The bounded cohomology of transformation groups of Euclidean spaces and discs

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Abstract

We prove that the groups of orientation-preserving homeomorphisms and diffeomorphisms of \mathbb{R}^n are boundedly acyclic, in all regularities. This is the first full computation of the bounded cohomology of a transformation group that is not compactly supported, and it implies that many characteristic classes of flat \mathbb{R}^n - and S^n -bundles are unbounded. We obtain the same result for the group of homeomorphisms of the disc that restrict to the identity on the boundary, and for the homeomorphism group of the non-compact Cantor set. In the appendix, Alexander Kupers proves a controlled version of the annulus theorem which we use to study the bounded cohomology of the homeomorphism group of the discs.

1 Introduction

1.1 Euclidean spaces vs. compactness

The topological group Homeo(\mathbb{R}^n) of all self-homeomorphisms of Euclidean space remains very mysterious to this day. Its homotopy type is not completely understood and neither is its group cohomology, despite many deep discoveries, especially in recent times (see [RW23], [GRW23] and the references therein). In particular, the results obtained so far indicate highly non-trivial higher homotopy groups.

By deep results of Thurston [Thu22] and McDuff [McD80], when M is the interior of a compact manifold, the cohomology of classifying space of the topological group Homeo(M)coincides with the group cohomology of the underlying "abstract" (discrete) group. In contrast to the mystery surrounding the group Homeo (\mathbb{R}^n) , Mather proved in 1971 [Mat71] that the group Homeo_c (\mathbb{R}^n) of *compactly supported* homeomorphisms of \mathbb{R}^n , as an abstract group, is acyclic. Moreover, when M itself is compact, the group cohomology of Homeo(M)is sometimes completely known thanks to Thurston's previously mentioned result. Thus, "mystery resides at infinity".

Our contribution in this article is to the bounded cohomology H_b^{\bullet} of homeomorphism and diffeomorphism groups (always as abstract groups). In the compactly supported case, the analogue of Mather's theorem was established in 1985 by Matsumoto–Morita [MM85]: Homeo_c(\mathbb{R}^n) is boundedly acyclic. More recently [MN23], the bounded cohomology of Homeo(M) and Diff(M) has been determined for the compact case of $M = S^1$. Our main result is the first complete determination of the bounded cohomology of transformation groups without any compactness assumption, answering [MN23, Question 7.6].

Theorem A (Theorems 3.4 and 3.12). For all $n \in \mathbb{N}^*$, the group Homeo₊(\mathbb{R}^n) is boundedly acyclic.

Some results in bounded cohomology are analogues of ordinary cohomological statements, e.g. Matsumoto–Morita's extension of Mather's result. By contrast, Theorem A is genuinely different: Even though the ordinary cohomology is not yet completely known, it is known to have a remarkably rich structure when the dimension n is high enough ([RW23, Section 3.3]).

Moreover, it is known that the group cohomology of transformations of the Euclidean space depends on regularity. For a manifold M as before, it is a consequence of a remarkable theorem of Tsuboi [Tsu89] that the cohomology of $BDiff^1(M)$, the classifying space of the group $Diff^1(M)$ with C^1 -topology, is the same as the group cohomology of $Diff^1(M)$ as a discrete group. However, in regularities r > 1, it is known that the group cohomology of $Diff^r(M)$ is different from the cohomology of $BDiff^r(M)$. In particular, when $M = \mathbb{R}^n$, the topological group $Diff^r(\mathbb{R}^n)$ for r > 0 is homotopy equivalent to O(n) but as a discrete group, its group cohomology is the same as the cohomology of $B\Gamma_n^r$, the classifying space of C^r -Haefliger structures of codimension n [Seg78, Proposition 3.1 and Proposition 1.3]. The cohomology of $B\Gamma_n^r$ is related to the secondary characteristic classes of foliations of codimension n and it is known to be highly nontrivial (see Section 5 and [Miz88]). Still, we also obtain the analogue of Theorem A for diffeomorphisms in all regularities.

Theorem B (Theorems 3.4 and 3.12). For all $r \in \mathbb{N}^* \cup \{\infty\}$ and all $n \in \mathbb{N}^*$, the group $\operatorname{Diff}_+^r(\mathbb{R}^n)$ is boundedly acyclic.

