# GELFAND PAIRS ADMIT AN IWASAWA DECOMPOSITION

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Abstract. Every Gelfand pair $(G, K)$ admits a decomposition $G = KP$, where $P < G$ is an amenable subgroup. In particular, the Furstenberg boundary of $G$ is homogeneous.

Applications include the complete classification of non-positively curved Gelfand pairs, relying on earlier joint work with Caprace, as well as a canonical family of pure spherical functions in the sense of Gelfand–Godement for general Gelfand pairs.

Let $G$ be a locally compact group. The space $\mathscr{M}^{\mathrm{b}}(G)$ of bounded measures on $G$ is an algebra for convolution, which is simply the push-forward of the multiplication map $G \times G \to G$.

**Definition.** Let $K < G$ be a compact subgroup. The pair $(G, K)$ is a **Gelfand pair** if the algebra $\mathscr{M}^{\mathrm{b}}(G)^{K,K}$ of bi-$K$-invariant measures is commutative.

This definition, rooted in Gelfand's 1950 work [14], is often given in terms of algebras of *functions* [12]. This is equivalent, by an approximation argument in the narrow topology, but has the inelegance of requiring the choice (and existence) of a Haar measure on $G$.

Examples of Gelfand pairs include notably all connected semi-simple Lie groups $G$ with finite center, where $K$ is a maximal compact subgroup. Other examples are provided by their analogues over local fields [18], and non-linear examples include automorphism groups of trees [23],[2].

All these "classical" examples also have in common another very useful property: they admit a *co-compact amenable subgroup* $P < G$. In the semi-simple case, $P$ is a minimal parabolic subgroup. Moreover, the Iwasawa decomposition implies that $G$ can be written as $G = KP$. This note shows that this situation is not a coincidence:

**Theorem.** *Let $(G, K)$ be a Gelfand pair. Then $G$ admits a co-compact amenable subgroup $P < G$ such that $G = KP$.*

The mere existence of $P$ has a number of strong consequences discussed below. Most immediate is that $G$ belongs to the exclusive club whose members boast a homogeneous Furstenberg boundary:

**Corollary 1.** *Choose a* maximal *subgroup $P < G$ as in the Theorem.*
*Then the Furstenberg boundary of $G$ is the homogeneous space $\partial G = G/P \cong K/(K \cap P)$.*
*In particular, $P$ is unique up to conjugacy.*

Another general consequence is that $G$ is *exact* in the sense of C*-algebras [20, §7.1].

**Remark.** The proof of the Theorem is easy. What surprises us (besides the fact that it went unnoticed during decades of harmonic analysis on Gelfand pairs) is that the unique group $P$ of Corollary 1 is obtained by purely existential methods. Indeed, the author is unaware of a constructive proof — *or even of a heuristic based on the classical Iwasawa decomposition*, explaining $(G, K) \mapsto P$.

We next derive a more geometric illustration of how consequential the existence of $P$ is. The above classical examples of Gelfand pairs are all **CAT(0) groups** in the sense that they occur as co-compact isometry groups of non-positively curved spaces: either *Riemannian symmetric spaces* or

*Euclidean buildings*. General CAT(0) groups constitute a much more cosmopolitan category populated by all sorts of exotic spaces hailing from combinatorial group theory, Kac–Moody theory, etc. Using the "indiscrete Bieberbach theorem" established with P.-E. Caprace [9], the Theorem of this note leads to a complete classification of CAT(0) Gelfand pairs:

**Corollary 2.** *Let $(G,K)$ be a Gelfand pair and assume that $G < \mathrm{Isom}(X)$ acts co-compactly on a geodesically complete locally compact* CAT(0) *space $X$.*

*Then $X$ is a product of Euclidean spaces, Riemannian symmetric spaces of non-compact type, Bruhat–Tits buildings and biregular trees.*

*In particular, $G$ lies in a product of Gelfand pairs belonging to the classical sets of examples above.*

This statement contains for instance a result by Caprace–Ciobotaru [5], namely: let $X$ be an irreducible locally finite thick Euclidean building. If $G = \mathrm{Aut}(X)$ (or any co-compact subgroup $G < \mathrm{Isom}(X)$) is a Gelfand pair for some compact $K < G$, then $X$ is Bruhat–Tits.

