < \mathrm{Is}(X)$ be a subgroup acting minimally without fixed point at infinity. Then the Γ-action is reduced.*

Proof. Suppose for a contradiction that the Γ-action on X is not reduced. Then there exists an unbounded closed convex subset $Y \subsetneq X$ such that $g.Y$ is at finite Hausdorff distance from Y for all $g \in \Gamma$. In particular ∂Y is Γ-invariant. By Proposition 3.1, it must have intrinsic circumradius $> \pi/2$. Proposition 3.6 therefore yields a canonical closed convex subset $Z_0 = Z \times Z'$ with $\partial(Z \times \{z'\}) = \partial Y$ for all $z' \in Z'$; clearly Z_0 is Γ-invariant and hence we have $Z_0 = X$ by minimality. Since X is irreducible by assumption, we deduce $X = Z$ and hence $X = Y$, as desired. $\qquad\square$

3.C. **Minimal actions and boundary-minimal spaces.** Boundary-minimality and minimality, as defined in the Introduction, are two possible ways for a $\mathrm{CAT}(0)$ space to be "non-degenerate", as illustrated by the following.

Lemma 3.9. *Let X be a complete $\mathrm{CAT}(0)$ space.*

(i) *A group $G < \mathrm{Is}(X)$ acts minimally if and only if any continuous convex G-invariant function on X is constant.*

(ii) *If X is boundary-minimal then any bounded convex function on X is constant.*

Proof. Necessity in the first assertion follows immediately by considering sub-level sets (see [Mon06, Lemma 37]). Sufficiency is due to the fact that the distance to a closed convex set is a convex continuous function [BH99, II.2.5]. The second assertion was established in the proof of Lemma 3.7. $\square$

   Proposition 3.6 has the following important consequence:

Corollary 3.10. *Let X be a proper $\mathrm{CAT}(0)$ space. If ∂X has circumradius $> \pi/2$, then X possesses a canonical closed convex subspace $Y \subseteq X$ such that Y is boundary-minimal and $\partial Y = \partial X$.*

Proof. Let $Z_0 = Z \times Z'$ be the product decomposition provided by Proposition 3.6. The group $\mathrm{Is}(X)$ permutes the elements of Z' and hence acts by isometries on Z'. Under the present hypotheses, the space Z' is bounded since $\partial Z = \partial X$. Therefore it has a circumcentre z', and the fibre $Y = Z \times \{z'\}$ is thus $\mathrm{Is}(X)$-invariant. $\square$

Proposition 3.11. *Let X be a proper $\mathrm{CAT}(0)$ space which is minimal. Assume that ∂X has finite dimension. Then ∂X has circumradius $> \pi/2$ (unless X is reduced to a point). In particular, X is boundary-minimal.*

   The proof of Proposition 3.11 requires some preliminaries. Given a point at infinity ξ, consider the Busemann function B_ξ; the cocycle property (recalled in Section 2) implies in particular that for any isometry $g \in \mathrm{Is}(X)$ fixing ξ and any $x \in X$ the real number $B_{\xi,x}(g.x)$ is independent on the choice of x and yields a canonical homomorphism

$$\beta_\xi : \mathrm{Is}(X)_\xi \longrightarrow \mathbf{R} : g \longmapsto B_{\xi,x}(g.x)$$

called the **Busemann character** centred at ξ.

   Given an isometry g, it follows by the $\mathrm{CAT}(0)$ property that $\inf_{n \geq 0} d(g^n x, x)/n$ coincides with the translation length of g independently of x. We call an isometry **ballistic** when this number is positive. An important fact about a ballistic isometry g of any complete $\mathrm{CAT}(0)$ space X is that for any $x \in X$ the sequence $\{g^n.x\}_{n \geq 0}$ converges to a point $\eta_g \in \partial X$ independent of x; η_g is called the (canonical) **attracting fixed point** of g in ∂X. Moreover, this convergence holds also in angle, which means that $\lim \overline{\angle}_x(g^n x, r(t))$ vanishes as $n, t \to \infty$ when $r : \mathbf{R}_+ \to X$ is any ray pointing to η_g. This is a (very) special case of the results in [KM99].

Lemma 3.12. *Let $\xi \in X$ and $g \in \mathrm{Is}(X)_\xi$ be an isometry which is not annihilated by the Busemann character centred at ξ. Then g is ballistic. Furthermore, if $\beta_\xi(g) > 0$ then $\angle_\mathrm{T}(\xi, \eta_g) > \pi/2$.*

Proof. We have $\beta_\xi(g) = B_{\xi,x}(g.x) \leq d(x, g.x)$ for all $x \in X$. Thus g is ballistic as soon as $\beta_\xi(g)$ is non-zero.
   Assume $\beta_\xi(g) > 0$ and suppose for a contradiction that $\angle_\mathrm{T}(\xi, \eta_g) \leq \pi/2$. Choose $x \in X$ and let ϱ, σ be the rays issuing from x and pointing towards ξ and η_g respectively. Recall from [BH99, II.9.8] that $\angle_\mathrm{T}(\xi, \eta_g) = \lim_{t,s \to \infty} \overline{\angle}_x(\varrho(t), \sigma(s))$. The convergence in direction of $g^n x$ implies that this angle is also given by $\lim_{t,n \to \infty} \overline{\angle}_x(\varrho(t), g^n x)$. Since $\beta_\xi(g) > 0$ we can fix n large enough to have

$$\cos \liminf_{t \to \infty} \overline{\angle}_x(\varrho(t), g^n x) > -\frac{\beta_\xi(g)}{d(gx, x)}.$$

We now apply the asymptotic angle formula from Section 2 with $y = g^n x$ and deduce that the left hand side is $-\beta_\xi(g^n x)/d(g^n x, x)$. Since $\beta_\xi(g^n x) = n\beta_\xi(gx)$ and $d(g^n x, x) \leq nd(gx, x)$, we have a contradiction. $\qquad\square$

Proof of Proposition 3.11. We can assume that ∂X is non-empty since otherwise X is a point by minimality. Suppose for a contradiction that its circumradius is $\leq \pi/2$. Then $\mathrm{Is}(X)$ possesses a global fixed point $\xi \in \partial X$ and ξ is a circumcentre of ∂X, see Proposition 3.1. Lemma 3.12 implies that $\mathrm{Is}(X) = \mathrm{Is}(X)_\xi$ is annihilated by the Busemann character centred at ξ. Thus $\mathrm{Is}(X)$ stabilises every horoball, contradicting minimality. $\qquad\square$

We shall use repeatedly the following elementary fact.

Lemma 3.13. *Let G be a group with an isometric action on a proper geodesically complete* $\mathrm{CAT}(0)$ *space X. If G acts cocompactly or more generally has full limit set, then the action is minimal. (This holds more generally when $\Delta G = \partial X$ in the sense of Section 4.B below.)*

Proof. Let $Y \subseteq X$ be a non-empty closed convex invariant subset, choose $y \in Y$ and suppose for a contradiction that there is $x \notin Y$. Let $r : \mathbf{R}_+ \to X$ be a geodesic ray starting at y and going through x. By convexity [BH99, II.2.5(1)], the function $d(r(t), Y)$ tends to infinity and thus $r(\infty) \notin \partial Y$. This is absurd since $\Delta G \subseteq \partial Y$. $\qquad\square$

Proof of Proposition 1.5. (i) See Proposition 3.11.

(ii) Since $\mathrm{Is}(X)$ has full limit set, any $\mathrm{Is}(X)$-invariant subspace has full boundary. Minimality follows, since boundary-minimality ensures that X possesses no proper subspace with full boundary.

(iii) X is minimal by Lemma 3.13, hence boundary-minimal by (i), since any cocompact space has finite-dimensional boundary by [Kle99, Theorem C]. $\qquad\square$

4. Minimal invariant subspaces for subgroups

4.A. **Existence of a minimal invariant subspace.** For the record, we recall the following elementary dichotomy; a refinement will be given in Theorem 4.3 below:

Proposition 4.1. *Let G be a group acting by isometries on a proper* $\mathrm{CAT}(0)$. *Then either G has a global fixed point at infinity, or any filtering family of non-empty closed convex G-invariant subsets has non-empty intersection.*

(Recall that a family of sets is filtering if it is directed by containment $\supseteq$.)

Proof. (Remark 36 in [Mon06].) Suppose $\mathscr{Y}$ is such a family, choose $x \in X$ and let x_Y be its projection on each $Y \in \mathscr{Y}$. If the net $\{x_Y\}_{Y \in \mathscr{Y}}$ is bounded, then $\bigcap_{Y \in \mathscr{Y}} Y$ is non-empty. Otherwise it goes to infinity and any accumulation point in ∂X is G-fixed in view of $d(gx_Y, x_Y) \leq d(gx, x)$. $\qquad\square$

4.B. **Dichotomy.** Let G be a group acting by isometries on a complete $\mathrm{CAT}(0)$ space X.

Lemma 4.2. *Given any two $x, y \in X$, the convex closures of the respective G-orbits of x and y in X have the same boundary in ∂X.*

Proof. Let Y be the convex closure of the G-orbit of x. In particular Y is the minimal closed convex G-invariant subset containing x. Given any closed convex G-invariant subset Z, let $r = d(x, Z)$. Recall that the tubular closed neighbourhood $\mathcal{N}_r(Z)$ is convex [BH99, II.2.5(1)]. Since it is also G-invariant and contains x, the minimality of Y implies $Y \subseteq \mathcal{N}_r(Z)$. $\qquad\square$

This yields a canonical closed convex G-invariant subset of the boundary ∂X, which we denote by ΔG. It contains the limit set ΛG but is sometimes larger.

Combining what we established thus far with the splitting arguments from [Mon06], we obtain a dichotomy:

Theorem 4.3. *Let G be a group acting by isometries on a complete* CAT(0) *space X and $H < G$ any subgroup.*
If H admits no minimal non-empty closed convex invariant subset and X is proper, then:

(A.i) ΔH *is a non-empty closed convex subset of ∂X of intrinsic circumradius at most $\pi/2$.*

(A.ii) *If ∂X is finite-dimensional, then the normaliser $\mathscr{N}_G(H)$ of H in G has a global fixed point in ∂X.*

If H admits a minimal non-empty closed convex invariant subset $Y \subseteq X$, then:

(B.i) *The union Z of all such subsets is a closed convex $\mathscr{N}_G(H)$-invariant subset.*

(B.ii) *Z splits H-equivariantly and isometrically as a product $Z \simeq Y \times C$, where C is a complete* CAT(0) *space which admits a canonical $\mathscr{N}_G(H)/H$-action by isometries.*

(B.iii) *If the H-action on X is non-evanescent, then C is bounded and there is a canonical minimal non-empty closed convex H-invariant subset which is $\mathscr{N}_G(H)$-stable.*

(When X is proper, the *non-evanescence* condition of (iii) simply means that H has no fixed point in ∂X; see [Mon06].)

Proof. In view of Lemma 4.2, the set ΔH is contained in the boundary of any non-empty closed convex H-invariant set and is $\mathscr{N}_G(H)$-invariant. Thus the assertions (A.i) and (A.ii) follow from Proposition 3.2, noticing that in a proper space ΔH is non-empty unless H has bounded orbits, in which case it fixes a point, providing a minimal subspace. For (B.i), (B.ii) and (B.iii), see Remarks 39 in [Mon06]. □

4.C. **Normal subgroups.**

Proof of Theorem 1.10. We adopt the notation and assumptions of the theorem. By (A.ii), N admits a minimal non-empty closed convex invariant subset $Y \subseteq X$. This set is unbounded, since otherwise N fixes a point and thus by G-minimality $X^N = X$, hence $N = 1$. Since X is irreducible, points (B.i) and (B.ii) show $Y = X$ and thus N acts indeed minimally.

Since the displacement function of any $g \in \mathscr{Z}_G(N)$ is a convex N-invariant function, it is constant by minimality. Hence g is a Clifford translation and must be trivial since otherwise X splits off a Euclidean factor, see [BH99, II.6.15].

The derived subgroup $N' = [N, N]$ is also normal in G and therefore acts minimally by the previous discussion, noticing that N' is non-trivial since otherwise $N \subseteq \mathscr{Z}_G(N)$. If N fixed a point at infinity, N' would preserve all corresponding horoballs, contradicting minimality.

