Stabilization for SL_n in bounded cohomology

Nicolas Monod

ABSTRACT. We prove that for all local fields SL_n is stable over n in terms of continuous bounded cohomology. We complement this by various computations in low degree, showing notably $\mathrm{H}^3_{\mathrm{b}}(\mathrm{SL}_n(\mathbf{R})) = 0$ for all $n \in \mathbf{N}$. We link the corresponding vanishing for p-adic fields to a question on prime numbers.

1. Introduction

The continuous Eilenberg-MacLane cohomology of classical simple Lie groups is well known. By results going back to É. Cartan and W.T. van Est, it can notably be realized as relative Lie algebra cohomology or as the cohomology of the compact dual symmetric space [**4**]. For instance, the continuous cohomology $\mathrm{H}^\bullet_{\mathrm{c}}(\mathrm{SL}_n(\mathbf{C}))$ is an exterior algebra over the Borel classes. In this example, there is moreover a *stabilization*: the standard inclusions $\mathrm{SL}_n(\mathbf{C}) \hookrightarrow \mathrm{SL}_{n+1}(\mathbf{C})$ induce an isomorphism on $\mathrm{H}^q_{\mathrm{c}}$ for n large enough compared to q; in particular, one obtains a description of the cohomology of the direct limit $\mathrm{SL}_\infty(\mathbf{C})$ which is notably of relevance for Bott periodicity [**9**].

The continuous cohomology of semi-simple Lie groups is also essential for the understanding of the more complicated cohomology of their discrete subgroups [**4, 3**]. On the other hand, the theory of *bounded cohomology*, which plays an important role for discrete groups in a number of contexts [**14, 12, 6, 7, 16, 20, 21, 22**], is reputedly much less accessible to computation – and mostly unknown as yet.

However, we find that for the continuous bounded cohomology $\mathrm{H}^\bullet_{\mathrm{cb}}$ of Lie groups, more tools are available, at least in low degrees. Our first result is a stabilization statement:

THEOREM 1.1. *Let k be a local field and $0 \leq m \leq n$. Then the standard embeddings $\mathrm{GL}_m(k) \hookrightarrow \mathrm{GL}_n(k)$ induce isomorphisms*

$$\mathrm{H}^q_{\mathrm{cb}}(\mathrm{GL}_n(k)) \cong \mathrm{H}^q_{\mathrm{cb}}(\mathrm{GL}_m(k))$$

for any $0 \leq q \leq m$. Similarly, one has

$$\mathrm{H}^q_{\mathrm{cb}}(\mathrm{SL}_n(k)) \cong \mathrm{H}^q_{\mathrm{cb}}(\mathrm{GL}_m(k))$$

if $0 \leq q \leq m-1$.

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Theorem 1.1 makes it all the more desirable to gain an understanding of the low degree bounded cohomology for the low dimensional linear groups. Let us first review what is known. In degree one, $\mathrm{H}^{\bullet}_{\mathrm{cb}}$ always vanishes. In degree two, the situation is fully understood:

THEOREM (see [**6, 7**]). *Let G be a connected simple Lie group or more generally the group of k-points of a connected isotropic almost simple algebraic group over a local field k of any characteristic. Then $\mathrm{H}^2_{\mathrm{cb}}(G) \cong \mathrm{H}^2_{\mathrm{c}}(G)$. (In particular, this space is one dimensional if k is Archimedean and the associated symmetric space is of Hermitian type; it vanishes in all other cases).*

Beyond degree two, no complete results are known. For the case of $\mathrm{SL}_2(\mathbf{C})$, we have shown with M. Burger [**8**, Theorem 1.2] that a result of S. Bloch [**2**] implies

THEOREM (see [**8**]). *The space $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_2(\mathbf{C}))$ is naturally isomorphic to $\mathrm{H}^3_{\mathrm{c}}(\mathrm{SL}_2(\mathbf{C}))$ (and thus has dimension one).*

We have also proved the following [**8**, Theorem 1.5]

THEOREM (see [**8**]). $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_2(\mathbf{R})) = 0$.

Further, A. Goncharov's study of functional equations for the trilogarithm [**13**] shows that the Borel class generating $\mathrm{H}^5_{\mathrm{c}}(\mathrm{SL}_3(\mathbf{C}))$ can be represented by a bounded cocycle and hence $\mathrm{H}^5_{\mathrm{cb}} \to \mathrm{H}^5_{\mathrm{c}}$ is onto for this group. Goncharov gives also formulae expressing cocycles for higher Borel classes in terms of higher polylogarithms ; we could however not check whether theses cocycles are bounded (this would settle for $\mathrm{SL}_n(\mathbf{C})$ a conjecture of J.L. Dupont [**10**], see [**20**, 9.3.9]). The only other surjectivity results in higher rank that we are aware of are R.P. Savage's proof [**23**] that the volume form associated to $\mathrm{SL}_n(\mathbf{R})$ is in the image of $\mathrm{H}^d_{\mathrm{cb}}$ (for $d = n(n+1)/2 - 1$) and the fact that the Euler class for $\mathrm{GL}^+_n(\mathbf{R})$ is in the image of $\mathrm{H}^n_{\mathrm{cb}}$ as can be deduced *e.g.* from the Ivanov-Turaev cocycle [**17**].

However, none of this information can be immediately used in Theorem 1.1 as it stands because of the assumption $0 \leq q \leq m-1$ or m. We shall show:

THEOREM 1.2. $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_n(\mathbf{R})) = 0$ *and* $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_n(\mathbf{R})) = 0$ *for all* $n \in \mathbf{N}$.

It turns out that a similar investigation for the field $\mathbf{Q}$ (Proposition 4.1 below) leads to an intriguing question:

For a prime p, denote by v_p the valuation normalized by $v_p(p^n) = n$. If q is another prime, define $D_{p,q} : \mathbf{Q} \setminus \{0,1\} \to \mathbf{Z}$ by

$$D_{p,q}(x) = v_p(x)v_q(1-x) - v_q(x)v_p(1-x).$$

This function is obviously unbounded because of the Chinese remainder theorem; on the other hand, one can form arbitrary combinations of such $D_{p,q}$ by varying the primes p, q since the sum is finite at each x. We propose the following

QUESTION 1.3. Is the function $\sum_{p<q} \alpha_{p,q} D_{p,q}$ unbounded on $\mathbf{Q} \setminus \{0,1\}$ for every family of real numbers $\{\alpha_{p,q}\}$ (unless they are all zero)?