- **Remarks.** (i) The above formulation in terms of *orientation-preserving* homeomorphism and diffeomorphism groups $\text{Homeo}_+, \text{Diff}_+^r$ is more natural for our proofs, but it immediately implies the result for the entire groups (see [Mon01, Corollary 8.8.5]). It is standard that for r > 0, the derivative at the origin induces a homotopy equivalence between $\text{Diff}_+^r(\mathbb{R}^n)$ and $\text{GL}_n(\mathbb{R})_+$ as topological groups. So we have $\text{Diff}_+^r(\mathbb{R}^n) = \text{Diff}_\circ^r(\mathbb{R}^n)$ for r > 0, where Diff_\circ^r denotes the identity component. The same statement holds for r = 0by the deep theorem in [Kir69, Corollary of Theorem 2]. We will use the corresponding notation $\text{Homeo}_\circ, \text{Diff}_\circ^r$ when this point of view is more natural.
 - (ii) Ordinary acyclicity usually refers to the vanishing of homology with integral coefficients, which then implies the vanishing of cohomology with arbitrary trivial coefficients. *Bounded acyclicity* is defined directly as the vanishing of bounded cohomology with real coefficients, because fundamental techniques of the theory are unavailable over the integers. Nevertheless, it can be understood as a version of homological acyclicity but with a control of the norm in the spirit of homological isoperimetry, see Matsumoto– Morita [MM85].

As suggested by the fact that our results differ from the picture in ordinary cohomology, our proof will require a new device. That device can also be used in ordinary cohomology; as an illustration, the special case n = 1 of Theorem A actually goes through in ordinary cohomology and recovers the following result of McDuff (which she also attributes to Thurston and Segal). Contrary to the original proof, this does not rely on the topology of the group Homeo₊(\mathbb{R}), nor on foliation theory and Haefliger spaces.

Theorem C ([McD80]). The group Homeo₊(\mathbb{R}) is acyclic.

A theorem of Ghys [Ghy01] states that the bounded Euler class in $H_b^2(\text{Homeo}_+(S^1);\mathbb{Z})$ classifies actions on the circle by orientation-preserving homeomorphisms up to semiconjugacy. Ghys's Theorem can be used to prove that, given a countable discrete group Γ , the semiconjugacy relation on the representation variety $\text{Hom}_{irr}(\Gamma, \text{Homeo}_+(S^1))$ is *smooth*. Here Hom_{irr} denotes those representations without global fixed points, and smooth equivalence relations are the simplest ones from the point of view of Borel reducibility. On the other hand, the semiconjugacy relation on the representation variety $\text{Hom}_{irr}(\Gamma, \text{Homeo}_+(\mathbb{R}))$ is *essentially hyperfinite*, a strictly more complex type of equivalence relation, already when Γ is a free group [BMBRT21, Section 3.5]. This is in agreement with the fact that $\text{Homeo}_+(\mathbb{R})$ has vanishing bounded cohomology with integer coefficients in all positive degrees, which is a direct consequence of Theorems A and C (see e.g. [FFLM23, Corollary 5.8]).

1.2 Taming infinity: A sketch

The new method that we introduce to study the various transformation groups of \mathbb{R}^n is a general paradigm applicable in many other situations, some of which will be mentioned later in this article. We will now first sketch it in one of the simplest possible cases, namely for the group Homeo₊(\mathbb{R}) of (orientation-preserving) homeomorphisms of the line:

The most naive attempt to reduce $\operatorname{Homeo}_+(\mathbb{R})$ to the compactly supported case is to cut \mathbb{R} into an infinite sequence of compact intervals. Since we prefer usual sequences indexed by \mathbb{N} rather than bi-infinite sequences, we shall work with the subgroup G of elements that are trivial in a neighborhood of $-\infty$ and we consider a sequence of points that increases to $+\infty$. Once we prove that this group is boundedly acyclic, an elementary symmetry argument will allow us to take care of the germs at $-\infty$ in a similar way to conclude that $\operatorname{Homeo}_+(\mathbb{R})$ itself is boundedly acyclic.

To be more precise, we consider sequences $x = (x_n)$ of fat points x_n , which means (as in [MN23]) germs at 0 of embeddings of an interval (-1, 1) into \mathbb{R} . Thus by definition a homeomorphism fixing x_n must be trivial in a neighborhood of the core \dot{x}_n , which is the point $\dot{x}_n = x_n(0) \in \mathbb{R}$. In conclusion, the stabilizer in G of the entire sequence of fat points is the product group

Homeo_c((
$$-\infty, \dot{x}_1$$
)) × $\prod_{j=1}^{\infty}$ Homeo_c((\dot{x}_j, \dot{x}_{j+1})).

That stabilizer, a power of compactly supported groups, is known to be [boundedly] acyclic by previous work. Therefore a spectral sequence argument reduces the problem to the study of two semisimplicial sets: On the one hand, the collection of all fat sequences x as above. On the other hand, the quotient of this semisimplicial object by the *G*-action. An important aspect of our approach is that the semisimplicial structure is given by the *subsequence partial order* on spaces of sequences. We shall see that, in great generality, this is an acyclic and boundedly acyclic object.