Having established that N acts minimally and without fixed point at infinity, we can apply the splitting theorem (Corollary 10 in [Mon06]) and deduce from the irreducibility of X that N does not split.

Finally, let $R \lhd N$ be the amenable radical and observe that it is normal in G. The theorem of Adams–Ballmann [AB98a] states that R either (i) fixes a point at infinity or (ii) preserves a Euclidean flat in X. (Although their result is stated for amenable groups without mentioning any topology, the proof applies indeed to every topological group that preserves a probability measure whenever it acts continuously on a compact metrisable space.) If R is non-trivial, we know already from the above discussion that (i) is impossible

and that R acts minimally; it follows that X is a flat. By irreducibility and since $X \neq \mathbf{R}$, this forces X to be a point, contradicting $R \neq 1$. $\qquad\square$

Corollary 1.11 will be proved in Section 5.B. For Corollary 1.12, it suffices to observe that the centraliser of any element of a discrete normal subgroup is open. Next, we recall the following definition.

A subgroup N of a group G is **ascending** if there is a family of subgroups $N_\alpha < G$ indexed by the ordinals and such that $N_0 = N$, $N_\alpha \lhd N_{\alpha+1}$, $N_\alpha = \bigcup_{\beta < \alpha} N_\beta$ if α is a limit ordinal and $N_\alpha = G$ for α large enough. The smallest such ordinal is the **order**.

Proposition 4.4. *Consider a group acting minimally by isometries on a proper* $\mathrm{CAT}(0)$ *space. Then any ascending subgroup without global fixed point at infinity still acts minimally.*

Proof. We argue by transfinite induction on the order ϑ of ascending subgroups $N < G$, the case $\vartheta = 0$ being trivial. Let X be a space as in the statement. By Proposition 4.1, each N_α has a minimal set. If $\vartheta = \vartheta' + 1$, it follows from (B.iii) that $N_{\vartheta'}$ acts minimally and we are done by induction hypothesis. Assume now that ϑ is a limit ordinal. For all α, we denote as in (B.i) by $Z_\alpha \subseteq X$ the union of all N_α-minimal sets. The induction hypothesis implies that for all $\alpha \leq \beta < \vartheta$, any N_β-minimal set is N_α-minimal. Thus, if $Z_0 = Y_0 \times C_0$ is a splitting as in (B.ii) with a N-minimal set Y_0, we have a nested family of decompositions $Z_\alpha = Y_0 \times C_\alpha$ for a nested family of closed convex subspaces C_α of the compact $\mathrm{CAT}(0)$ space C_0, indexed by $\alpha < \vartheta$. Thus, for any $c \in \bigcap_{\alpha < \vartheta} C_\alpha$, the space $Y_0 \times \{c\}$ is G-invariant and hence $Y_0 = X$ indeed. $\qquad\square$

Remark 4.5. Proposition 4.4 holds more generally for complete $\mathrm{CAT}(0)$ spaces if N is non-evanescent. Indeed Proposition 4.1 hold in that generality (Remark 36 in [Mon06]) and C remains compact in a weaker topology (Theorem 14 in [Mon06]).

Proof of Theorem 1.13. In view of Theorem 1.10, it suffices to prove that any non-trivial ascending subgroup $N < G$ as in that statement still acts minimally and without global fixed point at infinity. We argue by induction on the order ϑ and we can assume that ϑ is a limit ordinal by Theorem 1.10. Then $\bigcap_{\alpha < \vartheta} (\partial X)^{N_\alpha}$ is empty and thus by compactness there is some $\alpha < \vartheta$ such that $(\partial X)^{N_\alpha}$ is empty. Now N_α acts minimally on X by Proposition 4.4 and thus we conclude using the induction hypothesis. $\qquad\square$

5. Algebraic and geometric product decompositions

5.A. **Preliminary decomposition of the space.** We shall prepare our spaces by means of a geometric decomposition. For any geodesic metric space with *finite affine rank*, Foertsch–Lytchak [FL06] established a canonical decomposition generalising the classical theorem of de Rham [dR52]. However, such a statement fails to be true for $\mathrm{CAT}(0)$ spaces that are merely proper, due notably to compact factors that can be infinite products. Nevertheless, using asymptotic $\mathrm{CAT}(0)$ geometry and Section 3.A, we can adapt the arguments from [FL06] and obtain:

Theorem 5.1. *Let X be a proper* $\mathrm{CAT}(0)$ *space with ∂X finite-dimensional and of circumradius $> \pi/2$. Then there is a canonical closed convex subset $Z \subseteq X$ with $\partial Z = \partial X$, invariant under all isometries, and admitting a canonical maximal isometric splitting*

$$(5.\mathrm{i}) \qquad\qquad Z \cong \mathbf{R}^n \times Z_1 \times \cdots \times Z_m \qquad (n, m \geq 0)$$

with each Z_i irreducible and $\neq \mathbf{R}^0, \mathbf{R}^1$. Every isometry of Z preserves this decomposition upon permuting possibly isometric factors Z_i.

Remark 5.2. It is well known that in the above situation the splitting (5.i) induces a decomposition

$$\mathrm{Is}(Z) \; = \; \mathrm{Is}(\mathbf{R}^n) \times \Big(\big(\mathrm{Is}(Z_1) \times \cdots \times \mathrm{Is}(Z_m) \big) \rtimes F \Big),$$

where F is the permutation group of $\{1, \ldots, d\}$ permuting possible isometric factors amongst the Y_j. Indeed, this follows from the statement that isometries preserve the splitting upon permutation of factors, see *e.g.* Proposition I.5.3(4) in [BH99]. Of course, this does not *a priori* mean that we have a unique, nor even canonical, splitting *in the category of groups*; this shall however be established for Theorem 1.6.

The hypotheses of Theorem 5.1 are satisfied in some naturally occurring situations:

Corollary 5.3. *Let X be a proper* CAT(0) *space with finite-dimensional boundary.*
  (i) *If* $\mathrm{Is}(X)$ *has no fixed point at infinity, then X possesses a subspace Z satisfying all the conclusions of Theorem 5.1.*
  (ii) *If* $\mathrm{Is}(X)$ *acts minimally, then X admits a canonical splitting as in 5.i.*

Proof of Corollary 5.3. By Proposition 3.1, if $\mathrm{Is}(X)$ has no fixed point at infinity, then ∂X has circumradius $> \pi/2$. By Proposition 3.11, the same conclusion holds is $\mathrm{Is}(X)$ acts minimally. $\qquad\square$

Proof of Theorems 1.9 and 5.1. For Theorem 5.1, we let $Z \subseteq X$ be the canonical boundary-minimal subset with $\partial Z = \partial X$ provided by Corollary 3.10; we shall not use the circumradius assumption any more. For Theorem 1.9, we let $Z = X$. The remainder of the argument is common for both statements.

Recalling that in complete generality all isometries preserving the Euclidean factor decomposition [BH99, II.6.15], we can assume that Z has no Euclidean factor and shall obtain the decomposition (5.i) with $n = 0$.

Since Z is minimal amongst closed convex subsets with $\partial Z = \partial X$, it has no non-trivial compact factor. On the other hand, any proper geodesic metric space admits some maximal product decomposition into non-compact factors. In conclusion, Z admits some maximal splitting $Z = Z_1 \times \cdots \times Z_m$ with each Z_i irreducible and $\neq \mathbf{R}^0, \mathbf{R}^1$. (This can fail in presence of compact factors).

It remains to prove that any other such decomposition $Z = Z'_1 \times \cdots \times Z'_{m'}$ coincides with the first one after possibly permuting the factors (in particular, $m' = m$). We now borrow from the argumentation in [FL06], indicating the steps and the necessary changes. It is assumed that the reader has a copy of [FL06] at hand but keeps in mind that our spaces might lack the finite affine rank condition assumed in that paper. We shall replace the notion of affine subspaces with a large-scale particular case: a **cône** shall be any subspace isometric to a closed convex cône in some Euclidean space. This includes the particular cases of a point, a ray or a full Euclidean space.

Whenever a space Y has some product decomposition and Y' is a factor, write $Y'_y \subseteq Y$ for the corresponding fibre $Y'_y \cong Y'$ through $y \in Y$. The following is an analogue of Corollary 1.2 in [FL06].

Lemma 5.4. *Let Y be a proper* CAT(0) *space with finite-dimensional boundary and without compact factors. Suppose given two decompositions $Y = Y_1 \times Y_2 = S_1 \times S_2$ with all four $(Y_i)_y \cap (S_j)_y$ reduced to $\{y\}$ for some $y \in Y$. Then Y is a Euclidean space.*

Proof of Lemma 5.4. Any $y \in Y$ is contained in a maximal cône based at y since ∂Y has finite dimension; by abuse of language we call such cônes maximal. The arguments of Sections 3 and 4 in [FL06] show that any maximal cône is *rectangular*, which means that it

inherit a product structure from any product decomposition of the ambient CAT(0) space. Specifically, it suffices to observe that the product of two cônes is a cône and that the projection of a cône along a product decomposition of CAT(0) spaces remains a cône. (In fact, the "equality of slopes" of Section 4.2 in [FL06], namely the fact that parallel geodesic segments in a CAT(0) space have identical slopes in product decompositions, is a general fact for CAT(0) spaces. It follows from the convexity of the metric, see for instance [Mon06, Proposition 49] for a more general statement.) The deduction of the statement of Lemma 5.4 from the rectangularity of maximal cônes following [FL06] is particularly short since all proper CAT(0) Banach spaces are Euclidean.   □

Lemma 5.5. *For a given $z \in Z$ and any product decomposition $Z = S \times S'$, the intersection $(Z_i)_z \cap S_z$ is either $\{z\}$ or $(Z_i)_z$.*

Proof of Lemma 5.5. Write $P^S : Z \to S$ and $P^{Z_i} : Z \to Z_i$ for the projections and set $F_z = S_z \cap (Z_i)_z$. Following [FL06], define $T \subseteq Z$ by $T = P^S(F_z) \times S'$. We contend that $P^{Z_i}(T)$ has full boundary in Z_i.

Indeed, given any point in $\partial(Z_i)_z$, we represent is by a ray r originating from z. We can choose a maximal cône in Z based at z and containing r. We know already that this cône is rectangular, and therefore the proof of Lemma 5.2 in [FL06] shows that $P^{Z_i}(r)$ lies in $P^{Z_i}(T)$, justifying our contention.

We observe that Z_i inherits from Z the property that it has no closed convex proper subset of full boundary. In conclusion, since $P^{Z_i}(T)$ is a convex set, it is dense in Z. However, according to Lemma 5.1 in [FL06], it splits as $P^{Z_i}(T) = P^{Z_i}(F_z) \times P^{Z_i}(S')$. Upon possibly replacing $P^{Z_i}(S')$ by its completion (whilst $P^{Z_i}(F_z)$ is already closed in Z_i since P^{Z_i} is isometric on $(Z_i)_z$), we obtain a splitting of the closure of $P^{Z_i}(T)$, and hence of Z_i. This completes the proof of the lemma since Z_i is irreducible.   □

Now the main argument runs by induction over $m \geq 2$. Lemma 5.5 identifies by induction Z_i with some Z'_j. Indeed, Lemma 5.4 excludes that all pairwise intersections reduce to a point since Z has no Euclidean factor.   □

5.B. **Proof of Theorem 1.6 and Addendum 1.8.** The following consequence of the solution to Hilbert's fifth problem belongs to the mathematical lore.

Theorem 5.6. *Let G be a locally compact group with trivial amenable radical. Then G possesses a canonical finite index open normal subgroup $G^\dagger$ such that $G^\dagger = L \times D$, where L is a connected semi-simple Lie group with trivial centre and no compact factors, and D is totally disconnected.*

Proof. This follows from the Gleason–Montgomery–Zippin solution to Hilbert's fifth problem and the fact that connected semi-simple Lie groups have finite outer automorphism groups. More details may be found for example in [Mon01, § 11.3].   □

Combining Theorem 5.6 with Theorem 1.10, we find the statement given as Corollary 1.11 in the Introduction.

Theorem 5.7. *Let $X \neq \mathbf{R}$ be an irreducible proper CAT(0) space with finite-dimensional Tits boundary and $G < \mathrm{Is}(X)$ any closed subgroup whose action is minimal and does not have a global fixed point in ∂X.*

Then G is either totally disconnected or an almost connected simple Lie group with trivial centre.