Using an adèle argument, the Bloch-Suslin complex and Theorem 1.1, we show:

THEOREM 1.4. *A positive answer to Question 1.3 implies* $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_n(\mathbf{Q}_p)) = 0$ *for all* $n \in \mathbf{N}$ *and all primes* p.

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2. Cohomology with Coefficients on Projective Spaces

Let k be a local field, *i.e.* a locally compact non-discrete field, and let $n \geq 1$. We write

$$G_n = \mathrm{GL}_n(k),\ G_n^+ = \mathrm{SL}_n(k),\ \mathbf{P}_{n-1} = \mathbf{P}^{n-1}(k) = (k^n \setminus \{0\})/k^*.$$

We endow $\mathbf{P}_{n-1}$ with the measure class determined by the Haar measures on k. The natural G_n-action preserves this class and therefore we obtain a coefficient module $L^\infty(\mathbf{P}_{n-1})$, and more generally for every $m \geq 0$ the diagonal action on the direct product turns $L^\infty(\mathbf{P}_{n-1}^{m+1})$ into a coefficient module (a *coefficient module* is a dual isometric Banach representation with separable continuous predual; for background see [**20**]). We agree that G_n, G_n^+ are trivial groups for $n \leq 0$.

An important ingredient for Theorem 1.1 is the following.

Proposition 2.1. *Let k be a local field and n, p, q non-negative integers. There are canonical isomorphisms*

(i) $\mathrm{H}^q_{\mathrm{cb}}\big(G_n, L^\infty(\mathbf{P}_{n-1}^{p+1})\big) \ \cong\ \mathrm{H}^q_{\mathrm{cb}}(G_{n-p-1})$ *for* $0 \leq p \leq n$.
(ii) $\mathrm{H}^q_{\mathrm{cb}}\big(G_n^+, L^\infty(\mathbf{P}_{n-1}^{p+1})\big) \ \cong\ \mathrm{H}^q_{\mathrm{cb}}(G_{n-p-1})$ *for* $0 \leq p \leq n-1$.
(iii) $\mathrm{H}^q_{\mathrm{cb}}\big(G_n, L^\infty(\mathbf{P}_{n-1}^{p+1})\big) \ =\ \mathrm{H}^q_{\mathrm{cb}}\big(G_n^+, L^\infty(\mathbf{P}_{n-1}^{p+1})\big) \ = 0$ *for* $p \geq n-2$ *and* $q \neq 0$.

Remark 2.2. The case $p \geq n-2, q = 0$ is not contained in the proposition because indeed the very definition of $\mathrm{H}^0_{\mathrm{cb}}$ yields $\mathrm{H}^0_{\mathrm{cb}}\big(G_n^+, L^\infty(\mathbf{P}_{n-1}^{p+1})\big) \cong L^\infty(\mathbf{P}_{n-1}^{p+1})^{G_n^+}$ and $\mathrm{H}^0_{\mathrm{cb}}\big(G_n, L^\infty(\mathbf{P}_{n-1}^{p+1})\big) \cong L^\infty(\mathbf{P}_{n-1}^{p+1})^{G_n}$ for all p. In particular, comparing with (i) and (ii), we recover the basic facts

$$L^\infty(\mathbf{P}_{n-1}^{p+1})^{G_n} \cong \mathbf{R} \qquad (\forall\, p \leq n), \tag{2.1}$$

$$L^\infty(\mathbf{P}_{n-1}^{p+1})^{G_n^+} \cong \mathbf{R} \qquad (\forall\, p \leq n-1). \tag{2.2}$$

Remark 2.3. An elementary consequence of the proposition is that for all $n, p \geq 0$ one has

$$\begin{aligned} \mathrm{H}^1_{\mathrm{cb}}\big(G_n, L^\infty(\mathbf{P}_{n-1}^{p+1})\big) &= 0,\\ \mathrm{H}^1_{\mathrm{cb}}\big(G_n^+, L^\infty(\mathbf{P}_{n-1}^{p+1})\big) &= 0. \end{aligned}$$

Indeed, for bounded cohomology with *trivial* coefficients one has always $\mathrm{H}^1_{\mathrm{cb}}(-) = 0$, so that for $p \leq n-1$ we are done by (i) and (ii); for other p, apply (iii).

We prepare now the proof of Proposition 2.1. For $v \in k^n \setminus \{0\}$ we denote by $\bar{v}$ the corresponding element of $\mathbf{P}_{n-1}$. We denote by $(e_1, \ldots, e_n)$ the canonical base of k^n and consider by abuse of notation the corresponding inclusion $k^n \subset k^{n+1}$. We let

$$Q_n = \mathrm{Stab}_{G_n}(\bar{e}_n), \qquad Q_n^+ = \mathrm{Stab}_{G_n^+}(\bar{e}_n).$$

If we write N_n and N_n^+ for the corresponding kernels of the Q_n- and Q_n^+-actions on k^{n-1}, we have semi-direct product decompositions $Q_n = N_n \rtimes G_{n-1}$ and $Q_n^+ = N_n^+ \rtimes G_{n-1}$.

LEMMA 2.4. *Let $\sigma : G_{n-1} \to Q_n$ be a section, E a coefficient Q_n-module and $\iota : E^{N_n} \to E$ the inclusion map. Then we have for all $q \in \mathbf{N}$ topological isomorphisms*

$$\mathrm{H}^q_{\mathrm{cb}}(G_{n-1}, E^{N_n}) \xleftarrow{\sigma^*} \mathrm{H}^q_{\mathrm{cb}}(Q_n, E^{N_n}) \xrightarrow{\iota_*} \mathrm{H}^q_{\mathrm{cb}}(Q_n, E).$$

Moreover, the restriction map σ^* does not depend on the choice of σ.