In any case, so far all this construction has merely kicked the difficulty further down the road: The "mystery at infinity" is now contained in the abstract semisimplicial set defined by G-orbits of increasing sequences under the order relation of taking subsequences.

At that point we observe that homeomorphisms of \mathbb{R} are transitive on sequences (of fat points) that increase to infinity and that, in fact, the entire structure of the quotient

semisimplicial object is determined by the abstract order relation on *indexing* the subsequences. In precise terms, we show that this quotient semisimplicial set can be identified with the nerve of the monoid $\text{Emb}_{<}(\mathbb{N})$ of order-preserving embeddings $\mathbb{N} \to \mathbb{N}$. The remaining part of the proof will therefore be to establish:

Theorem D. The monoid $\text{Emb}_{<}(\mathbb{N})$ is acyclic and boundedly acyclic.

It should appear plausible now that the above sketch can be implemented in many other situations. For instance, in the case of \mathbb{R}^n , we shall consider suitably concentric sequences of fat spheres so that the stabilizer becomes an infinite product of groups of compactly supported homeomorphisms of annuli (and one ball). Each aspect of the above outline will raise new difficulties in various situations, except the final step: The quotient structure will always be the same monoid $\text{Emb}_{<}(\mathbb{N})$.

It remains therefore to explain how we prove Theorem D, which encodes the common residual difficulty at infinity of the diverse situations. The above spectral sequence argument is entirely reversible: If we exhibit *some* [boundedly] acyclic group G acting on some [boundedly] acyclic space of sequences with [boundedly] acyclic stabilizers and the same quotient object $\text{Emb}_{<}(\mathbb{N})$, then the [bounded] acyclicity of $\text{Emb}_{<}(\mathbb{N})$ will follow.

This leaves us with the freedom to choose the most favorable example. One possibility is the group of all countably supported permutations of an uncountable set. This group is *dissipated*, a property well-known to imply acyclicity via the concept of binate (or pseudomitotic) group [Var85, Ber89]. This property also implies bounded acyclicity [FFLM23] and therefore we will obtain a proof of Theorem D. In closing, we observe that the choice of this group is not unnatural: There are uncountably many ways to pass to subsequences, but each one is of course a rearrangement of countably many items.

1.3 Beyond Euclidean spaces

As mentioned above, our proof of Theorems A and B relies on a general criterion for bounded acyclicity. This applies to a variety of groups, which were not approachable by the previous methods tailored to compact spaces.

An instance of this is in the zero-dimensional setting of Cantor sets. The acyclicity of the homeomorphism group of the Cantor set was shown by Tsuboi and Sergiescu [ST94] (and later generalized to the more general Menger spaces [ST96]). A similar strategy also proves bounded acyclicity, as shown by Andritsch [And22]. Our methods allow us to treat the non-compact Cantor set, which is uniquely defined as a locally compact, non-compact, metrizable, totally disconnected space without isolated points. To the best of our knowledge, also the acyclicity result is new.

Theorem E. The homeomorphism group of the non-compact Cantor set is acyclic and boundedly acyclic.

In another direction, we consider homeomorphism groups of discs fixing the boundary pointwise. While discs are compact, from the point of view of homology and bounded cohomology, their transformation groups are closer to the groups of Theorem A: The difficulties present at infinity are now present at the boundary. In this case, the bounded acyclicity was already shown (in all regularities) for D^2 [MN23, Theorems 1.3 and 1.4], however the proof is dependent on certain combinatorial properties of (fat) chords in the 2-disc that fail in higher dimensions. **Theorem F.** For all $n \in \mathbb{N}^*$, the group Homeo (D^n, ∂) of homeomorphisms of the disc D^n fixing the boundary pointwise, is boundedly acyclic.

We obtain the following consequence, which generalizes [MN23, Theorem 1.3] and answers [MN23, Question 7.5]:

Corollary G. For all $n \in \mathbb{N}^*$, the restriction $\operatorname{Homeo}_+(D^n) \to \operatorname{Homeo}_+(S^{n-1})$ induces an isomorphism in bounded cohomology in all degrees.

This allows to determine the bounded cohomology of $\text{Homeo}_+(D^n)$ in low degrees (Corollary 3.21).

These are just two more examples of the applicability of our methods, and we do not aim for a complete list. In Section 6 we discuss further instances in which our methods apply, and speculate about other settings where deploying our techniques seems to require additional work.

Other applications follow from our main results by some standard manipulationos. A notable instance is the case of compact annuli, where we prove:

Theorem H. For all $n \in \mathbb{N}^*$, the identity component Homeo_o $(S^n \times [0,1], \partial)$ of the group of homeomorphisms of the annulus $S^n \times [0,1]$ fixing both boundary components pointwise, is boundedly acyclic.