Proof. By Theorem 1.10, G has trivial amenable radical. Let $G^\dagger$ be as in Theorem 5.6. Applying Theorem 1.10 to this normal subgroup of G, deduce that we have either $G^\dagger = L$ with L simple or $G^\dagger = D$. $\square$

We can now complete the proof of Theorem 1.6 and Addendum 1.8 and we adopt their notation. Since $\mathrm{Is}(X)$ has no global fixed point at infinity, there is a canonical minimal non-empty closed convex $\mathrm{Is}(X)$-invariant subset $X' \subseteq X$ (Remarks 39 in [Mon06]). We apply Corollary 5.3 to $Z = X'$ and Remark 5.2 to $G = \mathrm{Is}(Z)$, setting

$$G^* = \mathrm{Is}(\mathbf{R}^n) \times \mathrm{Is}(Z_1) \times \cdots \times \mathrm{Is}(Z_m).$$

All the claimed properties of the resulting factor groups are established in Theorem 1.10, Theorem 1.13 and Theorem 1.1 in [CM08b] (the proof of which is completely independent from the present considerations). Finally, the claim that any product decomposition of G^* is a regrouping of the factors in (1.i) is established as follows. Notice that the G^*-action on Z is still minimal and without fixed point at infinity (this is almost by definition but alternatively also follows from Theorem 1.10). Therefore, given any product decomposition of G^*, we can apply the splitting theorem (Corollary 10 in [Mon06]) and obtain a corresponding splitting of Z. Now the uniqueness of the decomposition of the space Z (away from the Euclidean factor) implies that the given decomposition of G^* is a regrouping of the factors occurring in Remark 5.2. $\square$

5.C. **CAT(0) spaces without Euclidean factor.** For the sake of future references, we record the following consequence of the results obtained thus far:

Corollary 5.8. *Let X be a proper $\mathrm{CAT}(0)$ space with finite-dimensional boundary and no Euclidean factor, such that $G = \mathrm{Is}(X)$ acts minimally without fixed point at infinity. Then G has trivial amenable radical and any subgroup of G acting minimally on X has trivial centraliser. Furthermore, given a non-trivial normal subgroup $N \lhd G$, any N-minimal N-invariant closed subspace of X is a regrouping of factors in the decomposition of Addendum 1.8. In particular, if each irreducible factor of G is non-discrete, then G has no non-trivial finitely generated closed normal subgroup.*

Proof. The triviality of the amenable radical comes from the corresponding statement in irreducible factors of X, see Theorem 1.10. By the second paragraph of the proof of Theorem 1.10, any subgroup of G acting minimally has trivial centraliser. The fact that minimal invariant subspaces for normal subgroups are fibres in the product decomposition (1.ii) follows since any product decomposition of X is a regrouping of factors in (1.ii) and since any normal subgroup of G yields such a product decomposition by Theorem 4.3(B.i) and (B.ii). Assume finally that each irreducible factor in (1.i) is non-discrete and let $N < G$ be a finitely generated closed normal subgroup. Then N is discrete by Baire's category theorem, and N acts minimally on a fibre, say Y, of the space decomposition (1.ii). Therefore, the projection of N to $\mathrm{Is}(Y)$ has trivial centraliser, unless N is trivial. Since N is discrete, normal and finitely generated, its centraliser is open. Since $\mathrm{Is}(Y)$ is non-discrete by assumption, we deduce that N is trivial, as desired. $\square$

6. Totally disconnected group actions

6.A. **Smoothness.** When considering actions of totally disconnected groups, a desirable property is **smoothness**, namely that points have open stabilisers. This condition is important in representation theory, but also in our geometric context, see point (ii) of Corollary 6.3 below and [Cap09].

In general, this condition does not hold, even for actions that are cocompact, minimal and without fixed point at infinity. An example will be constructed in Section 6.C in [CM08b]. However, we establish it under a rather common additional hypothesis. Recall that a metric space X is called **geodesically complete** (or said to have **extensible geodesics**) if every geodesic segment of positive length may be extended to a locally isometric embedding of the whole real line. The following contains Theorem 1.2 from the Introduction.

Theorem 6.1. *Let G be a totally disconnected locally compact group with a minimal, continuous and proper action by isometries on a proper* CAT(0) *space X.*

If X is geodesically complete, then the action is smooth. In fact, the pointwise stabiliser of every bounded set is open.

Remark 6.2. In particular, the stabiliser of a point acts as a *finite* group of isometries on any given ball around this point in the setting of Theorem 6.1.

Corollary 6.3. *Let X be a proper* CAT(0) *space and G be a totally disconnected locally compact group acting continuously properly on X by isometries. Then:*

   (i) *If the G-action is cocompact, then every element of zero translation length is elliptic.*
   (ii) *If the G-action is cocompact and every point $x \in X$ has an open stabiliser, then the G-action is semi-simple.*
   (iii) *If the G-action is cocompact and X is geodesically complete, then the G-action is semi-simple.*

Proof of Corollary 6.3. Points (i) and (ii) follow readily from Theorem 6.1, see [Cap09, Corollary 3.3].

(iii) In view of Lemma 3.13, this follows from Theorem 6.1 and (ii).   □

The following is a key fact for Theorem 6.1:

Lemma 6.4. *Let X be a geodesically complete proper* CAT(0) *space. Let $(C_n)_{n \geq 0}$ be an increasing sequence of closed convex subsets whose union $C = \bigcup_n C_n$ is dense in X.*

Then every bounded subset of X is contained in some C_n; in particular, $C = X$.

Proof. Suppose for a contradiction that for some $r > 0$ and $x \in X$ the r-ball around x contains an element x_n not in C_n for each n. We shall construct inductively a sequence $\{c_k\}_{k \geq 1}$ of pairwise r-disjoint elements in C with $d(x, c_k) \leq 2r + 2$, contradicting the properness of X.

If $c_1, \ldots, c_{k-1}$ have been constructed, choose n large enough to that C_n contains them all and $d(x, C_n) \leq 1$. Consider the (non-trivial) geodesic segment from x_n to its nearest point projection $\overline{x_n}$ on C_n; by geodesic completeness, it is contained in a geodesic line and we choose y at distance $r + 1$ from C_n on this line. Notice that $x_n \in [\overline{x_n}, y]$ and hence $d(y, x) \leq 2r + 1$. Moreover, $d(y, c_i) \geq r + 1$ for all $i < k$. Since C is dense, we can choose c_k close enough to y to ensure $d(c_k, x) \leq 2r + 2$ and $d(c_k, c_i) \geq r$ for all $i < k$, completing the induction step.   □

End of proof of Theorem 6.1. The subset $C \subseteq X$ consisting of those points $x \in X$ such that the stabiliser G_x is open is clearly convex and G-stable. By [Bou71, III § 4 No 6], the group G contains a compact open subgroup and hence C is non-empty. Thus C is dense by minimality of the action. Since $\mathrm{Is}(X)$ is second countable, we can choose a descending chain $Q_n < G$ of compact open subgroups whose intersection acts trivially on X. Therefore, C may be written as the union of an ascending family of closed convex subsets $C_n \subseteq X$, where C_n is the fixed point set of Q_n. Now the statement of the theorem follows from Lemma 6.4.   □

6.B. **Locally finite equivariant partitions and cellular decompositions.** Let X be a locally finite cell complex and G be its group of cellular automorphisms, endowed with the topology of pointwise convergence on bounded subsets. Then G is a totally disconnected locally compact group and every bounded subset of X has an open pointwise stabiliser in G. One of the interest of Theorem 6.1 is that it allows for a partial converse to the latter statement:

Proposition 6.5. *Let X be a proper* CAT(0) *space and G be a totally disconnected locally compact group acting continuously properly on X by isometries. Assume that the pointwise stabiliser of every bounded subset of X is open in G. Then we have the following:*

(i) *X admits a canonical locally finite G-equivariant partition.*

(ii) *Denoting by $\sigma(x)$ the piece supporting the point $x \in X$ in that partition, we have* $\mathrm{Stab}_G(\sigma(x)) = \mathscr{N}_G(G_x)$ *and* $\mathscr{N}_G(G_x)/G_x$ *acts freely on $\sigma(x)$.*

(iii) *If $G\backslash X$ is compact, then so is* $\mathrm{Stab}_G(\sigma(x))\backslash\sigma(x)$ *for all $x \in X$.*

Proof. Consider the equivalence relation on X defined by

$$x \sim y \quad \Leftrightarrow \quad G_x = G_y.$$

This yields a canonical G-invariant partition of X. We need to show that it is locally finite. Assume for a contradiction that there exists a converging sequence $\{x_n\}_{n \geq 0}$ such that the subgroups G_{x_n} are pairwise distinct. Let $x = \lim_n x_n$.

We claim that $G_{x_n} < G_x$ for all sufficiently large n. Indeed, upon extracting there would otherwise exist a sequence $g_n \in G_{x_n}$ such that $g_n.x \neq x$ for all n. Upon a further extraction, we may assume that g_n converges to some $g \in G$. By construction g fixes x. Since G_x is open by hypothesis, this implies that g_n fixes x for sufficiently large n, a contradiction. This proves the claim.

By hypothesis the pointwise stabiliser of any ball centred at x is open. Thus G_x possesses a compact open subgroup U which fixes every x_n. This implies that we have the inclusion $U < G_{x_n} < G_x$ for all n. Since the index of U in G_x is finite, there are only finitely many subgroups of G_x containing U. This final contradiction finishes the proof of (i).

(ii) Straightforward in view of the definitions.

(iii) Suppose for a contradiction that $H\backslash\sigma(x)$ is not compact, where $H = \mathrm{Stab}_G(\sigma(x))$. Let then $y_n \in \sigma(x)$ be a sequence such that $d(y_n, H.x) > n$. Let now $g_n \in G$ be such that $\{g_n.y_n\}$ is bounded, say of diameter C. By (i), the set $\{g_n G_{y_n} g_n^{-1}\}$ is thus finite. Upon extracting, we shall assume that it is constant. Now, for all $n < k$, the element $g_n^{-1}g_k$ normalises $G_{y_k} = G_x$ and maps y_k to a point at distance $\leq C$ from y_n. In view of (ii), this is absurd. $\qquad\square$

Remark 6.6. The partition of X constructed above is non-trivial whenever G does not act freely. This is for example the case whenever G is non-discrete and acts faithfully.

The pieces in the above partition are generally neither bounded (even if $G\backslash X$ is compact), nor convex, nor even connected. However, if one assumes that the space admits a sufficiently large amount of symmetry, then one obtains a partition which deserves to be viewed as an equivariant cellular decomposition.

Corollary 6.7. *Let X be a proper* CAT(0) *space and G be a totally disconnected locally compact group acting continuously properly on X by isometries. Assume that the pointwise stabiliser of every bounded subset of X is open in G, and that no open subgroup of G fixes a point at infinity. Then X admits admits a canonical locally finite G-equivariant decomposition into compact convex pieces.*

Proof. For each $x \in X$, let $\tau(x)$ be the fixed-point-set of G_x. Then $\tau(x)$ is clearly convex; it is compact by hypothesis. Furthermore the map $x \mapsto \tau(x)$ is G-equivariant. The fact that the collection $\{\tau(x) \mid x \in X\}$ is locally finite follows from Proposition 6.5. $\qquad\square$

6.C. **Alexandrov angle rigidity.** A further consequence of Theorem 6.1 is a phenomenon of **angle rigidity**. Given an elliptic isometry g of complete a CAT(0) space X and a point $x \in X$, we denote by $c_{g,x}$ the projection of x on the closed convex set of g-fixed points.

Proposition 6.8. *Let G be a totally disconnected locally compact group with a continuous and proper cocompact action by isometries on a geodesically complete proper CAT(0) space X. Then there is $\varepsilon > 0$ such that for any elliptic $g \in G$ and any $x \in X$ with $gx \neq x$ we have $\angle_{c_{g,x}}(gx, x) \geq \varepsilon$.*

(We will later also prove an angle rigidity for the Tits angle, see Proposition 7.15.)