PROOF. Let $\pi : Q_n \to G_{n-1}$ be the quotient map. The amenability of N_n implies that ι_* as well as the *inflation* map

$$\pi^* : \ \mathrm{H}^q_{\mathrm{cb}}(G_{n-1}, E^{N_n}) \longrightarrow \mathrm{H}^q_{\mathrm{cb}}(Q_n, E^{N_n})$$

are topological isomorphisms, see Corollary 8.5.2 in [**20**]. The statements about σ^* now follow from $\sigma^*\pi^* = Id$. $\square$

Similarly, we have

LEMMA (2.4$^+$). *Let $\sigma : G_{n-1} \to Q_n^+$ be a section, E a coefficient Q_n^+-module and $\iota : E^{N_n^+} \to E$ the inclusion map. Then we have for all $q \in \mathbf{N}$ topological isomorphisms*

$$\mathrm{H}^q_{\mathrm{cb}}(G_{n-1}, E^{N_n^+}) \xleftarrow{\sigma^*} \mathrm{H}^q_{\mathrm{cb}}(Q_n^+, E^{N_n^+}) \xrightarrow{\iota_*} \mathrm{H}^q_{\mathrm{cb}}(Q_n^+, E).$$

Moreover, the restriction map σ^* does not depend on the choice of σ. $\square$

For $n \geq 2$, the canonical projection $k^n \to k^{n-1}$ which omits e_n induces a Q_n-equivariant map

$$\pi_n : \ \mathbf{P}_{n-1} \setminus \{\bar{e}_n\} \longrightarrow \mathbf{P}_{n-2}.$$

Since the N_n-action on $\mathbf{P}_{n-2}$ is trivial, π_n induces for every $p \geq 1$ a surjective Q_n-map

$$\overline{\pi_n} : \ \big(\mathbf{P}_{n-1} \setminus \{\bar{e}_n\}\big)^p \Big/ N_n \longrightarrow \mathbf{P}^p_{n-2}.$$

Since k is non-discrete, $\mathbf{P}_{n-1} \setminus \{\bar{e}_n\}$ has full measure in $\mathbf{P}_{n-1}$. Denoting by a double slash the standard quotients in the category of measure class spaces, we have

LEMMA 2.5. *For all $0 \leq p \leq n$, the map $\overline{\pi_n}$ yields a Q_n-equivariant isomorphism*

$$\mathbf{P}^p_{n-1} /\!\!/ N_n \longrightarrow \mathbf{P}^p_{n-2}.$$

of measure class spaces.

PROOF OF LEMMA 2.5. The case $p = 0$ is void, so let $p \geq 1$. It is enough to prove that the map $\overline{\pi_n}$ is injective outside a null set, which can be achieved by a dimension argument because k is non-discrete:

Take two families $u^1, \ldots u^p$ and $v^1, \ldots v^p$ of elements of $k^p \setminus ke_n$ and suppose $\pi_n(\overline{u^i}) = \pi_n(\overline{v^i})$ for all $1 \leq i \leq p$. Thus, there are $\lambda_1, \ldots, \lambda_p \in k^*$ such that, working in the canonical base (e_j), we have

$$u^i_j = \lambda_i v^i_j \qquad \forall\, 1 \leq i \leq p,\ 1 \leq j \leq n-1.$$

In this base, we have an identification $k^{n-1} \rtimes k^* \cong N_n$ which takes $\nu = (\nu_1, \dots, \nu_{n-1}; \nu_n)$ to $\begin{pmatrix} & & & 0 \\ & Id & & \vdots \\ & & & 0 \\ \nu_1 & \cdots & \nu_{n-1} & \nu_n \end{pmatrix}$. Thus the image of u^i under ν is

$$\begin{pmatrix} u^i_1 \\ \vdots \\ u^i_{n-1} \\ \langle \nu | u^i \rangle \end{pmatrix} = \begin{pmatrix} \lambda_i v^i_1 \\ \vdots \\ \lambda_i v^i_{n-1} \\ \langle \nu | u^i \rangle \end{pmatrix}$$

where $\langle \cdot | \cdot \rangle$ is the k-bilinear form associated to (e_j). In order to show that the p-tuples $(\overline{u^1}, \dots, \overline{u^p})$ and $(\overline{v^1}, \dots, \overline{v^p})$ are in the same N_n-orbit, one must solve simultaneously the equations $\langle \nu | u^i \rangle = \lambda_i v^i_n$ $(1 \le i \le p)$ in ν with $\nu_n \neq 0$. This system can be solved if the (u^i) are linearly independent in k^n; this condition fails only for a set of positive codimension in $\mathbf{P}^p_{n-1}$ since $p \le n$. The additional condition $\nu_n \neq 0$ is also generic in (u^i). □

We observe that with the above notations there is an identification $N_n^+ \cong k^{n-1}$ under which $\nu \in N_n^+$ is equivalent to $\nu_n = 1$. Repeating the preceding proof with this additional restriction decreasing by one the bound on p, we get:

Lemma 2.6. *For all $0 \le p \le n-1$, the map $\overline{\pi_n}$ yields a Q_n^+-equivariant isomorphism $\mathbf{P}^p_{n-1} /\!\!/ N_n^+ \to \mathbf{P}^p_{n-2}$ of measure class spaces.* □

Corollary 2.7. (i) *For all $0 \le p \le n$, the map $\overline{\pi_n}$ induces an identification $L^\infty(\mathbf{P}^p_{n-1})^{N_n} \cong L^\infty(\mathbf{P}^p_{n-2})$ of coefficient Q_n-modules (or G_{n-1}-modules).*

(ii) *For all $0 \le p \le n-1$, the map $\overline{\pi_n}$ induces an identification $L^\infty(\mathbf{P}^p_{n-1})^{N_n^+} \cong L^\infty(\mathbf{P}^p_{n-2})$ of coefficient Q_n^+-modules (or G_{n-1}-modules).* □

Proof of Proposition 2.1. We may assume $n \ge 2$ since otherwise all groups are Abelian and hence have vanishing bounded cohomology in all positive degrees.