As in the case of the disc, without imposing that the boundary is fixed we obtain a result on restrictions (Corollary 6.4) and computations in low degree (Corollary 6.5). Leveraging results from [MN23], we can treat all regularities in dimension 2:

Corollary I. For all $r \in \mathbb{N}$, the restriction $\operatorname{Diff}_{\circ}^{r}(S^{1} \times [0, 1]) \to \operatorname{Diff}_{\circ}^{r}(S^{1}) \times \operatorname{Diff}_{\circ}^{r}(S^{1})$ induces an isomorphism in bounded cohomology in all degrees. Explicitly, denoting by $\mathscr{E}_{i} \in H_{b}^{2}(\operatorname{Diff}_{\circ}^{r}(S^{1} \times [0, 1]))$ the bounded Euler class for the action of $\operatorname{Diff}_{\circ}^{r}(S^{1} \times [0, 1])$ on the boundary component $S^{1} \times \{i\}$, we have an isomorphism of graded \mathbb{R} -algebras:

$$H_b^*(\operatorname{Diff}^r_\circ(S^1 \times [0,1])) \cong \mathbb{R}[\mathscr{E}_0,\mathscr{E}_1].$$

In degree 2, we recover a theorem of Militon (see [Mil14] for the definition of the *torsion* number).

Corollary J ([Mil14]). For all $r \in \mathbb{N}$ and $G = \text{Diff}_{\circ}^{r}(S^{1} \times [0,1])$, the space $Q(G)/H^{1}(G)$ of non-trivial quasimorphisms is one-dimensional, spanned by the torsion number. In particular, for $r \neq 2, 3$, the space Q(G) of homogeneous quasimorphisms is one-dimensional, spanned by the torsion number.

1.4 Unboundedness of characteristic classes

The bounded acyclicity of $\text{Homeo}_+(\mathbb{R}^n)$ implies that the Milnor–Wood inequality fails for certain characteristic classes of flat \mathbb{R}^n -bundles and flat S^n -bundles. To describe these classes, first recall a theorem of Thurston ([Thu22, Corollary (b) of theorem 5] and see [McD80, Section 2, Theorem 2.5] for the proof) for homeomorphisms, we have the natural map between classifying spaces

$$B\operatorname{Homeo}^{\delta}(M) \to B\operatorname{Homeo}(M),$$
 (1)

where $\text{Homeo}^{\delta}(M)$ is the group of homeomorphisms of M with the discrete topology. This map is acyclic and in particular, it induces a homology isomorphism in all degrees. The same statement also holds in the relative case for manifolds with boundary and also for non-compactly supported homeomorphisms of an open manifold that is an interior of a compact manifold [McD80, Section 2, Theorem 2.5]. Therefore, the map

$$BHomeo^{\delta}(\mathbb{R}^n) \to BHomeo(\mathbb{R}^n),$$
 (2)

induces a homology isomorphism in all degrees. The same also holds for $\text{Homeo}_+(\mathbb{R}^n)$. There have been recent breakthroughs in understanding the homotopy type and the rational cohomology of $B\text{Homeo}(\mathbb{R}^n)$. In particular, there are topological Pontryagin classes p_i for Euclidean bundles that are defined as rational cohomology classes, and for oriented even dimensional Euclidean bundles, there is also an Euler class e. Galatius and Randal-Williams [GRW23] proved a remarkable result which implies that the map

$$\mathbb{R}[e, p_1, p_2, \dots] \to H^*(B\mathrm{Homeo}_+(\mathbb{R}^{2n}); \mathbb{R}),$$

is injective for $2n \ge 6$, and in the odd-dimensional case, it follows from [GRW23, Corollary 1.2] that

$$\mathbb{R}[p_1, p_2, \dots] \to H^*(B\mathrm{Homeo}_+(\mathbb{R}^{2n+1}); \mathbb{R}),$$

is injective for $2n + 1 \ge 7$. In low dimensions, we also know that $\operatorname{Homeo}_+(\mathbb{R}^3) \simeq \operatorname{SO}(3)$ and $\operatorname{Homeo}_+(\mathbb{R}^2) \simeq \operatorname{SO}(2)$ ([Ham74, Theorem 1.2.3] and [BH81]). Therefore $\mathbb{R}[p_1] \to H^*(B\operatorname{Homeo}_+(\mathbb{R}^3);\mathbb{R})$ and $\mathbb{R}[e] \to H^*(B\operatorname{Homeo}_+(\mathbb{R}^2);\mathbb{R})$ are isomorphisms. The homology isomorphism in (2) implies that the same holds for $H^*(B\operatorname{Homeo}^\delta_+(\mathbb{R}^n);\mathbb{R})$ for the corresponding n.