Similarly, the statement contains some cases of results by Abramenko–Parkinson–Van Maldeghem [1] and Lécureux [21, §7],[22] establishing the non-commutativity of Hecke algebras associated to certain Coxeter groups. Namely, when Kac–Moody theory associates to them a locally finite thick building, Corollary 2 implies that the Hecke algebra can only be commutative in the affine case.

The "Iwasawa decomposition" $G = KP$ is stronger yet than the existence of $P$. For instance, it is a key ingredient for results of Furman [13, Thm. 10] and it could shed some light on the *spherical dual* of $G$, see below. It should also impose further restrictions on the centraliser lattice in case $G$ is a compactly generated simple group, see [10]. Already the existence of $P$ implies that this lattice is at most countable: see [10, pp. 11–12] and use that $G/P$ is metrisable in this setting.

We now contemplate some of the analytic legacy that the decomposition $G = KP$ bestows upon a general Gelfand pair $(G,K)$. Following Gelfand and Godement [17], the fundamental building block of non-commutative Fourier–Plancherel theory is given by positive definite **spherical functions** on Gelfand pairs, namely continuous $\varphi\colon G \to \mathbf{C}$ satisfying

$$\varphi(x)\varphi(y) = \int_K \varphi(xky)\,dk \qquad \forall x,y \in G$$

where the integration is with respect to the unique Haar probability measure on $K$; see also [11] and [26]. This is the abstract generalisation of addition formulas for special functions such as Legendre functions [25].

Here is how $P$ enters the picture:

Let $\nabla_P$ be the modular function of $P$, which is non-trivial unless $G$ itself is amenable and $G = P$. Then $\varrho(kp) = \nabla_P(p)$ gives a well-defined continuous function $\varrho\colon G \to \mathbf{R}_{>0}$ when $k \in K$, $p \in P$ because $\nabla_P$ vanishes on $K \cap P$. For every parameter $s \in \mathbf{C}$, define

$$\varphi_s(g) = \int_K \varrho(g^{-1}k)^{\frac{1}{2}+is}\,dk.$$

In view of Corollary 1, $\varphi_s$ *is actually canonically attached to the pair* $(G,K)$ up to conjugation. On the other hand, $\varphi_s$ is the matrix coefficient of the (projectively) unique $K$-fixed vector in a parabolically induced representation from $P$. In particular, $\varphi_s$ is a pure positive definite spherical function on $G$ for each real $s$.

This is classical for semi-simple groups, where $\varphi_s$ above is the **Harish-Chandra formula**; the Theorem makes it available for general Gelfand pairs, as desired by Godement [16, §16]. Of course this only gives a principal series and suggests to investigate fully the characters of $P$.

*Proof of the Theorem and of Corollary 1.* We recall that an **affine $G$-flow** is a non-empty compact convex set $C$ in some locally convex topological vector space over $\mathbf{R}$, endowed with a jointly continuous $G$-action preserving the affine structure of $C$. An affine flow is called **irreducible** if it does not contain any proper affine subflow. An argument due to Furstenberg implies that $G$ admits an irreducible flow $\Delta G$ which is **universal** in the sense that it maps onto every irreducible flow. Moreover, $\Delta G$ is unique up to unique isomorphisms. It turns out that $\Delta G$ is the simplex of probability measures $\mathscr{P}(\partial G)$ over the Furstenberg boundary $\partial G$ of $G$, and that this is actually one of the possible *definitions* of $\partial G$. For all this, we refer to [15].

We shall be more interested in the convex subset $\mathscr{P}(G)$ of $\mathscr{M}^{\mathrm{b}}(G)$ consisting of the probability measures, as well as in the corresponding subset $\mathscr{P}(G)^{K,K}$. We note the following straightforward facts:

- $\mathscr{P}(G)$ is closed under the multiplication given by convolution.
- The monoid $\mathscr{P}(G)$ contains $G$ via the identification of points with Dirac masses.
- The normalised Haar measure $\kappa$ of $K$ is an idempotent belonging to $\mathscr{P}(G)^{K,K}$.
- $\mathscr{P}(G)^{K,K} = \kappa\mathscr{P}(G)\kappa$; it is a monoid with $\kappa$ as identity.