(i) and (ii): The identifications

$$\mathbf{P}_{n-1} \cong G_n/Q_n, \quad \mathbf{P}_{n-1} \cong G_n^+/Q_n^+ \tag{2.3}$$

yield coefficient modules identifications

$$L^\infty(\mathbf{P}^{p+1}_{n-1}) \cong L^\infty_{\mathrm{w}*}\big(G_n/Q_n, L^\infty(\mathbf{P}^p_{n-1})\big),$$

where $L^\infty_{\mathrm{w}*}$ denotes the space of weak-* measurable bounded function classes; for background on this and the following, we refer to [**20**]. The right hand side above is isomorphic to the induced module $\mathrm{Ind}_{Q_n}^{G_n} L^\infty(\mathbf{P}^p_{n-1})$, see point (v) in Examples 10.1.2 in [**20**]. Therefore there is a natural *induction* isomorphism

$$\mathrm{H}^q_{\mathrm{cb}}\big(G_n, L^\infty(\mathbf{P}^{p+1}_{n-1})\big) \cong \mathrm{H}^q_{\mathrm{cb}}\big(Q_n, L^\infty(\mathbf{P}^p_{n-1})\big)$$

by Propositions 10.1.3 and 10.1.5 in [**20**]. Similarly,

$$\mathrm{H}^q_{\mathrm{cb}}\big(G_n^+, L^\infty(\mathbf{P}^{p+1}_{n-1})\big) \cong \mathrm{H}^q_{\mathrm{cb}}\big(Q_n^+, L^\infty(\mathbf{P}^p_{n-1})\big).$$

By Lemma 2.4 and Lemma 2.4^+, we have restriction isomorphisms

$$\mathrm{H}^q_{\mathrm{cb}}\big(Q_n, L^\infty(\mathbf{P}^p_{n-1})\big) \cong \mathrm{H}^q_{\mathrm{cb}}\big(G_{n-1}, L^\infty(\mathbf{P}^p_{n-1})^{N_n}\big),$$
$$\mathrm{H}^q_{\mathrm{cb}}\big(Q_n^+, L^\infty(\mathbf{P}^p_{n-1})\big) \cong \mathrm{H}^q_{\mathrm{cb}}\big(G_{n-1}, L^\infty(\mathbf{P}^p_{n-1})^{N_n^+}\big).$$

Since we have assumed $p \le n$ resp. $p \le n-1$, we may apply both cases of Corollary 2.7 and deduce

$$\begin{aligned} \mathrm{H}^q_{\mathrm{cb}}\big(G_n, L^\infty(\mathbf{P}^{p+1}_{n-1})\big) &\cong \mathrm{H}^q_{\mathrm{cb}}\big(G_{n-1}, L^\infty(\mathbf{P}^{p}_{n-2})\big), \\ \mathrm{H}^q_{\mathrm{cb}}\big(G^+_n, L^\infty(\mathbf{P}^{p+1}_{n-1})\big) &\cong \mathrm{H}^q_{\mathrm{cb}}\big(G_{n-1}, L^\infty(\mathbf{P}^{p}_{n-2})\big). \end{aligned}$$

We may now repeat the argument ℓ times on the common right hand side above, thus decreasing simultaneously n and p to get

$$\mathrm{H}^q_{\mathrm{cb}}\big(G_{n-1-\ell}, L^\infty(\mathbf{P}^{p-\ell}_{n-2-\ell})\big).$$

If $p \le n-2$ we stop at $\ell = p$ and obtain the term

$$\mathrm{H}^q_{\mathrm{cb}}\big(G_{n-p-1}, L^\infty(\mathbf{P}^{0}_{n-2-p})\big) = \mathrm{H}^q_{\mathrm{cb}}(G_{n-p-1})$$

If $p = n-1$ we stop at $\ell = p-1$ and obtain

$$\mathrm{H}^q_{\mathrm{cb}}\big(G_1, L^\infty(\mathbf{P}_0)\big) = \mathrm{H}^q_{\mathrm{cb}}(G_1) = \mathrm{H}^q_{\mathrm{cb}}(G_0)$$

(G_1 is amenable), whilst if $p = n$ we stop at $\ell = p-2$ and obtain likewise

$$\mathrm{H}^q_{\mathrm{cb}}\big(G_1, L^\infty(\mathbf{P}^2_0)\big) = \mathrm{H}^q_{\mathrm{cb}}(G_1) = \mathrm{H}^q_{\mathrm{cb}}(G_{-1}).$$

In all three cases we have the right hand side of (i) and (ii).

(iii): We claim that for $p \ge n-2$ the diagonal G_n-action on $\mathbf{P}^{p+1}_{n-1}$ is amenable (in the sense of R. Zimmer [**25**]). Indeed (see [**1**]), the amenability of the action is equivalent to the conjunction of

- the equivalence relation associated to the action is amenable,
- the stabilizer of almost every point is an amenable group.

The first condition is satisfied because the orbits are locally closed. As for the second, the identification (2.3) shows that the stabilizer of a point in $\mathbf{P}^{p+1}_{n-1}$ is the intersection of $p+1$ conjugates of Q_n. Since $p+1 \ge n-1$, a generic such intersection is contained in a minimal parabolic subgroup, hence is amenable. This proves the claim.

The amenability of the action is equivalent to the relative injectivity of the coefficient G_n-module $L^\infty(\mathbf{P}^{p+1}_{n-1})$, see Theorem 5.7.1 in [**20**]. Similarly, $L^\infty(\mathbf{P}^{p+1}_{n-1})$ is relatively injective as a coefficient G^+_n-module[1]. This implies the vanishing of the bounded cohomology in every positive degree (Proposition 7.4.1 in [**20**]). □

3. A Double Complex

Keep the notation of the previous section. We define a fist quadrant double complex $(L^{\bullet,\bullet}, {}^{I}d, {}^{II}d)$ by

$$L^{p,q} = L^\infty(G^{p+1}_n \times \mathbf{P}^{q+1}_{n-1})^{G_n} \qquad (p, q \ge 0)$$

with ${}^{I}d : L^{p,q} \to L^{p+1,q}$ defined by the homogeneous coboundary d^p associated to G^{p+1}_n; we recall for later use the definition $d^p = \sum_{j=0}^{p} (-1)^j d^p_j$, where d^p_j omits the jth variable. As for ${}^{II}d : L^{q,p} \to L^{q,p+1}$ it is defined by the homogeneous coboundary on $\mathbf{P}^{p+1}_{n-1}$ affected with the sign $(-1)^{(p+1)}$. To such a complex are associated two spectral sequences ${}^{I}\mathrm{E}$, ${}^{II}\mathrm{E}$; for spectral sequences, we follow standard notations, see *e.g.* [**11**, III.7], [**5**, III §14] or the Section 12.1 in [**20**]. The following lemma is straightforward except for a technical point:

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[1]One can also argue that for coefficient modules, relative injectivity is preserved by passing to closed subgroups of second countable locally compact groups, see Proposition 5.8.1 in [**20**].