Proof. First we observe that this bound on the Alexandrov angle is really a local property at $c_{g,x}$ of the germ of the geodesic $[c_{g,x}, x]$ since for any $y \in [c_{g,x}, x]$ we have $c_{g,y} = c_{g,x}$.

Next, we claim that for any $n \in \mathbf{N}$, any isometry of order $\leq n$ of any complete CAT(0) space B satisfies $\angle_{c_{g,x}}(gx, x) \geq 1/n$ for all $x \in B$ that are not g-fixed. Indeed, it follows from the definition of Alexandrov angles (see [BH99, II.3.1]) that for any $y \in [c_{g,x}, x]$ we have

$$d(gy, y) \leq d(c_{g,x}, y)\angle_{c_{g,x}}(gx, x).$$

Therefore, if $\angle_{c_{g,x}}(gx, x) < 1/n$, the entire g-orbit of y would be contained in a ball around y not containing $c_{g,x} = c_{g,y}$. This is absurd since the circumcentre of this orbit is a g-fixed point.

In order to prove the proposition, we now suppose for a contradiction that there are sequences $\{g_n\}$ of elliptic elements in G and $\{x_n\}$ in X with $g_n x_n \neq x_n$ and $\angle_{c_n}(g_n x_n, x_n) \to 0$, where $c_n = c_{g_n, x_n}$. Since the G-action is cocompact, there is (upon extracting) a sequence $\{h_n\}$ in G such that $h_n c_n$ converges to some $c \in X$. Upon conjugating g_n by h_n, replacing x_n by $h_n x_n$ and c_n by $h_n c_n$, we can assume $c_n \to c$ without loosing any of the conditions on g_x, x_n and c_n, including the relation $c_n = c_{g_n, x_n}$.

Since $d(g_n c, c) \leq 2d(c_n, c)$, we can further extract and assume that $\{g_n\}$ converges to some limit $g \in G$; notice also that g fixes c. By Lemma 3.13, the action is minimal and hence Theorem 6.1 applies. Therefore, we can assume that all g_n coincide with g on some ball B around c and in particular preserve B. Using Remark 6.2, this provides a contradiction. $\quad\square$

A first consequence is an analogue of a result that E. Swenson proved for discrete groups (Theorem 11 in [Swe99]).

Corollary 6.9. *Let G be a totally disconnected locally compact group with a continuous and proper cocompact action by isometries on a geodesically complete proper CAT(0) space X not reduced to a point.*

Then G contains hyperbolic elements (thus in particular elements of infinite order).

Beyond the totally disconnected case, we can appeal to Theorem 1.6 and Addendum 1.8 and state the following.

Corollary 6.10. *Let G be any locally compact group with a continuous and proper cocompact action by isometries on a geodesically complete proper CAT(0) space X not reduced to a point.*

Then G contains elements of infinite order; if moreover $(\partial X)^G = \varnothing$, then G contains hyperbolic elements.

Proof of Corollary 6.9. Proposition 6.8 allows us use the argument form [Swe99]: We can choose a geodesic ray $r : \mathbf{R}_+ \to X$, an increasing sequence $\{t_i\}$ going to infinity in $\mathbf{R}_+$ and $\{g_i\}$ in G such that the function $t \mapsto g_i r(t + t_i)$ converges uniformly on bounded intervals (to a geodesic line). For $i < j$ large enough, the angle $\angle_{h(r(t_i))}(r(t_i), h^2(r(t_i)))$ defined with $h = g_i^{-1} g_j$ is arbitrarily close to π. In order to prove that h is hyperbolic, it suffices to show that this angle will eventually equal π. Suppose this does not happen; by Corollary 6.3(iii), we can assume that h is elliptic. We set $x = r(t_i)$ and $c = c_{h,x}$. Considering the congruent triangles (c, x, hx) and $(c, hx, h^2 x)$, we find that $\angle_c(x, hx)$ is arbitrarily small. This is in contradiction with Proposition 6.8. $\qquad\square$

Proof of Corollary 6.10. If the connected component G° is non-trivial, then it contains elements of infinite order; if it is trivial, we can apply Corollary 6.9.

Assume now $(\partial X)^G = \varnothing$. Then Theorem 1.6 and Addendum 1.8 apply. Therefore, we obtain hyperbolic elements either from Corollary 6.9 or from the fact that any non-compact semi-simple group contains elements that are algebraically hyperbolic, combined with the fact that the latter act as hyperbolic isometries. That fact is established in Theorem 7.4(i) below, the proof of which is independent of Corollary 6.10. $\qquad\square$

6.D. **Algebraic structure.** Given a topological group G, we define its **socle** $\mathrm{soc}(G)$ as the subgroup generated by all minimal non-trivial closed normal subgroups of G. Notice that G might have no minimal non-trivial closed normal subgroup, in which case its socle is trivial.

We also recall that the **quasi-centre** of a locally compact group G is the subset $\mathscr{QZ}(G)$ consisting of all those elements possessing an open centraliser. Clearly $\mathscr{QZ}(G)$ is a (topologically) characteristic subgroup of G. Since any element with a discrete conjugacy class possesses an open centraliser, it follows that the quasi-centre contains all discrete normal subgroups of G.

Proposition 6.11. *Let X be a proper $\mathrm{CAT}(0)$ space without Euclidean factor and $G < \mathrm{Is}(X)$ be a closed subgroup acting minimally cocompactly without fixed point at infinity. If G has trivial quasi-centre, then $\mathrm{soc}(G^*)$ is direct product of r non-trivial characteristically simple groups, where r is the number of irreducible factors of X and G^* is the canonical finite index open normal subgroup acting trivially on the set of factors of X.*

The proof will use the following general fact inspired by a statement for tree automorphisms, Lemma 1.4.1 in [BM00].

Proposition 6.12. *Let G be a compactly generated totally disconnected locally compact group without non-trivial compact normal subgroups. Then any filtering family of non-discrete closed normal subgroups has non-trivial (thus non-compact) intersection.*

A variant of this proposition is proved in [CM08a]; since the proof is short, we give it for the sake of completeness.

Proof of Proposition 6.12. Let $\mathfrak{g}$ be a **Schreier graph** for G. We recall that it consists in choosing any open compact subgroup $U < G$ (which exists by [Bou71, III §4 No 6]), defining the vertex set of $\mathfrak{g}$ as G/U and drawing edges according to a compact generating set which is a union of double cosets modulo U; see [Mon01, §11.3]. Since G has no non-trivial compact normal subgroup, the continuous G-action on $\mathfrak{g}$ is faithful. Let v_0 be a vertex of $\mathfrak{g}$ and denote by $v_0^{\perp}$ the set of neighbouring vertices. Since G is vertex-transitive on $\mathfrak{g}$, it follows that for any normal subgroup $N \lhd G$, the N_{v_0}-action on $v_0^{\perp}$ defines a finite permutation group $F_N < \mathrm{Sym}(v_0^{\perp})$ which, as an abstract permutation group, is independent of the choice of v_0. Therefore, if N is non-discrete, this permutation group F_N has to be non-trivial since

U is open and $\mathfrak{g}$ connected. Now a filtering family $\mathscr{F}$ of non-discrete normal subgroups yields a filtering family of non-trivial finite subgroups of $\mathrm{Sym}(v_0^\perp)$. Thus the intersection of these finite groups is non-trivial. Let g be a non-trivial element in this intersection. For any $N \in \mathscr{F}$, let N_g be the inverse image of $\{g\}$ in N_{v_0}. Thus N_g is a non-empty compact subset of N for each $N \in \mathscr{F}$. Since the family $\mathscr{F}$ is filtering, so are $\{N_{v_0} \mid N \in \mathscr{F}\}$ and $\{N_g \mid N \in \mathscr{F}\}$. The result follows, since a filtering family of non-empty closed subsets of the compact set G_{v_0} has a non-empty intersection. $\qquad\square$

Evidently open normal subgroups form a filtering family; we can thus deduce:

Corollary 6.13. *Let G be a compactly generated locally compact group without any non-trivial compact normal subgroup. If G is residually discrete, then it is discrete.* $\qquad\square$

Proof of Proposition 6.11. We first observe that G^* has no non-trivial discrete normal subgroup. Indeed, such a subgroup has finitely many G-conjugates, which implies that each of its elements has discrete G-conjugacy class and hence belongs to $\mathscr{QZ}(G)$, which was assumed trivial.

Let now $\{N_i\}$ be a chain of non-trivial closed normal subgroups of G^*. If N_i is totally disconnected for some i, then the intersection $\bigcap_i N_i$ is non-trivial by Proposition 6.12. Otherwise N_i° is non-trivial and normal in $(G^*)^\circ$ for each i, and the intersection $\bigcap_i N_i$ is non-trivial by Theorem 1.6 (since the latter describes in particular the possible normal connected subgroups of G^*). In all cases, Zorn's lemma implies that the ordered set of non-trivial closed normal subgroups of G^* possesses minimal elements.

Given two minimal closed normal subgroups M, M', the intersection $M \cap M'$ is thus trivial and, hence, so is $[M, M']$. Thus minimal closed normal subgroups of G^* centralise one another. We deduce from Corollary 5.8 that the number of minimal closed normal subgroups is at most r.

Consider now an irreducible totally disconnected factor H of G^*. We claim that the collection of non-trivial closed normal subgroups of H forms a filtering family. Indeed, given two such normal subgroup N_1, N_2, then $N_1 \cap N_2$ is again a closed normal subgroup of H. It is is trivial, then the commutator $[N_1, N_2]$ is trivial and, hence, the centraliser of N_1 in H is non-trivial, contradicting Theorem 1.10. This confirms the claim. Thus the intersection of all non-trivial closed normal subgroups of H is non-trivial by Proposition 6.12. Clearly this intersection is the socle of H; it is clear we have just established that it is contained in every non-trivial closed normal subgroup of H. In particular $\mathrm{soc}(H)$ is characteristically simple. The desired result follows, since $\mathrm{soc}(H)$ is clearly a minimal closed normal subgroup of G^*. $\qquad\square$

Theorem 6.14. *Let X be a proper irreducible geodesically complete $\mathrm{CAT}(0)$ space. Let $G < \mathrm{Is}(X)$ be a closed totally disconnected subgroup acting cocompactly, in such a way that no open subgroup fixes a point at infinity. Then we have the following:*

(i) *Every compact subgroup of G is contained in a maximal one; the maximal compact subgroups fall into finitely many conjugacy classes.*

(ii) $\mathscr{QZ}(G) = 1$.

(iii) $\mathrm{soc}(G)$ *is a non-discrete characteristically simple group.*

Proof. (i) By Lemma 3.13, the action is minimal and hence Theorem 6.1 applies. In particular, we can apply Corollary 6.7 and consider the resulting equivariant decomposition. Let $Q < G$ be a compact subgroup and x be a Q-fixed point. If G_x is not contained in a maximal compact subgroup of G, then there is an infinite sequence $(x_n)_{n \geq 0}$ such that $x_0 = x$ and $G_{x_n} \subseteq G_{x_{n+1}}$. By Corollary 6.7, the sequence x_n leaves every bounded subset.

Since the fixed points $X^{G_{x_n}}$ form a nested sequence, it follows that X^{G_x} is unbounded. In particular its visual boundary $\partial(X^{G_x})$ is non-empty and the open subgroup G_x has a fixed point at infinity. This contradicts the hypotheses, and the claim is proved. Notice that a similar argument shows that for each $x \in X$, there are finitely many maximal compact subgroups $Q_i < G$ containing G_x.

The fact that G possesses finitely many conjugacy classes of maximal compact subgroups now follows from the compactness of $G \backslash X$.

(ii) We claim that $\mathscr{QZ}(G)$ is **topologically locally finite**, which means that every finite subset of it is contained in a compact subgroup. The desired result follows since it is then amenable but G has trivial amenable radical by Theorem 1.10. Let $S \subseteq \mathscr{QZ}(G)$ be a finite subset. Then G possesses a compact open subgroup U centralising S. By hypothesis the fixed point set of U is compact. Since $\langle S \rangle$ stabilises X^U, it follows that $\overline{\langle S \rangle}$ is compact, whence the claim.