If a nontrivial class in $H^*(B\text{Homeo}^{\delta}(M);\mathbb{R})$ is not in the image of the comparison map

$$H_h^*(B\mathrm{Homeo}^{\delta}(M);\mathbb{R}) \to H^*(B\mathrm{Homeo}^{\delta}(M);\mathbb{R}),$$

we say that class is *unbounded*. A corollary of our main theorem is that any nontrivial class in $H^*(B\text{Homeo}^{\delta}_+(\mathbb{R}^n);\mathbb{R})$ is unbounded. More specifically we have the following.

Corollary K. For C^0 -flat oriented \mathbb{R}^{2n} -bundles, all classes in $\mathbb{R}[e, p_1, p_2, ...]$ are unbounded for $2n \ge 6$ and for C^0 -flat \mathbb{R}^2 -bundles, all the powers of the Euler class are unbounded. For C^0 -flat \mathbb{R}^{2n+1} -bundles, all classes in $\mathbb{R}[p_1, p_2, ...]$ are unbounded for $2n+1 \ge 7$ and for C^0 -flat \mathbb{R}^3 -bundles, all powers of p_1 are unbounded.

Similarly, for C^r -flat \mathbb{R}^n -bundles when r > 0, every nontrivial class in $H^*(BDiff^{r,\delta}_+(\mathbb{R}^n);\mathbb{R})$ is unbounded. To detect nontrivial classes in $H^*(BDiff^{r,\delta}_+(\mathbb{R}^n);\mathbb{R})$, we shall use a deep theorem of Segal [Seg78, Prop. 1.3 and 3.1], that there exists a map

$$BDiff_{+}^{r,\delta}(\mathbb{R}^n) \to BS\Gamma_n^r,$$

which is a homology isomorphism, where $BS\Gamma_n^r$ is the classifying space of Haefliger structures for codimension n foliations that are transversely oriented. There is extensive literature on secondary characteristic classes of foliations that give nontrivial classes in $H^*(BS\Gamma_n^r; \mathbb{R})$ when r > 1. In particular, there is a Godbillon–Vey class in $H^{2n+1}(BS\Gamma_n^r; \mathbb{R})$ that is nontrivial (see [Miz88] for the corresponding cocycle formula in $H^{2n+1}(BDiff_+^{r,\delta}(\mathbb{R}^n); \mathbb{R})$). To detect the nontriviality of classical characteristic classes, we consider a map

$$\nu \colon BS\Gamma_n^r \to BGL_n(\mathbb{R})_+,$$

which classifies oriented normal bundles to the codimension n foliations where $\operatorname{GL}_n(\mathbb{R})_+$ is the group invertible matrices with positive determinant. For all regularities, it is known that the map ν is at least (n + 1)-connected, see [Hae71, Remark 1, Section II.6]. Therefore, the map

$$\nu^* \colon H^*(B\mathrm{GL}_n(\mathbb{R})_+;\mathbb{R}) \to H^*(B\mathrm{S\Gamma}_n^r;\mathbb{R}),$$

is injective in degrees $* \le n + 1$ for all regularities. For r = 1, it is an isomorphism in all degrees by a remarkable theorem of Tsuboi [Tsu89] and for r > 1, it is conjecturally injective in degrees below 2n [Hur06, Problem 14.5]. So we obtain the following corollary about the unboundedness of the Euler class and Pontryagin classes.

Corollary L. For C^r -flat oriented \mathbb{R}^n -bundles when r > 1, the polynomials of degree less than n + 2 on Pontryagin classes and the Euler class (when n is even) are all unbounded and when r = 1, the same statement holds without any condition on the degree of the polynomials.

Remark. Calegari in [Cal04] proved that the Euler class is unbounded for oriented C^r -flat \mathbb{R}^2 -bundles.

These unboundedness results for classical characteristic classes of flat \mathbb{R}^n -bundles are in contrast to Milnor, Sullivan, Smillie, and Gromov's results on the boundedness of these classes for flat *linear* \mathbb{R}^n -bundles. For example, Milnor showed that the Euler class is bounded for flat linear \mathbb{R}^2 -bundles [Mil58]; Sullivan [Sul76] and Smillie [Smi] generalized it to flat linear \mathbb{R}^{2n} -bundles. Gromov further generalized it [Gro82, Page 23] by proving that if G is an algebraic subgroup in the linear group $\operatorname{GL}_n(\mathbb{R})$, then any class in the image of the map

$$H^*(BG;\mathbb{R}) \to H^*(BG^{\delta};\mathbb{R}),$$

is a bounded class.