LEMMA 3.1. *The first spectral sequence converges to* $\mathrm{H}^\bullet_{\mathrm{cb}}(G_n)$. *More precisely,* ${}^{I}\mathrm{E}_r^{p,q} = 0$ $(\forall\, p \geq 0, q \geq 1, r \geq 1)$ *and* ${}^{I}\mathrm{E}_r^{p,0} \cong \mathrm{H}^p_{\mathrm{cb}}(G_n)$ $(\forall\, p \geq 0, r \geq 2)$.

PROOF. Since integration over the first variable provides a homotopy (cf. proof of [**7**, 1.5.6] or [**20**, 7.5.5]), the cohomology of the complex

$$0 \longrightarrow L^\infty(\mathbf{P}_{n-1}) \longrightarrow L^\infty(\mathbf{P}^2_{n-1}) \longrightarrow L^\infty(\mathbf{P}^3_{n-1}) \longrightarrow \cdots$$

is concentrated in degree zero, where it is $\mathbf{R}$. We have shown in Lemma 8.2.5 of [**20**] that $L^\infty_{\mathrm{w}*}(G_n^{p+1}, -)^{G_n}$ is exact for all $p \geq 0$ with respect to *adjoint* short exact sequences of coefficient G_n-modules. Therefore, since ${}^{I}\mathrm{E}_1^{p,q}$ is defined by

$$\begin{aligned} L^\infty_{\mathrm{w}*}(G_n^{p+1}, L^\infty(\mathbf{P}^q_{n-1}))^{G_n} &\longrightarrow L^\infty_{\mathrm{w}*}(G_n^{p+1}, L^\infty(\mathbf{P}^{q+1}_{n-1}))^{G_n} \longrightarrow \\ &\longrightarrow L^\infty_{\mathrm{w}*}(G_n^{p+1}, L^\infty(\mathbf{P}^{q+2}_{n-1}))^{G_n} \end{aligned}$$

for $q > 0$ and

$$0 \longrightarrow L^\infty_{\mathrm{w}*}(G_n^{p+1}, L^\infty(\mathbf{P}_{n-1}))^{G_n} \longrightarrow L^\infty_{\mathrm{w}*}(G_n^{p+1}, L^\infty(\mathbf{P}^2_{n-1}))^{G_n}$$

when $q = 0$, we deduce canonical identifications ${}^{I}\mathrm{E}_1^{p,0} \cong L^\infty(G_n^{p+1})^{G_n}$, ${}^{I}\mathrm{E}_1^{p,q} = 0$ for all $p \geq 0, q \geq 1$. Thus ${}^{I}\mathrm{E}_2^{p,0}$ is defined by the L^∞ homogeneous resolution for $\mathrm{H}^p_{\mathrm{cb}}(G_n)$ and the statement follows. $\square$

LEMMA 3.2. *There is a canonical identification* ${}^{II}\mathrm{E}_1^{p,q} \cong \mathrm{H}^q_{\mathrm{cb}}\big(G_n, L^\infty(\mathbf{P}^{p+1}_{n-1})\big)$ *for all* $p, q \geq 0$.

PROOF. In view of the identification $L^{q,p} \cong L^\infty_{\mathrm{w}*}\big(G_n^{q+1}, L^\infty(\mathbf{P}^{p+1}_{n-1})\big)^{G_n}$, the spaces ${}^{II}\mathrm{E}_1^{p,q}$ are indeed defined by the L^∞ homogeneous resolution for the continuous bounded cohomology of G_n with coefficients in $L^\infty(\mathbf{P}^{p+1}_{n-1})$. $\square$

Recall that we have given in Proposition 2.1 an interpretation of the right hand side in Lemma 3.2. However, in order to compute the second tableau ${}^{II}\mathrm{E}_2$ we need to understand the differentials ${}^{II}\mathrm{E}_1^{p,q} \to {}^{II}\mathrm{E}_1^{p+1,q}$, induced (up to the sign) by the homogeneous coboundary $d^{p+1} : L^\infty(\mathbf{P}^{p+1}_{n-1}) \to L^\infty(\mathbf{P}^{p+2}_{n-1})$.

LEMMA 3.3. *Let* n, p, q *be non-negative integers with* $p \leq n-2$. *If* p *is even, then the differential* ${}^{II}\mathrm{E}_1^{p,q} \to {}^{II}\mathrm{E}_1^{p+1,q}$ *vanishes. If* p *is odd, then the isomorphisms of* (*the proof of*) *Proposition 2.1* (i) *conjugate the differential to the* restriction *map*

$$\mathrm{res}:\ \mathrm{H}^q_{\mathrm{cb}}(G_{n-p-1}) \longrightarrow \mathrm{H}^q_{\mathrm{cb}}(G_{n-p-2})$$

induced by the upper left inclusion $G_{n-p-2} \to G_{n-p-1}$.