(iii) Follows from (ii) and Proposition 6.11. $\square$

7. Cocompact CAT(0) spaces

7.A. **Fixed points at infinity.** We begin with a simple observation. We recall that two points at infinity are **opposite** if they are the two endpoints of a geodesic line. We denote by ξ^{op} the set of points opposite to ξ. Recall from [Bal95, Theorem 4.11(i)] that, if X is proper, then two points $\xi, \eta \in \partial X$ at Tits distance $> \pi$ are necessarily opposite. (Recall that Tits distance is by definition the length metric associated to the Tits angle.) However, it is not true in general that two points at Tits distance π are opposite.

Proposition 7.1. *Let X be a proper* CAT(0) *space and $H < \mathrm{Is}(X)$ a closed subgroup acting cocompactly. If H fixes a point ξ at infinity, then $\xi^{\mathrm{op}} \neq \varnothing$ and H acts transitively on ξ^{op}.*

Proof. First we claim that there is a geodesic line $\sigma : \mathbf{R} \to X$ with $\sigma(\infty) = \xi$. Indeed, let $r : \mathbf{R}_+ \to X$ be a ray pointing to ξ and $\{g_n\}$ a sequence in H such that $g_n r(n)$ remains bounded. The Arzelà–Ascoli theorem implies that $g_n r(\mathbf{R}_+)$ subconverges to a geodesic line in X. Since ξ is fixed by all g_n, this line has an endpoint at ξ.

Let now $\sigma' : \mathbf{R} \to X$ be any other geodesic with $\sigma'(\infty) = \xi$ and choose a sequence $\{h_n\}_{n \in \mathbf{N}}$ in H such that $d(h_n \sigma(-n), \sigma'(-n))$ remains bounded. By convexity and since all h_n fix ξ, $d(h_n \sigma(t), \sigma'(t))$ is bounded in n for all t and thus subconverges (uniformly for t in bounded intervals). On the one hand, it implies that $\{h_n\}$ has an accumulation point h. On the other hand, it follows that $h\sigma(-\infty) = \sigma'(-\infty)$. $\square$

Recall that any complete CAT(0) space X admits a canonical splitting $X = X' \times V$ preserved by all isometries, where V is a (maximal) Hilbert space called the Euclidean factor of X, see [BH99, II.6.15(6)]. Furthermore, there is a canonical embedding $X' \subseteq X'' \times V'$, where V' is a Hilbert space generated by all directions in X' pointing to "flat points" at infinity, namely points for which the Busemann functions are affine on X'; moreover, every isometry of X' extends uniquely to an isometry of $X'' \times V'$ which preserves that splitting. This is a result of Adams–Ballmann [AB98a, Theorem 1.6], who call V' the **pseudo-Euclidean factor** (one could also propose "Euclidean pseudo-factor").

Corollary 7.2. *Let X be a proper* CAT(0) *space with a cocompact group of isometries. Then the pseudo-Euclidean factor of X is trivial.*

Proof. In view of the above discussion, X' is also a proper CAT(0) space with a cocompact group of isometries. The set of flat points in $\partial X'$ admits a canonical (intrinsic) circumcentre

ξ by Lemma 1.7 in [AB98a]. In particular, ξ is fixed by all isometries and therefore, by Proposition 7.1, it has an opposite point, which is impossible for a flat point unless it lies already in the Euclidean factor (see [AB98a]). $\square$

Proposition 7.3. *Let G be a group acting cocompactly by isometries on a proper $\mathrm{CAT}(0)$ space X without Euclidean factor and assume that the stabiliser of every point at infinity acts minimally on X. Then G has no fixed point at infinity.*

Proof. If G has a global fixed point ξ, then the stabiliser G_η of an opposite point $\eta \in \xi^{\mathrm{op}}$ (which exists by Proposition 7.1) preserves the union $Y \subseteq X$ of all geodesic lines connecting ξ to η. By [BH99, II.2.14], this space is convex and splits as $Y = \mathbf{R} \times Y_0$. Since G_η acts minimally, we deduce $Y = X$ which provides a Euclidean factor. $\square$

7.B. **Actions of simple algebraic groups.** Let X be a $\mathrm{CAT}(0)$ space and $\mathbf{G}$ be an algebraic group defined over the field k. An isometric action of $\mathbf{G}(k)$ on X is called **algebraic** if every (algebraically) semi-simple element $g \in \mathbf{G}(k)$ acts as a semi-simple isometry.

When $\mathbf{G}$ is semi-simple, we denote by X_{model} the Riemannian symmetric space or Bruhat–Tits building associated with $\mathbf{G}(k)$.

Theorem 7.4. *Let k be a local field and $\mathbf{G}$ be an absolutely almost simple simply connected k-group. Let X be a non-compact proper $\mathrm{CAT}(0)$ space on which $G = \mathbf{G}(k)$ acts continuously by isometries.*

Assume either: (a) the action is cocompact; or: (b) it has full limit set, is minimal and ∂X is finite-dimensional. Then:

(i) *The G-action is algebraic.*

(ii) *There is a G-equivariant bijection $\partial X \cong \partial X_{\mathrm{model}}$ which is an isometry with respect to Tits' metric and a homeomorphism with respect to the cône topology. This bijection extends to a G-equivariant rough isometry $\beta : X_{\mathrm{model}} \to X$.*

(iii) *If X is geodesically complete, then X is isometric to X_{model}.*

(iv) *For any semi-simple k-subgroup $\mathbf{L} < \mathbf{G}$, there non-empty closed convex subspace $Y \subseteq X$ minimal for $L = \mathbf{L}(k)$; moreover, there is no L-fixed point in ∂Y.*

In the above point (ii), a **rough isometry** refers to a map $\beta : X_{\mathrm{model}} \to X$ such that there is a constant C with

$$d_{X_{\mathrm{model}}}(x,y) - C \ \leq \ d_X(\beta(x), \beta(y)) \ \leq \ d_{X_{\mathrm{model}}}(x,y) + C$$

for all $x, y \in X_{\mathrm{model}}$ and such that $\beta(X_{\mathrm{model}})$ has finite codiameter in X. Such a map is also called a $(1, C)$-quasi-isometry.

Remarks 7.5.

(i) Notice that there is no assumption on the k-rank of $\mathbf{G}$ in this result.

(ii) We recall for (b) that minimality follows from full limit set in the geodesically complete case (Lemma 3.13).

(iii) In the context of (ii), we recall that in general two $\mathrm{CAT}(0)$ spaces with the same cocompact isometry group need not have homeomorphic boundaries [CK00].

(iv) *A posteriori*, point (ii) shows in particular that the action is also cocompact under the assumption (b).

Before proceeding to the proof, we give two examples showing that the assumptions made in Theorem 7.4 are necessary.

Example 7.6. Without the assumption of geodesic completeness, it is not true in general that, in the setting of the theorem, the space X contains a closed convex G-invariant subspace which is isometric to X_{model}. A simple example of this situation may obtained as follows. Consider the case where k is non-Archimedean and G has k-rank one. Let $0 < r < 1/2$ and let X be the space obtained by replacing the r-ball centred at each vertex in the tree X_{model} by an isometric copy of a given Euclidean n-simplex, where $n + 1$ is valence of the vertex. In this way, one obtains a CAT(0) space which is still endowed with an isometric G-action that is cocompact and minimal, but clearly X is not isometric to X_{model}.

We do not know whether such a construction may also be performed in the Archimedean case (see Problem 7.2 in [CM08b]).

Example 7.7. Under the assumptions (b), minimality is needed. Indeed, we claim that for any CAT(0) space X_0 there is a canonical CAT(-1) space X (in particular X is a CAT(0) space) together with a canonical map $i : \mathrm{Is}(X_0) \hookrightarrow \mathrm{Is}(X)$ with the following properties: The boundary ∂X is reduced to a single point; X non-compact; X is proper if and only if X_0 is so; the map i is an isomorphism of topological groups onto its image. This claim justifies that minimality is needed since we can apply it to the case where X_0 is the symmetric space or Bruhat–Tits building associated to $\mathbf{G}(k)$. (In that case the action has indeed full limit set, a cheap feat as the isometry group is non-compact and the boundary rather incapacious.)

To prove the claim, consider the **parabolic cône** Y associated to X_0. This is the metric space with underlying set $X_0 \times \mathbf{R}_+^*$ where the distance is defined as follows: given two points (x, t) and (x', t') of Y, identify the interval $[x, x'] \subseteq X_0$ with an interval of corresponding length in $\mathbf{R}$ and measure the length from the resulting points (x, t) and (x', t') in the upper half-plane model for the hyperbolic plane. This is a particular case of the synthetic version ([Che99], [AB98b]) of the Bishop–O'Neill "warped products" [BO69] and its properties are described in [BGP92], [AB04, 1.2(2A)] and [HLS00, §2]. In particular, Y is CAT(-1).

We now let $\xi \in \partial Y$ be the point at infinity corresponding to $t \to \infty$ and define $X \subseteq Y$ to be an associated horoball; for definiteness, set $X = X_0 \times [1, \infty)$. We now have $\partial X = \{\xi\}$ by the CAT(-1) property or alternatively by the explicit description of geodesic rays (*e.g.* 2(iv) in [HLS00]). The remaining properties follow readily.

Proof of Theorem 7.4. We start with a few preliminary observations. Finite-dimensionality of the boundary always holds since it is automatic in the cocompact case. Since X is non-compact, the action is non-trivial, because it has full limit set. It is well known that every non-trivial continuous homomorphism of G to a locally compact second countable group is proper [BM96, Lemma 5.3]. Thus the G-action on X is proper.

We claim that the stabiliser of any point $\xi \in \partial X$ contains the unipotent radical of some proper parabolic subgroup of G. Indeed, fix a polar decomposition $G = KTK$. Let $x_0 \in X$ be a K-fixed point. Choose a sequence $\{g_n\}_{n \geq 0}$ of elements of G such that $g_n.x_0$ converges to ξ. Write $g_n = k_n.a_n.k_n'$ with $k_n, k_n' \in K$ and $a_n \in T$. We may furthermore assume, upon replacing $\{g_n\}$ by a subsequence, that $\{k_n\}$ converges to some $k \in K$, that $\{a_n.x_0\}$ converges in $X \cup \partial X$ and that $\{a_n.p\}$ converges in $X_{\mathrm{model}} \cup \partial X_{\mathrm{model}}$, where $p \in X_{\mathrm{model}}$ is some base point. Let $\eta = \lim_{n \to \infty} a_n.x_0$ and observe $\eta = k^{-1}\xi$. Furthermore, the stabiliser of η contains the group

$$U = \{g \in G \mid \lim_{n \to \infty} a_n^{-1}ga_n = 1\}.$$

The convergence in direction of $\{a_n\}$ in T implies that U contains the unipotent radical U_Q of the parabolic subgroup $Q < G$ corresponding to $\lim_{n \to \infty} a_n.p \in \partial X_{\mathrm{model}}$. (In fact, the arguments for Lemma 2.4 in [Pra77] probably show $U = U_Q$; this follows *a posteriori*

from (ii) below.) Therefore, the stabiliser of $\xi = k.\eta$ in G contains the unipotent radical of kQk^{-1}, proving the claim.

Notice that we have seen in passing that any point at infinity lies in the limit set of some torus; in the above notation, ξ is in the limit set of kTk^{-1}.

(i) Every element of G which is algebraically elliptic acts with a fixed point in X, since it generates a relatively compact subgroup. We need to show that every non-trivial element of a maximal split torus $T < G$ acts as a semi-simple isometry. Assume for a contradiction that some element $t \in T$ acts as a parabolic isometry. Since X has finite-dimensional boundary and we can apply Corollary 3.3(ii). It follows that the Abelian group T has a canonical fixed point at infinity ξ fixed by the normaliser $\mathscr{N}_G(T)$. By the preceding paragraph, we know furthermore that the stabiliser of ξ in G also contains the unipotent radical of some parabolic subgroup of G. Recall that G is generated by $\mathscr{N}_G(T)$ together with any such unipotent radical: this follows from the fact that $\mathscr{N}_G(T)$ has no fixed point at infinity in X_{model} and that G is generated by the unipotent radicals of any two distinct parabolic subgroups. Therefore ξ is fixed by the entire group G. Since G has trivial Abelianisation, its image under the Busemann character centred at ξ vanishes, thereby showing that G must stabilise every horoball centred at ξ. This is absurd both in the minimal and the cocompact case.