For flat S^n -bundles, we also have topological Pontryagin classes and the Euler class (if it is oriented and n is odd). These classes are induced by the following composition

$$BHomeo^{\delta}(S^n) \to BHomeo^{\delta}(D^{n+1}) \to BHomeo^{\delta}(\mathbb{R}^{n+1}),$$

where the first map is induced by coning the sphere to a disc and the second map is induced by the restriction to the interior of the disc. So we can pull the classes in $H^*(B\text{Homeo}^{\delta}_{\circ}(\mathbb{R}^{n+1});\mathbb{R})$ to $H^*(B\text{Homeo}^{\delta}_{\circ}(S^n);\mathbb{R})$. As a consequence of Galatius–Randal-Williams's result [GRW23, Theorem 1.4], we shall prove in Section 5 that

$$\mathbb{R}[p_1, p_2, \dots] \to H^*(B\mathrm{Homeo}^{\delta}(S^n); \mathbb{R}),$$

is injective for $n \ge 6$. We then use the bounded acyclicity of $\text{Homeo}(\mathbb{R}^n)$ to obtain the following unboundedness result for the invariants of flat sphere bundles.

Theorem M. For C^0 -flat S^n -bundles, the classes in $\mathbb{R}[p_1, p_2, ...]$ are unbounded for $n \ge 7$ and for n = 4, 5 and 6 the classes $\mathbb{R}[p_1, p_2]$ are unbounded. For n = 2, 3, all the powers of p_1 are unbounded.

Remark. The unboundedness of p_1 in $H^4(B\text{Homeo}^{\delta}(S^3);\mathbb{R})$ was first shown in [MN23, Theorem 1.8], answering a question of Ghys [Lan94, F.1]. Theorem M partially answers [MN23, Questions 7.3 and 7.4].

Outline. We begin by stating the criteria for bounded acyclicity and acyclicity in Section 2. In Section 3 we go through the list of groups from the introduction, and verify that they satisfy the criteria. The case of the disc presents some additional technical difficulties which are solved by a controlled annulus theorem, proven in Appendix A. We prove the criteria in Section 4. Section 5 is dedicated to unboundedness results for characteristic classes of flat \mathbb{R}^n -and S^n -bundles. Finally in Section 6, we present further computations (including the results on compact annuli), and some speculations.

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2 A criterion for bounded acyclicity

We start by formulating a general criterion for bounded acyclicity of groups admitting a certain action on a poset. This criterion will be the common denominator of all our proofs.

Definition 2.1. Let \mathcal{P} be a poset. We say that \mathcal{P} satisfies the **W** property, if for every finite subposet \mathcal{Q} with minimal elements $\{x_1, \ldots, x_k\}$ the following holds. For every subset $I \subseteq \{1, \ldots, k\}$ there exists $y_I \in \mathcal{P}$ such that

- 1. if $I \subseteq J$ then $y_I \preceq y_J$;
- 2. for $x \in \mathcal{Q}$, if $x_i \leq x$ for all $i \in I$ then $y_I \leq x$.

The diagram below exhibits this property for the case $Q = \{x_1, x_2\}$, which justifies the name.



Let G be a group acting on a set X. The set of all sequences $\mathbf{x} = (x_i)_{i \in \mathbb{N}^*}, x_i \in X$ of pairwise distinct elements x_i has a natural poset structure, where $\mathbf{x} \preceq \mathbf{y}$ if \mathbf{x} is a subsequence of \mathbf{y} . The action of G of X induces an order-preserving action on this poset of sequences. Any G-invariant subposet \mathcal{X} will be called a G-poset of sequences in X.

Theorem 2.2. Let G be a group acting on a set X and let \mathcal{X} be a G-poset of sequences in X. Suppose that the following hold:

1. The poset \mathcal{X} satisfies the **W** property;

- 2. The action of G on \mathcal{X} is transitive;
- 3. The stabilizer in G of some (equivalently: every) $\mathbf{x} \in \mathcal{X}$ is boundedly acyclic.

Then G is boundedly acyclic.

We will prove this theorem in Section 4. In the next section, we will see that this criterion is very powerful in proving bounded acyclicity of transformation groups.

There is a corresponding criterion for acyclicity, which will also be proved in Section 4.

Theorem 2.3. Let G be a group acting on a set X and let \mathcal{X} be a G-poset of X-sequences. Suppose that the following hold:

- 1. The poset \mathcal{X} satisfies the W property;
- 2. The action of G on \mathcal{X} is transitive;
- 3. The stabilizer in G of some (equivalently: every) $\mathbf{x} \in \mathcal{X}$ is acyclic.

Then G is acyclic.