PROOF. We consider for $0 \leq j \leq p+1$ the map $A^{p+1}_{n,j}$ defined by the commutative diagram

$$\begin{tikzcd} \mathrm{H}^q_{\mathrm{cb}}(G_n, L^\infty(\mathbf{P}^{p+1}_{n-1})) \arrow[r, "(d^{p+1}_j)_*"] & \mathrm{H}^q_{\mathrm{cb}}(G_n, L^\infty(\mathbf{P}^{p+2}_{n-1})) \\ \mathrm{H}^q_{\mathrm{cb}}(Q_n, L^\infty(\mathbf{P}^{p}_{n-1})) \arrow[u, "\cong"', "\mathrm{ind}"] \arrow[r, "A^{p+1}_{n,j}"] & \mathrm{H}^q_{\mathrm{cb}}(Q_n, L^\infty(\mathbf{P}^{p+1}_{n-1})). \arrow[u, "\cong"', "\mathrm{ind}"] \end{tikzcd}$$

We recall from [**20**, p. 134] that the left hand side induction can be defined *e.g.* as follows. Represent $[\alpha] \in \mathrm{H}^q_{\mathrm{cb}}(Q_n, L^\infty(\mathbf{P}^p_{n-1}))$ by a bounded measurable Q_n-equivariant cocycle $\alpha : G_n^{q+1} \to L^\infty(\mathbf{P}^p_{n-1})$ and define

$$\mathbf{i}\alpha :\ G_n^{q+1} \longrightarrow L^\infty(\mathbf{P}^{p+1}_{n-1}) \cong L^\infty_{\mathrm{w}*}(G_n/Q_n, L^\infty(\mathbf{P}^p_{n-1}))$$

by setting for $g_i, h \in G_n, x \in \mathbf{P}^p_{n-1}$

$$\mathbf{i}\alpha(g_0, \cdots, g_q)(hQ_n)(x) = \alpha(h^{-1}g_0, \cdots, h^{-1}g_q)(h^{-1}x). \tag{3.1}$$

A similar formula defines the right hand side induction. A computation with (3.1) shows that for $j \neq 0$ the map $A^{p+1}_{n,j}$ is but $(d^p_{j-1})_*$ induced by $L^\infty(\mathbf{P}^p_{n-1}) \to L^\infty(\mathbf{P}^{p+1}_{n-1})$. It follows that, for the isomorphisms of the proof of Proposition 2.1, we have a commutative diagram for $j \neq 0$

$$\begin{tikzcd}
\mathrm{H}^q_{\mathrm{cb}}(G_n, L^\infty(\mathbf{P}^{p+1}_{n-1})) \arrow[r, "(d^{p+1}_j)_*"] \arrow[d, "\cong"'] & \mathrm{H}^q_{\mathrm{cb}}(G_n, L^\infty(\mathbf{P}^{p+2}_{n-1})) \arrow[d, "\cong"'] \\
\mathrm{H}^q_{\mathrm{cb}}(G_{n-1}, L^\infty(\mathbf{P}^{p}_{n-2})) \arrow[r, "(d^{p}_{j-1})_*"] & \mathrm{H}^q_{\mathrm{cb}}(G_{n-1}, L^\infty(\mathbf{P}^{p+1}_{n-2})).
\end{tikzcd}$$

In the case $j = 0$, a computation with (3.1) shows that $A^{p+1}_{n,0}$ is the map $\mathrm{res} \circ \mathrm{ind}$ such that

$$\begin{tikzcd}
\mathrm{H}^q_{\mathrm{cb}}(G_n, L^\infty(\mathbf{P}^{p+1}_{n-1})) \arrow[r, "(d^{p+1}_0)_*"] \arrow[dr, "\mathrm{res}"] & \mathrm{H}^q_{\mathrm{cb}}(G_n, L^\infty(\mathbf{P}^{p+2}_{n-1})) \\
\mathrm{H}^q_{\mathrm{cb}}(Q_n, L^\infty(\mathbf{P}^{p}_{n-1})) \arrow[r, "A^{p+1}_{n,0}"] \arrow[u, "\cong"', "\mathrm{ind}"] & \mathrm{H}^q_{\mathrm{cb}}(Q_n, L^\infty(\mathbf{P}^{p+1}_{n-1})) \arrow[u, "\cong"', "\mathrm{ind}"]
\end{tikzcd}$$

commutes. One checks further that the relevant isomorphisms from the proof of Proposition 2.1 intertwine $A^{p+1}_{n,0}$ with $A^p_{n-1,0}$. Since d^{p+1} is an alternating sum of the d^{p+1}_j, the statement follows by induction on p. In fact the differential is induced by d^{p+1} up to $(-1)^{p+1}$ only, but this is irrelevant since we established that it vanishes for $p+1$ odd. $\square$

Proof of Theorem 1.1 for G_n. It is enough to show that the upper left embedding $\mathrm{GL}_{n-1}(k) \hookrightarrow \mathrm{GL}_n(k)$ induces isomorphisms

$$\mathrm{H}^q_{\mathrm{cb}}(\mathrm{GL}_n(k)) \cong \mathrm{H}^q_{\mathrm{cb}}(\mathrm{GL}_{n-1}(k))$$

for any $0 \leq q \leq n-1$. We do this by induction on n. The statement is trivial for $n \leq 2$. Take $n \geq 3$ and assume the statement true for all smaller values.

Lemma 3.2 and Proposition 2.1 (i) show that ${}^{II}\mathrm{E}^{p,q}_2$ is computed by

$$0 \longrightarrow \mathrm{H}^q_{\mathrm{cb}}(G_{n-1}) \longrightarrow \mathrm{H}^q_{\mathrm{cb}}(G_{n-2}) \longrightarrow \cdots \longrightarrow \mathrm{H}^q_{\mathrm{cb}}(G_0)$$

for $0 \leq p \leq n-2$ and $q \geq 1$. If we add the induction assumption to the conclusion of Lemma 3.3, we deduce that ${}^{II}\mathrm{E}^{0,q}_2 = \mathrm{H}^q_{\mathrm{cb}}(G_{n-1})$ and that ${}^{II}\mathrm{E}^{p,q}_2$ vanishes for $1 \leq p \leq n-2$ and $q \geq 1$. Lemma 3.2 and Proposition 2.1 (iii) imply now that ${}^{II}\mathrm{E}^{p,q}_2$ vanishes for all p, q provided $pq \neq 0$. In particular we get an exact sequence

$${}^{II}\mathrm{E}^{0,q-1}_2 \longrightarrow {}^{II}\mathrm{E}^{q,0}_2 \longrightarrow {}^{II}\mathrm{E}^{q}_\infty \longrightarrow {}^{II}\mathrm{E}^{0,q}_2 \longrightarrow {}^{II}\mathrm{E}^{q+1,0}_2. \tag{3.2}$$