(ii) Let $T < G$ be a maximal split torus. Let $F_{\mathrm{model}} \subseteq X_{\mathrm{model}}$ be the (maximal) flat stabilised by T. In view of (i) and the properness of the T-action, we know that T also stabilises a flat $F \subseteq X$ with $\dim F = \dim T$, see [BH99, II.7.1]. Choose a base point $p_0 \in F_{\mathrm{model}}$ in such a way that its stabiliser $K := G_{p_0}$ is a maximal compact subgroup of G. The union of all T-invariant flats which are parallel to F is $\mathscr{N}_G(T)$-invariant. Therefore, upon replacing F by a parallel flat, we may – and shall – assume that F contains a point x_0 which is stabilised by $N_K := \mathscr{N}_G(T) \cap K$. Note that, since $\mathscr{N}_G(T) = \langle N_K \cup T \rangle$, the flat F is $\mathscr{N}_G(T)$-invariant. Therefore, there is a well defined $\mathscr{N}_G(T)$-equivariant map α of the $\mathscr{N}_G(T)$-orbit of p_0 to F, defined by $\alpha(g.p_0) = g.x_0$ for all $g \in \mathscr{N}_G(T)$.

We claim that, up to a scaling factor, the map α is isometric and induces an $\mathscr{N}_G(T)$-equivariant isometry $\alpha : F_{\mathrm{model}} \to F$. In order to establish this, remark that the Weyl group $W := \mathscr{N}_G(T)/\mathscr{Z}_G(T)$ acts on F, since $W = N_K/T_K$, where $T_K := \mathscr{Z}_G(T) \cap K$ acts trivially on F. The group N_K normalises the coroot lattice $\Lambda < T$. Furthermore $N_K.\Lambda$ acts on F_{model} as an affine Weyl group since $N_K.\Lambda/T_K \cong W \ltimes \Lambda$. Moreover, since any reflection in W centralises an Abelian subgroup of corank 1 in Λ, it follows that $N_K.\Lambda$ acts on F as a discrete reflection group. But a given affine Weyl group has a unique (up to scaling factor) discrete cocompact action as a reflection group on Euclidean spaces, as follows from [Bou68, Ch. VI, § 2, Proposition 8]. Therefore the restriction of α to $\Lambda.x_0$ is a homothety. Since Λ is a uniform lattice in T, the claim follows.

At this point, it follows that α induces an $\mathscr{N}_G(T)$-equivariant map $\partial\alpha : \partial F_{\mathrm{model}} \to \partial F$, which is isometric with respect to Tits' distance. We recall that $\mathscr{N}_G(T)$ is the stabiliser of $\partial F_{\mathrm{model}}$ in G. Morover, for any $\eta \in \partial F_{\mathrm{model}}$, the stabilisers in G of η is contained in that of $\alpha(\eta)$ because of the geometric description of parabolic subgroups alluded to in the preliminary observation: see the argument for Lemma 2.4 in [Pra77]. Therefore, $\partial\alpha$ extends to a well defined G-equivariant map $\partial X_{\mathrm{model}} \to \partial X$, which we denote again by $\partial\alpha$. Since any two points of $\partial X_{\mathrm{model}}$ are contained in a common maximal sphere (i.e. an apartment), and since G acts transitively on these spheres, the map $\partial\alpha$ is isometric, because so is its restriction to the sphere $\partial F_{\mathrm{model}}$. Note that $\partial\alpha$ is surjective: indeed, this follows

from the last preliminary observation, which, combined with (i), shows in particular that $\partial X = K.\partial F$.

We now show that $\partial \alpha$ is a homeomorphism with respect to the cône topology. Since $\partial X_{\mathrm{model}}$ is compact, it is enough to show that $\partial \alpha$ is continuous. Now any convergent sequence in $\partial X_{\mathrm{model}}$ may be written as $\{k_n.\xi_n\}_{n \geq 0}$, where $\{k_n\}_{n \geq 0}$ (resp. $\{\xi_n\}_{n \geq 0}$) is a convergent sequence of elements of K (resp. $\partial F_{\mathrm{model}}$). On the sphere $\partial F_{\mathrm{model}}$, the cône topology coincides with the one induced by Tits' metric. Therefore, the equivariance of the Tits' isometry $\partial \alpha$ shows that $\{\partial \alpha(k_n.\xi_n)\}_{n \geq 0}$ is a convergent sequence in ∂X, as was to be proved.

We next claim that that G-action on X is cocompact even under the assumption (b). Towards a contradiction, assume otherwise. Choose a sequence $\{y_n\}$ in X with y_0 a K-fixed point and such that $d(y_n, g.y_0) \geq n$ for all $g \in G$. Upon replacing y_n ($n \geq 1$) by an appropriate G-translate, we can and shall assume that moreover

(7.i) $$d(y_n, y_0) \leq d(y_n, g.y_0) + c \qquad \forall g \in G, n \geq 1,$$

where c is some constant. Upon extracting a subsequence, the sequence $\{y_n\}$ converges to some point $\eta \in \partial X$. It was established above that $\partial X = K.\partial F$; in particular there exists $k \in K$ such that $k.\eta \in \partial F$. Now, upon replacing y_n by $k.y_n$, we obtain a sequence $\{y_n\}$ which still satisfies all above conditions but which converges to a boundary point η' of the flat F. Let $r : \mathbf{R}_+ \to F$ be a geodesic ray pointing towards η'. Since $\mathscr{N}_G(T)$ acts cocompactly on the flat F, it follows from (7.i) that for some constant c', we have

(7.ii) $$d(y_n, y_0) \leq d(y_n, r(t)) + c' \qquad \forall t \geq 0, n \geq 1.$$

Fix now $s > d(y_0, r(0)) + c'$. For n sufficiently large, let z_n be the point on $[r(0), y_n]$ at distance s of $r(0)$. We have

$$
\begin{aligned}
d(y_n, r(s)) &\leq d(y_n, z_n) + d(z_n, r(s)) \\
&= d(y_n, r(0)) - s + d(z_n, r(s)) \\
&< d(y_n, r(0)) - d(y_0, r(0)) - c' + d(z_n, r(s)) \\
&\leq d(y_n, y_0) - c' + d(z_n, r(s)).
\end{aligned}
$$

As n goes to infinity, this provides a contradiction to (7.ii) since z_n converges to $r(s)$; thus cocompactness is established.

It remains for (ii) to prove that $\partial \alpha$ extends to a G-equivariant rough isometry $\beta : X_{\mathrm{model}} \to X$. The orbital map $g \mapsto g.y_0$ associated to y_0 yields a map $\beta : G/K \to X$; when k is Archimedean, $X_{\mathrm{model}} = G/K$ whereas we extend β linearly to each chamber of the building X_{model} in the non-Archimedean case. It is a well-known consequence of cocompactness that the G-equivariant map $\beta : X_{\mathrm{model}} \to X$ is a quasi-isometry (see $e.g.$ the proof of the Milnor–Švarc lemma given in [BH99, I.8.19]). For our stronger statement, it suffices, in view of the KTK decomposition and of equivariance, to prove that there is a constant C' such that

$$d_{X_{\mathrm{model}}}(a.p_0, p_0) - C' \leq d_X(a.y_0, y_0) \leq d_{X_{\mathrm{model}}}(a.p_0, p_0) + C'$$

for all $a \in T$. This follows from the fact that β and α are at bounded distance from each other on F_{model} (indeed, at distance $d(y_0, x_0)$) and that β is isometric on F_{model}.

(iii) In the higher rank case, assertion (iii) follows from (ii) and the main result of [Lee00]. However, the full strength of $loc.\ cit.$ is really not needed here, since the main difficulty there is precisely the absence of any group action, which is part of the hypotheses in our

setting. For example, when the ground field k is the field of real numbers, the arguments may be dramatically shortened as follows; they are valid without any rank assumption.

Given any $\xi \in \partial X$, the unipotent radical of the parabolic subgroup G_ξ acts sharply transitively on the boundary points opposite to ξ. In view of this and of the properness of the G-action, the arguments of [Lee00, Proposition 4.27] show that geodesic lines in X do not branch; in other words X has uniquely extensible geodesics. From this, it follows that the group $N_K = \mathcal{N}_G(T) \cap K$ considered in the proof of (ii) has a unique fixed point in X, since otherwise it would fix pointwise a geodesic line, and hence, by (ii), opposite points in $\partial X_{\mathrm{model}}$. The fact that this is impossible is purely a statement on the classical symmetric space X_{model}; we give a proof for the reader's convenience:

Let F_{model} be the flat corresponding to T and $p_0 \in F_{\mathrm{model}}$ be the K-fixed point. If N_K fixed a point $\xi \in \partial X_{\mathrm{model}}$, then the ray $[p_0, \xi)$ would be pointwise fixed and, hence, the group N_K would fix a non-zero vector in the tangent space of X_{model} at p_0. A Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G yields an isomorphism between the isotropy representation of N_K on $T_{p_0} X_{\mathrm{model}}$ and the representation of the Weyl group W on $\mathfrak{p}$. An easy explicit computation shows that the latter representation has no non-zero fixed vector.

Since N_K has a unique fixed point, the latter is stabilised by the entire group K. Hence K fixes a point lying on a flat F stabilised by T. From the KTK-decomposition, it follows that the G-orbit of this fixed point is convex. Since the G-action on X is minimal by geodesic completeness (Lemma 3.13), we deduce that G is transitive on X. In particular X is covered by flats which are G-conjugate to F, and the existence of a G-equivariant homothety $X_{\mathrm{model}} \to X$ follows from the existence of a $\mathcal{N}_G(T)$-equivariant homothety $F_{\mathrm{model}} \to F$, which has been established above. It remains only to choose the right scale on X_{model} to make it an isometry.

In the non-Archimedean case, we consider only the rank one case, referring to [Lee00] for higher rank. Let K be a maximal compact subgroup of G and $x_0 \in X$ be a K-fixed point. By (ii), the group K acts transitively on ∂X. Since X is geodesically complete, it follows that the K-translates of any ray emanating from x_0 cover X entirely. On the other hand every point in X has an open stabiliser by Theorem 6.1, any point in X has a finite K-orbit. This implies that the space of directions at each point $p \in X$ is finite. In other words X is 1-dimensional. Since X is CAT(0) and locally compact, it follows that X is a locally finite metric tree. As we have just seen, the group K acts transitively on the geodesic segments of a given length emanating from x_0. One deduces that G is transitive on the edges of X. In particular all edges of X have the same length, which we can assume to be as in X_{model}, G has at most two orbits of vertices, and X is either regular or bi-regular. The valence of any vertex p equals

$$\min_{g \notin \mathcal{N}_G(G_p)} [G_p : G_p \cap gG_pg^{-1}],$$

and coincides therefore with the valence of X_{model}. It finally follows that X and X_{model} are isometric, as was to be proved.

(iv) Let $\mathbf{P}$ be a k-parabolic subgroup of $\mathbf{G}$ that is minimal amongst those containing $\mathbf{L}$. We may assume $\mathbf{P} \neq \mathbf{G}$ since otherwise L has no fixed point in ∂X and the conclusion holds in view of Proposition 4.1. It follows that $\mathbf{L}$ centralises a k-split torus $\mathbf{T}$ of positive dimension d. It follows from (i) that $T = \mathbf{T}(k)$ acts by hyperbolic isometries, and thus there is a T-invariant closed convex subset $Z \subseteq X$ of the form $Z = Z_1 \times \mathbf{R}^d$ such that the T-action is trivial on the Z_1 factor; this follows from Theorem II.6.8 in [BH99] and the properness of the action. Moreover, L preserves Z and its decomposition $Z = Z_1 \times \mathbf{R}^d$, acting by translations on the $\mathbf{R}^d$ factor (*loc. cit.*). Since $\mathbf{L}$ is semi-simple, this translation

action is trivial and thus L preserves any Z_1 fibre, say for instance $Z_0 := Z_1 \times \{0\} \subseteq Z$. For both the existence of a minimal set Y and the condition $(\partial Y)^L = \varnothing$, it suffices to show that L has no fixed point in ∂Z_0 (Proposition 4.1).