By generalised vector-valued integration [4, IV§7.1], any affine $G$-flow $C$ is endowed with an action of the monoid $\mathscr{P}(G)$ which is affine in both variables. It will be crucial below that this action is moreover continuous for the variable in $C$. One way to see this is to check first that any $\mu \in \mathscr{P}(G)$ induces a continuous map $C \to \mathscr{P}(C)$ by push-forward on orbits, using that the $G$-action on $C$ is equicontinuous over compact subsets of $G$. Then observe that the action of $\mu$ is obtained by composing this map $C \to \mathscr{P}(C)$ with the continuous barycenter map $\mathscr{P}(C) \to C$.

Since $K$ is compact, it has a non-empty fixed-point set $C^K$; better yet, the idempotent $\kappa$ provides a continuous projection $\kappa\colon C \to C^K$. In particular, the monoid $\mathscr{A} = \kappa\mathscr{P}(G)\kappa$ preserves the convex compact set $C^K$.

Only now do we use the assumption that we have a Gelfand pair: the monoid $\mathscr{A}$ is commutative. Since $\mathscr{A}$ acts by continuous operators, the Markov–Kakutani theorem therefore implies that $\mathscr{A}$ fixes a point $p$ in $C^K$. From now on, we assume that $C$ is irreducible. The convex set $\mathscr{P}(G)p$ is $G$-invariant and hence must be dense. It follows that $\kappa\mathscr{P}(G)p$ is dense in $C^K$, but $\kappa\mathscr{P}(G)p$ is $\mathscr{A}p$ which is reduced to $p$. In conclusion, we have shown that $K$ has a unique fixed point in $C$.

We now apply this to the case where $C = \Delta G$ is the simplex of probability measures on $\partial G$ and deduce that $K$ fixes a unique such measure on $\partial G$. Since $K$ is compact, every $K$-orbit supports an invariant measure: the push-forward of $\kappa$. This implies that $K$ has a single orbit in $\partial G$. In particular, $\partial G = G/P$ for some co-compact subgroup $P < G$ and moreover $G = KP$.

Next, we observe that $P$ is **relatively amenable** in $G$, which means by definition that every affine $G$-flow has a $P$-fixed point. Indeed, this property characterises the subgroups that fix a point in $\Delta G$: this follows from the universal property of $\Delta G$. This characterisation also implies that this $P$ is already *maximal* relatively amenable. Indeed, if $P' < G$ is relatively amenable and contains $P$, it also fixes a point in $\Delta G$; this induces an affine $G$-map $\mathscr{P}(G/P') \to \mathscr{P}(G/P)$, which must be the identity by universality of $\mathscr{P}(G/P) = \Delta G$.

We recall that relative amenability is equivalent to amenability in a wide class of ambient locally compact groups $G$ including all exact groups, but it is only *a posteriori* that the Theorem implies that $G$ is exact, see [20, §7.1]. In the locally compact setting, it is still an open question to exhibit an example where the weaker relative notion does not coincide with amenability [8]. In the co-compact case, however, we can settle the question with the Proposition below and conclude that $P$ is amenable. Thus the Proposition will complete the proof.

The following statement is a very basic case of much more general results by Andy Zucker [27, Thm. 7.5]; the elementary proof below is inspired by reading his preprint.

**Proposition.** *Let $G$ be a Hausdorff topological group and $P < G$ a closed subgroup such that $\partial G = G/P$. Then $P$ is amenable.*

**Warning.** A subgroup of $G$ fixing a point in $\partial G$ is not necessarily amenable. However, in the locally compact case and assuming $\partial G$ homogeneous, this follows from the Proposition because amenability of locally compact groups passes to subgroups.