On the other hand, ${}^{II}\mathrm{E}^{p,0}_2$ is computed by the complex

$$0 \longrightarrow L^\infty(\mathbf{P}_{n-1})^{G_n} \longrightarrow L^\infty(\mathbf{P}^2_{n-1})^{G_n} \longrightarrow L^\infty(\mathbf{P}^3_{n-1})^{G_n} \longrightarrow \cdots \tag{3.3}$$

By Remark 2.2 and in view of the definition of the coboundary, this complex has the form

$$0 \longrightarrow \mathbf{R} \xrightarrow{0} \mathbf{R} \xrightarrow{=} \mathbf{R} \xrightarrow{0} \mathbf{R} \xrightarrow{=} \cdots$$

for $0 \leq p \leq n$. Therefore, ${}^{II}\mathrm{E}^{p,0}_2$ vanishes for $1 \leq p \leq n$. Putting everything together in (3.2), we deduce ${}^{II}\mathrm{E}^{q}_\infty \cong \mathrm{H}^q_{\mathrm{cb}}(G_{n-1})$ for all $q \leq n-1$. This, together with Lemma 3.1, finishes the proof upon checking that the isomorphism is indeed induced by the inclusion. □

PROOF OF THEOREM 1.1 FOR G_n^+. We define a similar double complex by

$$L^{p,q} = L^\infty\big((G_n^+)^{p+1} \times \mathbf{P}_{n-1}^{q+1}\big)^{G_n^+}.$$

Then the proof is identical to the case of G_n except that one has to keep track of the restriction $p \le n-1$ of Proposition 2.1, Lemma 2.6 and Corollary 2.7 throughout the argument. This results in the restriction $q \le m-1$ in Theorem 1.1. □

As a by-product of the previous arguments, we obtain also:

PROPOSITION 3.4. *The restriction maps*

$$\mathrm{H}^n_{\mathrm{cb}}(G_n) \longrightarrow \mathrm{H}^n_{\mathrm{cb}}(G_{n-1}) \quad \textit{and} \quad \mathrm{H}^{n-1}_{\mathrm{cb}}(G_n^+) \longrightarrow \mathrm{H}^{n-1}_{\mathrm{cb}}(G_{n-1})$$

are injective for all n. If $k^*/(k^*)^n$ is trivial, then the restriction map

$$\mathrm{H}^n_{\mathrm{cb}}(G_n^+) \longrightarrow \mathrm{H}^n_{\mathrm{cb}}(G_{n-1})$$

is also injective.

PROOF. Our discussion showed that (3.2) yields an exact sequence

$${}^{II}\mathrm{E}_2^{n,0} \longrightarrow \mathrm{H}^n_{\mathrm{cb}}(G_n) \longrightarrow \mathrm{H}^n_{\mathrm{cb}}(G_{n-1}).$$

The first term is computed by (3.3) which vanishes as pointed out in Remark 2.2 – case $p = n$ of (2.1). Likewise, the spectral sequence for G_n^+ yields

$${}^{II}\mathrm{E}_2^{n-1,0} \longrightarrow \mathrm{H}^{n-1}_{\mathrm{cb}}(G_n^+) \longrightarrow \mathrm{H}^{n-1}_{\mathrm{cb}}(G_{n-1}^+),$$

wherein this time ${}^{II}\mathrm{E}_2^{\bullet,0}$ is realized by

$$0 \longrightarrow L^\infty(\mathbf{P}_{n-1})^{G_n^+} \longrightarrow L^\infty(\mathbf{P}_{n-1}^2)^{G_n^+} \longrightarrow L^\infty(\mathbf{P}_{n-1}^3)^{G_n^+} \longrightarrow \cdots$$

Here the first term vanishes by the case $p = n-1$ of (2.1).

Suppose now that $k^*/(k^*)^n$ is trivial. All we have to show is that ${}^{II}\mathrm{E}_2^{n,0}$ vanishes for the spectral sequence associated to G_n^+. Thus it suffices to show that $L^\infty(\mathbf{P}_{n-1}^{n+1})^{G_n^+}$ consist only of constant function classes. The G_n- and G_n^+-actions on $\mathbf{P}_{n-1}$ factor through $\mathrm{PGL}_n(k)$ and $\mathrm{PSL}_n(k)$ respectively. The condition on k implies that these groups coincide and thus we are done by (2.1). □

PROOF OF THEOREM 1.2. First, we observe that the restriction map

$$\mathrm{H}^\bullet_{\mathrm{cb}}(\mathrm{GL}_n(\mathbf{R})) \longrightarrow \mathrm{H}^\bullet_{\mathrm{cb}}(\mathrm{SL}_n(\mathbf{R}))$$

is injective for all n. Indeed, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet_{\mathrm{cb}}\big(\mathrm{PGL}_n(\mathbf{R})\big) \arrow[r] \arrow[d, "\mathrm{inf}"'] & \mathrm{H}^\bullet_{\mathrm{cb}}\big(\mathrm{PSL}_n(\mathbf{R})\big) \arrow[d, "\mathrm{inf}"] \\
\mathrm{H}^\bullet_{\mathrm{cb}}\big(\mathrm{GL}_n(\mathbf{R})\big) \arrow[r] & \mathrm{H}^\bullet_{\mathrm{cb}}\big(\mathrm{SL}_n(\mathbf{R})\big)
\end{tikzcd}$$

in which both lateral arrows are isomorphisms [**20**, 8.5.2] and the upper arrow in injective since $\mathrm{PSL}_n(\mathbf{R})$ is closed of finite index in $\mathrm{PGL}_n(\mathbf{R})$, see [**20**, 8.6.2].