We claim that ∂Z_0 is Tits-isometric to the spherical building of the Lévi subgroup $\mathscr{L}_G(\mathbf{T})$. Indeed, we know from (ii) that ∂X is equivariantly isometric to $\partial X_{\mathrm{model}}$ and the building of $\mathscr{L}_G(\mathbf{T})$ is characterised as the points at distance $\pi/2$ from the boundary of the T-invariant flat in $\partial X_{\mathrm{model}}$.

On the other hand, $\mathbf{L}$ has maximal semi-simple rank in $\mathscr{L}_G(\mathbf{T})$ by the choice of $\mathbf{P}$ and therefore cannot be contained in a proper parabolic subgroup of $\mathscr{L}_G(\mathbf{T})$. This shows that L has no fixed point in ∂Z_0 and completes the proof. $\square$

7.C. **No branching geodesics.** Recall that in a geodesic metric space X, the **space of directions** Σ_x at a point x is the completion of the space $\widetilde{\Sigma}_x$ of geodesic germs equipped with the Alexandrov angle metric at x. If X has *uniquely* extensible geodesics, then $\widetilde{\Sigma}_x = \Sigma_x$.

The following is a result of V. Berestovskii [Ber02] (we read it in [Ber, §3]; it also follows from A. Lytchak's arguments in [Lyt05, §4]).

Theorem 7.8. *Let X be a proper* $\mathrm{CAT}(0)$ *space with uniquely extensible geodesics and* $x \in X$. *Then* $\widetilde{\Sigma}_x = \Sigma_x$ *is isometric to a Euclidean sphere.* $\square$

We use this result to establish the following.

Proposition 7.9. *Let X be a proper* $\mathrm{CAT}(0)$ *space with uniquely extensible geodesics. Then any totally disconnected closed subgroup* $D < \mathrm{Is}(X)$ *is discrete.*

Proof. There is some compact open subgroup $Q < D$, see [Bou71, III §4 No 6]. Let x be a Q-fixed point. The isometry group of Σ_x is a compact Lie group by Theorem 7.8 and thus the image of the profinite group Q in it is finite. Let thus $K < Q$ be the kernel of this representation, which is open. Denote by $S(x, r)$ the r-sphere around x. The Q-equivariant "visual" map $S(x, r) \to \Sigma_x$ is a bijection by unique extensibility. It follows that K is trivial. $\square$

We are now ready for:

End of proof of Theorem 1.1. Since the action is cocompact, it is minimal by Lemma 3.13. The fact that extensibility of geodesics is inherited by direct factors of the space follows from the characterisation of geodesics in products, see [BH99, I.5.3(3)]. Each factor X_i is thus a symmetric space in view of Theorem 7.4(iii). By virtue of Corollary 6.3(iii), the totally disconnected factors D_j act by semi-simple isometries.

Assume now that X has uniquely extensible geodesics. For the same reason as before, this property is inherited by each direct factor of the space. Thus each D_j is discrete by Proposition 7.9. $\square$

Theorem 7.10. *Let X be a proper irreducible* $\mathrm{CAT}(0)$ *space with uniquely extensible geodesics. If X admits a non-discrete group of isometries with full limit set but no global fixed point at infinity, then X is a symmetric space.*

The condition on fixed points at infinity is necessary in view of E. Heintze's examples [Hei74] of negatively curved homogeneous manifolds which are not symmetric spaces. In fact these spaces consist of certain simply connected soluble Lie groups endowed with a left-invariant negatively curved Riemannian metric.

Proposition 7.11. *Let X be a proper* $\mathrm{CAT}(0)$ *space with uniquely extensible geodesics. Then ∂X has finite dimension.*

Proof. Let $x \in X$ and recall that by Berestovskii's result quoted in Theorem 7.8 above, Σ_x is isometric to a Euclidean sphere. By definition of the Tits angle, the "visual" map $\partial X \to \Sigma_x$ associating to a geodesic ray its germ at x is Tits-continuous (in fact, 1-Lipschitz). It is furthermore injective (actually, bijective) by unique extensibility. Therefore, the topological dimension of any *compact* subset of ∂X is bounded by the dimension of the sphere Σ_x. The claim follows now from Kleiner's characterisation of the dimension of spaces with curvature bounded above in terms of the topological dimension of compact subsets (Theorem A in [Kle99]). $\qquad\square$

Proof of Theorem 7.10. By Lemma 3.13, the action of $G := \mathrm{Is}(X)$ is minimal. In view of Proposition 7.11, the boundary ∂X is finite-dimensional. Thus we can apply Theorem 1.6 and Addendum 1.8. Since X is irreducible and non-discrete, Proposition 7.9 implies that G is an almost connected simple Lie group (unless $X = \mathbf{R}$, in which case X is indeed a symmetric space). We conclude by Theorem 7.4. $\qquad\square$

7.D. **No open stabiliser at infinity.** The following statement sums up some of the preceding considerations:

Corollary 7.12. *Let X be a proper geodesically complete* $\mathrm{CAT}(0)$ *space without Euclidean factor such that some closed subgroup $G < \mathrm{Is}(X)$ acts cocompactly. Suppose that no open subgroup of G fixes a point at infinity. Then we have the following:*

  (i) *X admits a canonical equivariant splitting*

$$X \cong \ X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q$$

  where each X_i is a symmetric space and each Y_j possesses a G-equivariant locally finite decomposition into compact convex cells.
  (ii) *G possesses hyperbolic elements.*
  (iii) *Every compact subgroup of G is contained in a maximal one; the maximal compact subgroups fall into finitely many conjugacy classes.*
  (iv) *$\mathscr{QZ}(G) = 1$; in particular G has no non-trivial discrete normal subgroup.*
  (v) *$\mathrm{soc}(G^*)$ is a direct product of $p + q$ non-discrete characteristically simple groups.*

Proof. (i) Follows from Theorem 1.1 and Corollary 6.7.

(ii) Clear from Corollary 6.10.

(iii) and (iv) Immediate from (i) and Theorem 6.14(i) and (ii).

(v) Follows from (i), (iv) and Proposition 6.11. $\qquad\square$

7.E. **Cocompact stabilisers at infinity.** We undertake the proof of Theorem 1.3 which describes isometrically any geodesically complete proper $\mathrm{CAT}(0)$ space such that the stabiliser of every point at infinity acts cocompactly.

Remark 7.13. (i) The formulation of Theorem 1.3 allows for symmetric spaces of Euclidean type. (ii) A **Bass–Serre tree** is a tree admitting an edge-transitive automorphism group; in particular, it is regular or bi-regular (the regular case being a special case of Euclidean buildings).

Lemma 7.14. *Let X be a proper* $\mathrm{CAT}(0)$ *space such that the stabiliser of every point at infinity acts cocompactly on X. For any $\xi \in \partial X$, the set of $\eta \in \partial X$ with $\angle_{\mathrm{T}}(\xi, \eta) = \pi$ is contained in a single orbit under $\mathrm{Is}(X)$.*

Proof. Write $G = \mathrm{Is}(X)$. In view of Proposition 7.1 applied to G_ξ, it suffices to prove that the G-orbit of any such η contains a point opposite to ξ. By definition of the Tits angle, there is a sequence $\{x_n\}$ in X such that $\angle_{x_n}(\xi, \eta)$ tends to π. Since G_ξ acts cocompactly, it contains a sequence $\{g_n\}$ such that, upon extracting, $g_n x_n$ converges to some $x \in X$ and $g_n \eta$ to some $\eta' \in \partial X$. The angle semi-continuity arguments given in the proof of Proposition II.9.5(3) in [BH99] show that $\angle_x(\xi, \eta') = \pi$, recalling that all g_n fix ξ. This means that there is a geodesic $\sigma : \mathbf{R} \to X$ through x with $\sigma(-\infty) = \xi$ and $\sigma(\infty) = \eta'$. On the other hand, since G_η is cocompact in G, the G-orbit of η is closed in the cône topology. This means that there is $g \in G$ with $\eta' = g\eta$, as was to be shown. □

We shall need another form of angle rigidity (compare Proposition 6.8), this time for Tits angles.

Proposition 7.15. *Let X be a geodesically complete proper* $\mathrm{CAT}(0)$ *space, $G < \mathrm{Is}(X)$ a closed totally disconnected subgroup and $\xi \in \partial X$. If the stabiliser G_ξ acts cocompactly on X, then the G-orbit of ξ is discrete in the Tits topology.*

Proof. Suppose for a contradiction that there is a sequence $\{g_n\}$ such that $g_n \xi \neq \xi$ for all n but $\angle_{\mathrm{T}}(g_n \xi, \xi)$ tends to zero. Since G_ξ is cocompact, we can assume that g_n converges in G; since the Tits topology is finer than the cône topology for which the G-action is continuous, the limit of g_n must fix ξ and we can therefore assume $g_n \to 1$. Let $B \subseteq X$ be an open ball large enough so that $G_\xi.B = X$. Since by Lemma 3.13 we can apply Theorem 6.1, there is no loss of generality in assuming that each g_n fixes B pointwise.

Let $c : \mathbf{R}_+ \to X$ be a geodesic ray pointing towards ξ with $c(0) \in B$. For each n there is $r_n > 0$ such that c and $g_n c$ branch at the point $c(r_n)$. In particular, g_n fixes $c(r_n)$ but not $c(r_n + \varepsilon)$ no matter how small $\varepsilon > 0$. We now choose $h_n \in G_\xi$ such that $x_n := h_n c(r_n) \in B$ and notice that the sequence $k_n := h_n g_n h_n^{-1}$ is bounded since k_n fixes x_n. We can therefore assume upon extracting that it converges to some $k \in G$; in view of Theorem 6.1, we can further assume that all k_n coincide with k on B and in particular k fixes all x_n. Since $\angle_{\mathrm{T}}(k_n \xi, \xi) = \angle_{\mathrm{T}}(g_n \xi, \xi)$, we also have $k \in G_\xi$. Considering any given n, it follows now that k fixes the ray from x_n to ξ. Thus k_n fixes an initial segment of this ray at x_n. This is equivalent to g_n fixing an initial segment at $c(r_n)$ of the ray from $c(r_n)$ to ξ, contrary to our construction. □

Here is a first indication that our spaces might resemble symmetric spaces or Euclidean buildings:

Proposition 7.16. *Let X be a proper* $\mathrm{CAT}(0)$ *space such that that the stabiliser of every point at infinity acts cocompactly on X. Then any point at infinity is contained in an isometrically embedded standard n-sphere in ∂X, where $n = \dim \partial X$.*

Proof. Let $\eta \in \partial X$. There is some standard n-sphere S isometrically embedded in ∂X because X is cocompact (Theorem C in [Kle99]). By Lemma 3.1 in [BL05], there is $\xi \in S$ with $\angle_{\mathrm{T}}(\xi, \eta) = \pi$. Let $\vartheta \in S$ be the antipode in S of ξ. In view of Lemma 7.14, there is an isometry sending ϑ to η. The image of S contains η. □

We need one more fact for Theorem 1.3. The boundary of a $\mathrm{CAT}(0)$ space need not be complete, regardless of the geodesic completeness of the space itself; however, this is the case in our situation in view of Proposition 7.16:

Corollary 7.17. *Let X be a proper* $\mathrm{CAT}(0)$ *space such that that the stabiliser of every point at infinity acts cocompactly on X. Then ∂X is geodesically complete.*

Proof. Suppose for a contradiction that some Tits-geodesic ends at $\xi \in \partial X$ and let $B \subseteq \partial X$ be a small convex Tits-neighbourhood of ξ; in particular, B is contractible. Since by Proposition 7.16 there is an n-sphere through ξ for $n = \dim \partial X$, the relative homology $\mathrm{H}_n(B, B \setminus \{\xi\})$ is non-trivial. Our assumption implies that $B \setminus \{\xi\}$ is contractible by using the geodesic contraction to some point $\eta \in B \setminus \{\xi\}$ on the given geodesic ending at ξ. This implies $\mathrm{H}_n(B, B \setminus \{\xi\}) = 0$, a contradiction. (This argument is adapted from [BH99, II.5.12].) □

End of proof of Theorem 1.3. We shall use below that product decompositions preserve geodesic completeness (this follows *e.g.* from [BH99, I.5.3(3)]). We can reduce to the case where X has no Euclidean factor. By Lemma 3.13, the group $G = \mathrm{Is}(X)$ as well as all stabilisers of points at infinity act minimally. In particular, Proposition 7.3 ensures that G has no fixed point at infinity and we can apply Theorem 1.6 and Addendum 1.8. Therefore, we can from now on assume that X is irreducible. If the identity component G° is non-trivial, then Theorem 1.1 (see also Theorem 7.4(iii)) ensures that X is a symmetric space, and we are done. We assume henceforth that G is totally disconnected.