2.1 Bounded acyclicity for compactly supported groups

When applying the criterion from Theorem 2.2, we will need a way to prove bounded acyclicity of the stabilizers. These will always be *compactly supported* transformation groups,, which are known to be boundedly acyclic in great generality [MM85, FFLM23, Mon22, CFFLM23]. For instance, to deduce Theorems A and B from the criterion, we will only need the following statement:

Theorem 2.4 ([MN23, Theorem 1.6, Lemma 2.3]). Let $n \in \mathbb{N}^*, r \in \mathbb{N}^* \cup \{\infty\}$, let M be a closed C^r -manifold and let Z be a C^r -manifold diffeomorphic to $M \times \mathbb{R}^n$. Then the groups $\operatorname{Homeo}_{c}(Z), \operatorname{Diff}_{c}^{r}(Z)$, are boundedly acyclic.

Moreover, any (possibly infinite) direct product of such groups is boundedly acyclic.

In more delicate cases, such as for discs, we will need a more general criterion (which actually implies Theorem 2.4).

Definition 2.5 ([CFFLM23, Definition 4.2]). We say that the group G has commuting \mathbb{Z} -conjugates if for every finitely generated subgroup $H \leq G$ there exists $t \in G$ such that $[H, t^p H t^{-p}] = 1$ for all $p \in \mathbb{N}^*$.

Theorem 2.6 ([CFFLM23, Theorem 1.3]). If the group G has commuting \mathbb{Z} -conjugates, then G is boundedly acyclic.

Another useful feature of bounded acyclicity that we will use in the sequel is the following:

Proposition 2.7 ([MN23, Proposition 2.4], see also [MR23, Theorem 4.1.1]). Consider a short exact sequence $1 \to N \to G \to Q \to 1$, where N is boundedly acyclic. Then the quotient $G \to Q$ induces an isomorphism $H_b^n(G) \cong H_b^n(Q)$ for all $n \in \mathbb{N}$.

The criteria presented so far only give bounded acyclicity. To apply Theorem 2.3 we use another property which guarantees both acyclicity and bounded acyclicity. **Definition 2.8.** We say that a group G is *binate* if for every finitely generated subgroup $H \leq G$ there exist a homomorphism $\psi \colon H \to G$ such that $[H, \psi(H)] = 1$ and an element $t \in G$ such that $t^{-1}\psi(h)t = h\psi(h)$ for all $h \in H$.

Theorem 2.9 ([Var85, Ber89]). Binate groups are acyclic.

Theorem 2.10 ([FFLM23]). *Binate groups are boundedly acyclic.*

Again, the main example of binate groups comes from compactly supported groups, although these are now more restricted:

Proposition 2.11 ([Var85, Ber89]). The groups Homeo_c(\mathbb{R}^n) are binate.

This implies that the groups $\text{Homeo}_{c}(\mathbb{R}^{n})$ are acyclic, which was already proved by Mather [Mat71]. An advantage of using the binate property is that it passes to products:

Lemma 2.12 ([SV87]). Any (possibly infinite) direct product of binate groups is binate.

2.2 Interweaving subsequences

We now present a common framework that will allow us to interweave subsequences in various settings, in order to verify the **W** property.

Let X be any set endowed with a preorder relation \sqsubseteq and an equivalence relation \simeq contained in \sqsubseteq (i.e. $x \simeq y \Rightarrow x \sqsubseteq y$). Define \mathcal{X} to be the collection of all cofinal increasing sequences of pairwise inequivalent elements of X. Explicitly, this means:

- $\forall n \leq m : x_n \sqsubseteq x_m;$
- $\forall x \in X \exists n : x \sqsubseteq x_n;$
- $\forall n \neq m : x_n \not\simeq x_m$.

Or course we shall only be interested in this definition for examples where there exist cofinal sequences at all. We record the following elementary observation.

Lemma 2.13. The set \mathcal{X} is closed under passage to subsequences.

We will prove that the subsequence order relation \leq has the desired property by cyclically weaving subsequences together.

Proposition 2.14. In the above setting, the poset \mathcal{X} has the W property.

Proof. Let $\mathcal{Q} \subseteq \mathcal{X}$ be a finite subposet, with minimal elements $\mathbf{x}^1, \ldots, \mathbf{x}^k$. We write $\mathbf{x}^j = \{x_n^j\}_{n \in \mathbb{N}^*}$. The main step in the construction of the various \mathbf{y}^J is the following inductive construction of a single sequence $\mathbf{y} = (y_n)_{n \in \mathbb{N}^*}$.