As mentioned in the Introduction, the case of $\mathrm{SL}_2(\mathbf{R})$ is a joint result with M. Burger [**8**, 1.5]; therefore $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_2(\mathbf{R}))$ vanishes as well. Now Proposition 3.4

implies that both $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_3(\mathbf{R}))$ and $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_3(\mathbf{R}))$ vanish since $(\mathbf{R}^*)^3 = \mathbf{R}^*$. Theorem 1.1 implies the result for $\mathrm{GL}_n(\mathbf{R})$. The case of $\mathrm{SL}_4(\mathbf{R})$ is handled by the injectivity of the second restriction map in Proposition 3.4. The remaining cases $\mathrm{SL}_n(\mathbf{R})$ for $n \geq 5$ follow from Theorem 1.1. □

REMARK 3.5. With the above arguments, we see that $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_n(\mathbf{C}))$ is at most one dimensional for all n.

4. p-adic fields and $\mathrm{GL}_2(\mathbf{Q})$

The first step for Theorem 1.4 is the following "global" statement:

PROPOSITION 4.1. *A positive answer to Question 1.3 implies* $\mathrm{H}^3_{\mathrm{b}}(\mathrm{GL}_2(\mathbf{Q})) = 0$.

PROOF. Write $G = \mathrm{GL}_2(\mathbf{Q})$ and $\mathbf{P} = \mathbf{P}(\mathbf{Q})$. Since $\mathbf{P}$ is the quotient of G by an amenable subgroup, the complex

$$0 \longrightarrow \ell^\infty(\mathbf{P})^G \longrightarrow \ell^\infty(\mathbf{P}^2)^G \longrightarrow \ell^\infty(\mathbf{P}^3)^G \longrightarrow \cdots$$

computes $\mathrm{H}^\bullet_{\mathrm{b}}(G)$, see [**20**, 7.5.9]. Thus any class in $\mathrm{H}^3_{\mathrm{b}}(\mathrm{GL}_2(\mathbf{Q})) = 0$ can be represented by a bounded G-invariant cocycle $\omega : \mathbf{P}^4 \to \mathbf{R}$ (which we may choose alternating). The cross-ratio maps the space of G-orbits of fourtuples of distinct points in $\mathbf{P}$ onto $\mathbf{Q}^{**} = \mathbf{Q} \setminus \{0, 1\}$ and ω descends to a map $f : \mathbf{Q}^{**} \to \mathbf{R}$. The cocycle relation $d\omega = 0$ translates into the Abel-Spence relation

$$f\left(\frac{y-x}{1-x}\right) - f(y) + f(x) - f\left(\frac{x}{y}\right) + f\left(\frac{x(1-y)}{y(1-x)}\right) = 0. \tag{4.1}$$

On the other hand (see [**24**] for details), one has for any infinite field F an exact sequence

$$K_3^{\mathrm{ind}}(F)_{\mathbf{Q}} \longrightarrow \mathbf{Q}[F^{**}]/R \xrightarrow{\delta} F^*_{\mathbf{Q}} \wedge F^*_{\mathbf{Q}} \longrightarrow K_2(F)_{\mathbf{Q}},$$

where $\delta([x]) = x \wedge (1-x)$, the subscript $\mathbf{Q}$ means tensorization $\otimes_{\mathbf{Z}} \mathbf{Q}$ and R is the subgroup generated by all

$$\left[\frac{y-x}{1-x}\right] - [y] + [x] - \left[\frac{x}{y}\right] + \left[\frac{x(1-y)}{y(1-x)}\right]$$

(with $x, y \in F^{**}$, $y \neq x, x-1$). In the case $F = \mathbf{Q}$, both K_2 and K_3^{ind} are torsion groups (see [**19**, 11.6] for K_2 and [**15**] for K_3; recall that K_3^{ind} is a quotient of K_3). Thus the map δ above is an isomorphism in the case at hand. Extending f linearly to $\mathbf{Q}[\mathbf{Q}^{**}]$, the relation (4.1) means that it descends to a map f_* modulo R. Now the induced map

$$\mathbf{Q}^* \wedge \mathbf{Q}^* \longrightarrow \mathbf{Q}^*_{\mathbf{Q}} \wedge \mathbf{Q}^*_{\mathbf{Q}} \xrightarrow{\delta^{-1}} \mathbf{Q}[\mathbf{Q}^{**}]/R \xrightarrow{f_*} \mathbf{R}$$

is of the form $\sum \alpha_{p,q} D_{p,q}$ as in Question 1.3 and thus a positive answer to Question 1.3 implies $f = 0$, whence $\omega = 0$. □

PROOF OF THEOREM 1.4. Let $\mathbf{A}$ be the ring of adèles of $\mathbf{Q}$ and $G_{\mathbf{A}} = \mathrm{PGL}_2(\mathbf{A})$. Since $G = \mathrm{PGL}_2(\mathbf{Q})$ embeds as a lattice in $G_{\mathbf{A}}$ (see *e.g.* Theorem 3.2.2 in [**18**, chap. I]), the restriction $\mathrm{H}^3_{\mathrm{cb}}(G_{\mathbf{A}}) \to \mathrm{H}^3_{\mathrm{b}}(G)$ is injective (Example 8.6.3 in [**20**]). The above proposition now implies $\mathrm{H}^3_{\mathrm{cb}}(G_{\mathbf{A}}) = 0$. Since $\mathrm{PGL}_2(\mathbf{Q}_p)$ is a direct factor of $G_{\mathbf{A}}$, the inflation $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{PGL}_2(\mathbf{Q}_p)) \to \mathrm{H}^3_{\mathrm{cb}}(G_{\mathbf{A}})$ is injective by functoriality and thus $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{PGL}_2(\mathbf{Q}_p))$ vanishes. The inflation associated to $\mathrm{GL}_2(\mathbf{Q}_p) \to \mathrm{PGL}_2(\mathbf{Q}_p)$ is an isomorphism because the kernel is amenable [**20**, 8.5.2]. Now we have the vanishing of $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_2(\mathbf{Q}_p))$; Proposition 3.4 yields the case of $\mathrm{GL}_3(\mathbf{Q}_p)$ and Theorem 1.1 handles $\mathrm{GL}_n(\mathbf{Q}_p)$ for $n \geq 4$. □

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University of Chicago
E-mail address: monod@math.uchicago.edu