For any $\xi \in \partial X$, the collection $\mathrm{Ant}(\xi) = \{\eta : \angle_\mathrm{T}(\xi, \eta) = \pi\}$ of **antipodes** is contained in a G-orbit by Lemma 7.14 and hence is Tits-discrete by Proposition 7.15. This discreteness and the geodesic completeness of the boundary (Corollary 7.17) are the assumptions needed for Proposition 4.5 in [Lyt05], which states that ∂X is a building. Since X is irreducible, ∂X is not a (non-trivial) spherical join, see Theorem II.9.24 in [BH99]. Thus, if this building has non-zero dimension, we conclude from the main result of [Lee00] that X is a Euclidean building of higher rank.

If on the other hand ∂X is zero-dimensional, then we claim that it is homogeneous under G. Indeed, we know already that for any given $\xi \in \partial X$, the set $\mathrm{Ant}(\xi)$ lies in a single orbit. Since in the present case $\mathrm{Ant}(\xi)$ is simply $\partial X \setminus \{\xi\}$, the claim follows from the fact that G has no fixed point at infinity.

We have to show that X is an edge-transitive tree. To this end, consider any point $x \in X$. The isotropy group G_x is open by Theorem 6.1. In particular, since G acts transitively on ∂X and since G_ξ is cocompact, it follows that G_x has finitely many orbits in ∂X. Let $\rho_1, \ldots, \rho_k$ be geodesic rays emanating from x and pointing towards boundary points which provide a complete set of representatives for the G_x-orbits. For $r > 0$ sufficiently large, the various intersections of the rays $\rho_1, \ldots, \rho_k$ with the r-sphere $S_r(x)$ centred at x forms a set of k distinct points. This set is a fundamental domain for the G_x-action on $S_r(x)$. Since G_x has discrete orbits on $S_r(x)$, we deduce from Theorem 6.1 that the sphere $S_r(x)$ is finite. Since this holds for any $r > 0$ sufficiently large and any $x \in X$, it follows that every sphere in X is finite. This implies that X is 1-dimensional (see [Kle99]). In other words X is a metric tree. We denote by V the set of branch points which we shall call the vertices. It remains to show that G has at most two orbits on V.

Given $\xi' \in \partial X$, let $\beta_{\xi'} : G_{\xi'} \to \mathbf{R}$ denote the Busemann character centred at ξ' (see §2). Since X is a cocompact tree, it follows that $\beta_{\xi'}$ has discrete image. Let $g \in G_{\xi'}$ be an element such that $\beta_{\xi'}(g)$ is positive and minimal. Then g is hyperbolic and translates a geodesic line L. Let ξ'' denote the endpoint of L distinct from ξ.

Let $v \in L$ be any vertex. We denote by e' and e'' the edges of L containing v and pointing respectively to ξ' and ξ''. Give any edge e containing v with $e' \neq e \neq e''$, we prolong e to a geodesic ray ρ whose intersection with L is reduced to $\{v\}$. Since $G_{\xi'}$ is transitive on $\partial X \setminus \{\xi'\}$ there exists $g' \in G_{\xi'}$ such that $g'.\xi'' = \rho(\infty)$. Upon pre-composing g' with a

suitable power of g, we may assume that $\beta_{\xi'}(g') = 0$. In other words g' fixes v. This shows that $G_{\xi',v}$ is transitive on the edges containing v and different from e'.

The same argument with ξ' and ξ'' interchanged shows that $G_{\xi'',v}$ is transitive on the edges containing v and different from e''. In particular G_v is transitive on the edges containing v.

A straightforward induction on the distance to v now shows that for any vertex $w \in V$, the isotropy group G_w is transitive on the edges containing w. This implies that G is indeed edge-transitive. □

8. A FEW CASES OF CAT(0) SUPERRIGIDITY

This section demonstrates that certain forms of superrigidity can be obtained by combining the structure results of this paper with known superrigidity techniques. Much more general results will be established in the companion paper [CM08b].

8.A. CAT(0) superrigidity for some classical non-uniform lattices. Let Γ be a non-uniform lattice in a simple (real) Lie group G of rank at least 2. By [LMR00, Theorem 2.15], unipotent elements of Γ are exponentially distorted. This means that, with respect to any finitely generating set of Γ, the word length of $|u^n|$ is an $O(\log n)$ when u is a unipotent. More generally an element u is called *distorted* if $|u^n|$ is sublinear. If Γ is virtually boundedly generated by unipotent elements, one can therefore apply the following fixed point principle:

Lemma 8.1. *Let Γ be a group which is virtually boundedly generated by distorted elements. Then any isometric Γ-action on a complete $\mathrm{CAT}(0)$ space such that elements of zero translation length are elliptic has a global fixed point.*

Proof. For any Γ-action on a $\mathrm{CAT}(0)$ space, the translation length of a distorted element is zero. Thus every such element has a fixed point; the assumption on Γ now implies that all orbits are bounded, thus providing a fixed point [BH99, II.2.8(1)]. □

Bounded generation is a strong property, which conjecturally holds for all (non-uniform) lattices of a higher rank semi-simple Lie group. It is known to hold for arithmetic groups in split or quasi-split algebraic groups over a number field K of K-rank ≥ 2 by [Tav90], as well as in a few cases of isotropic but non-quasi-split groups [ER06].

As noticed in a conversation with Sh. Mozes, Lemma 8.1 yields the following elementary superrigidity statement.

Proposition 8.2. *Let $\Lambda = \mathrm{SL}_n(\mathbf{Z}[\frac{1}{p_1 \cdots p_k}])$ with $n \geq 3$ and p_i distinct primes and set $H = \mathrm{SL}_n(\mathbf{Q}_{p_1}) \times \cdots \times \mathrm{SL}_n(\mathbf{Q}_{p_k})$.*
Given any isometric Λ-action on any complete $\mathrm{CAT}(0)$ space such that every element of zero translation length is elliptic, there exists a Λ-invariant closed convex subspace on which the given action extends uniquely to a continuous H-action by isometries.

Proof. Let X be a complete $\mathrm{CAT}(0)$ space endowed with a Λ-action as in the statement. The subgroup $\Gamma = \mathrm{SL}_n(\mathbf{Z}) < \Lambda$ fixes a point by Lemma 8.1. The statement now follows because Γ is the intersection of Λ with the *open* subgroup $\mathrm{SL}_n(\mathbf{Z}_{p_1}) \times \cdots \times \mathrm{SL}_n(\mathbf{Z}_{p_k})$ of H; for later use, we isolate this elementary fact as Lemma 8.3 below. □

Lemma 8.3. *Let H be a topological group, $U < H$ an open subgroup, $\Lambda < H$ a dense subgroup and $\Gamma = \Lambda \cap U$. Any Λ-action by isometries on a complete $\mathrm{CAT}(0)$ space with a Γ-fixed point admits a Λ-invariant closed convex subspace on which the action extends continuously to H.*

Proof. Let X be the CAT(0) space and $x_0 \in X$ a Γ-fixed point. For any finite subset $F \subseteq \Lambda$, let $Y_F \subseteq X$ be the closed convex hull of Fx_0. The closed convex hull Y of Λx_0 is the closure of the union Y_∞ of the directed family $\{Y_F\}$. Therefore, since the action is isometric and Y is complete, it suffices to show that the Λ-action on Y_∞ is continuous for the topology induced on Λ by H. Equivalently, it suffices to prove that all orbital maps $\Lambda \to Y_\infty$ are continuous at $1 \in \Lambda$. This is the case even for the discrete topology on Y_∞ because the pointwise fixator of each Y_F is an intersection of finitely many conjugates of Γ, the latter being open by definition. □

The same arguments as below show that Theorem 1.14 holds for any lattice of a higher-rank semi-simple Lie group which is boundedly generated by distorted elements (and accordingly Theorem 1.15 generalises to suitable (S-)arithmetic groups).

Proof of Theorems 1.14 and 1.15. We start with the case $\Gamma = \mathrm{SL}_n(\mathbf{Z})$. By Theorem 1.1, we obtain a closed convex subspace X' which splits as a direct product

$$X' \cong X_1 \times \cdots \times X_p \times Y_0 \times Y_1 \times \cdots \times Y_q$$

in an $\mathrm{Is}(X')$-equivariant way, where $Y_0 \cong \mathbf{R}^n$ is the Euclidean factor. Each totally disconnected factor D_i of $\mathrm{Is}(X')^*$ acts by semi-simple isometries on the corresponding factor Y_i of X' by Corollary 6.3. Therefore, by Lemma 8.1 for each $i = 0, \dots, q$, the induced Γ-action on Y_i has a global fixed point, say y_i. In other words Γ stabilises the closed convex subset

$$Z := X_1 \times \cdots \times X_p \times \{y_0\} \times \cdots \times \{y_q\} \subseteq X.$$

Note that the isometry group of Z is an almost connected semi-simple real Lie group L. Combining Lemma VII.5.1 and Theorems VII.5.15 and VII.6.16 from [Mar91], it follows that the Zariski closure of the image of Γ in L is a commuting product $L_1.L_2$, where L_1 is compact, such that the corresponding homomorphism $\Gamma \to L_2$ extends to a continuous homomorphism $G \to L_2$. We define $Y \subseteq Z$ as the fixed point set of L_1. Now L_2, and hence Γ, stabilises Y. Therefore the continuous homomorphism $G \to L_2$ yields a G-action on Y which extends the given Γ-action, as desired.

Applying Theorem 7.4 point (iv) to the pair $L_2 < L$ acting on Z, we find in particular that L_2 has no fixed point at infinity in Y. Thus, upon replacing Y by a subspace, it is L_2-minimal. Now Theorem 2.4 in [CM08b] (which is completely independent of the present considerations) implies that the Γ- and G-actions on Y are minimal and without fixed point in ∂Y (although there might be fixed points in ∂X).

Turning to Theorem 1.15, the only change is that one replaces Lemma 8.1 by Proposition 8.2. □

8.B. CAT(0) superrigidity for irreducible lattices in products.

The aim of this section is to state a version of the superrigidity theorem [Mon06, Theorem 6] with CAT(0) targets. The original statement from *loc. cit.* concerns actions of lattices on *arbitrary* CAT(0) spaces, with reduced unbounded image. The following statement shows that, when the underlying CAT(0) space is nice enough, the assumption on the action can be considerably weakened.

We recall for the statement that any isometric action on a proper CAT(0) space without globel fixed point at infinity admits a *canonical* minimal non-empty closed convex invariant subspace, see Remarks 39 in [Mon06].

Theorem 8.4. *Let Γ be an irreducible uniform (or square-integrable weakly cocompact) lattice in a product $G = G_1 \times \cdots \times G_n$ of $n \geq 2$ locally compact σ-compact groups. Let X be a proper CAT(0) space with finite-dimensional boundary.*

Given any Γ-action on X without fixed point at infinity, if the canonical Γ-minimal subset $Y \subseteq X$ has no Euclidean factor, then the Γ-action on Y extends to a continuous G-action by isometries.

Remark 8.5. Although the above condition on the Euclidean factor in the Γ-minimal subspace Y might seem awkward, it cannot be avoided, as illustrated by Example 64 in [Mon06]. Notice however that if Γ has the property that any isometric action on a finite-dimensional Euclidean space has a global fixed (for example if Γ has Kazhdan's property (T)), then any minimal Γ-invariant subspace has no Euclidean factor.

Proof of Theorem 8.4. Let $Y \subseteq X$ be the canonical subspace recalled above. Then $\mathrm{Is}(Y)$ acts minimally on Y, without fixed point at infinity. In particular we may apply Theorem 1.6 and Addendum 1.8. In order to show that the Γ-action on Y extends to a continuous G-action, it is sufficient to show that the induced Γ-action on each irreducible factor of Y extends to a continuous G-action, factoring through some G_i. But the induced Γ-action on each irreducible factor of Y is reduced by Corollary 3.8. Thus the result follows from [Mon06, Theorem 6]. $\square$

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