*Proof of the Proposition.* We know that $P$ is co-compact and relatively amenable. The latter is equivalent to the existence of a $P$-invariant mean $\mu$ on the space $\mathrm{C}^{\mathrm{b}}_{\mathrm{ru}}(G)$ of right uniformly continuous bounded functions (cf. Thm. 5 in [8]). It suffices to show that $\mu$ descends to $\mathrm{C}^{\mathrm{b}}_{\mathrm{ru}}(P)$, viewed as a quotient of $\mathrm{C}^{\mathrm{b}}_{\mathrm{ru}}(G)$ under restriction (by Katetov extension [19]). Let thus $f \in \mathrm{C}^{\mathrm{b}}_{\mathrm{ru}}(G)$ be any map vanishing on $P$; we need to show $\mu(f) = 0$ and can assume $f \geq 0$. Given $\epsilon > 0$ there is an identity neighbourhood $U$ in $G$ such that $f \leq \epsilon$ on $UP$. By Urysohn's lemma in $G/P$, there is $h \in \mathrm{C}(G/P)$ vanishing on a neighbourhood of $P$ but taking constant value $\|f\|_\infty$ outside $UP$. Viewing $h$ as an element of $\mathrm{C}^{\mathrm{b}}_{\mathrm{ru}}(G)$, we thus have $f \leq \epsilon \mathbf{1}_G + h$. We now claim $\mu(h) = 0$, which finishes the proof since $\epsilon$ is arbitrary. The claim follows from the fact that $\mu$ is mapped to a $P$-invariant probability measure on $G/P$ under the inclusion of $\mathrm{C}(G/P)$ in $\mathrm{C}^{\mathrm{b}}_{\mathrm{ru}}(G)$. Indeed, the only $P$-invariant probability measure on $G/P \cong \partial G$ is the Dirac mass at $P$ by strong proximality of $P$ on $\partial G$, see [15, II.3.1]. □

*Proof of Corollary 2.* Consider $G < \mathrm{Isom}(X)$ as in the statement. We first recall that $X$ is *minimal* in the sense that it does not contain a closed convex $G$-invariant proper subset, see [6, 3.13]. Next, we recall that general splitting results (1.9 together with 1.5(iii) in [6]) allow us to reduce to the case where $X$ has no Euclidean factor. In any Gelfand pair, $G$ is unimodular [24, 24.8.1]; this, together with the elements collected thus far, allows us to apply Theorem M in [7]. That result states that $G$ has no fixed point at infinity. On the other hand, our Theorem above provides a subgroup $P < \mathrm{Isom}(X)$ acting co-compactly on $X$. We are now in position to apply the indiscrete Bieberbach theorem [9, Thm. B], which identifies $X$ with a product of classical spaces as desired. □

We now justify our claims concerning the functions $\varphi_s$ on $G$. Since $G$ is unimodular (reference above), Weil's integration formula [3, VII§2.5] implies that the push-forward of $\kappa$ on $G/P$ has a Radon–Nikodým cocycle given at $(g, xP)$ by $\rho(g^{-1}x)/\rho(x)$. Therefore, the unitary induction $\pi_s$ of the character $\nabla_P^{is}$ is given on various spaces of functions $f$ on $G/P$ by

$$(\pi_s(g)f)(xP) = f(g^{-1}xP)\left(\frac{\rho(g^{-1}x)}{\rho(x)}\right)^{\frac{1}{2}+is}.$$

The only $K$-invariant vectors $v$ are constant functions on $G/P$ and hence the associated matrix coefficient $\varphi_s(g) = \langle \pi_s(g)v, v\rangle$ is uniquely defined once $v$ has unit norm. The fact that $\varphi_s$ is pure and spherical (for $s \in \mathbf{R}$) now follows from the general theory of Gelfand pairs, specifically I.II.6 and I.III.2 in [12].

**Remark.** A part of the proof of the Theorem is reminiscent of the fact that any irreducible *unitary* representation of $G$ has at most a one-dimensional subspace of $K$-fixed vectors, a fact that actually characterises Gelfand pairs. We recall that the corresponding statement fails for *real* Hilbert spaces, whereas our affine flows are always over the reals.

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## References

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