We start by setting $y_1 \coloneqq x_1^1$. Next, we will choose y_2 among the elements of \mathbf{x}^2 as follows. Since \mathcal{Q} is a finite set of sequences of pairwise inequivalent elements, there is $q \in \mathbb{N}$ such that $\forall \mathbf{x} \in \mathcal{Q} \forall r > q : x_r \not\simeq y_1$. Since \mathbf{x}^2 is increasing and cofinal, we can choose p > q with $y_1 \sqsubseteq x_p^2$ and we define $y_2 = x_p^2$. The general inductive step is analogous: Suppose that y_1, \ldots, y_n have been defined, with each y_i an element of \mathbf{x}^j for $j \equiv i \mod k$. Then we choose y_{n+1} to be an element x_p^{j+1} of \mathbf{x}^{j+1} (or \mathbf{x}^1 if $j \equiv 0 \mod k$) with the following property. If q is an upper

bound for the index of any element of any $\mathbf{x} \in \mathcal{Q}$ that is equivalent to some y_1, \ldots, y_n , then we take p > r such that $y_i \sqsubseteq x_p^{j+1}$ (respectively $y_i \sqsubseteq x_p^1$) for all $i = 1, \ldots, n$. The sequence \mathbf{y} is by definition an increasing sequence of pairwise inequivalent elements and it is cofinal because it shares a subsequence with a cofinal increasing sequence (any of the \mathbf{x}^i).

Turning to the requirements of the **W** property, we now define \mathbf{y}^I for $I \subseteq \{1, \ldots, k\}$ to be the subsequence of \mathbf{y} that selects only those indices that are congruent to an element of Imodulo k; in particular $\mathbf{y}^{\{1,\ldots,k\}} = \mathbf{y}$. Then by definition $\mathbf{y}^I \preceq \mathbf{y}^J$ whenever $I \subseteq J$, and the fact that $\mathbf{y}^I \preceq \mathbf{x}$ if $\mathbf{x}^i \preceq \mathbf{x}$ for all $i \in I$, is the point of the construction above. \Box

Remark 2.15. Since the above proof verifies the **W** property by explicitly interweaving subsequences of the finite set of sequences given for **W**, it holds verbatim for sub-posets $\mathcal{Y} \subseteq \mathcal{X}$ as soon as \mathcal{Y} has the following two closure properties:

- \mathcal{Y} is closed under passing to subsequences;
- If x ∈ X can be written as the disjoint union of two subsequences, each in Y, then x ∈ Y.

3 Verifying the criterion

We now go through the groups from the introduction (except for transformation groups of compact annuli, which will be treated in Section 6), justifying why they satisfy the criterion from Theorem 2.2 (and in some cases, from Theorem 2.3 as well). We start with the case of the line, which is the easiest, and the only one among the transformation groups of Euclidean spaces where we are able to establish ordinary acyclicity as well. Then we move on to transformation groups of \mathbb{R}^n , and then to discs, where the proof of transitivity will be especially involved. Next, we study the non-compact Cantor set, where the combinatorics of sequences feels different, but still fits into our framework; in this case we will also be able to establish ordinary acyclicity. In the last section, we show that the auxiliary group of countably supported permutations of an uncountable set satisfies the criteria: Although it may seem unrelated, it will feature an important role in the proof itself.

3.1 Transformation groups of the line

Let $r \in \mathbb{N} \cup \{\infty\}$. In this section we prove that the groups $\operatorname{Diff}_{+}^{r}(\mathbb{R})$ are boundedly acyclic, our proof will depend on Theorem 2.2. This includes also $\operatorname{Homeo}_{+}(\mathbb{R})$ which corresponds to the case r = 0. In fact, we will work with the subgroup $G \leq \operatorname{Diff}_{+}^{r}(\mathbb{R})$ of elements that fix a neighborhood of $(-\infty, 0]$ pointwise. This can be seen as a subgroup of the group of homeomorphisms of $\mathbb{R}_{>0} \coloneqq (0, +\infty)$. The bounded acyclicity of $\operatorname{Diff}_{+}^{r}(\mathbb{R})$ will follow by some standard manipulations.

The objects that make up the G-poset of X-sequences to which Theorem 2.2 applies are fat points.

Definition 3.1. A fat point in $\mathbb{R}_{>0}$ is a germ at 0 of an orientation-preserving C^r -embedding $(-1,1) \to \mathbb{R}_{>0}$. The image of 0 is the core \dot{x} of the fat point x.

In order to define a poset \mathcal{F} of fat sequences and to deduce from Proposition 2.14 that this poset has the **W** property, it suffices to define relations \sqsubseteq and \simeq satisfying the condition of Subsection 2.2. We define $x \simeq y \Leftrightarrow \dot{x} = \dot{y}$ and $x \sqsubseteq y \Leftrightarrow \dot{x} \leq \dot{y}$. Thus in particular we see that the poset \mathcal{F} of increasing cofinal sequences of mutually inequivalent elements can be described concretely as the set of sequences $\mathbf{x} = (x_i)_{i \in \mathbb{N}^*}$ of fat points such that the cores $\dot{\mathbf{x}} = (\dot{x}_i)_{i \in \mathbb{N}^*}$ form a strictly increasing diverging sequence in $\mathbb{R}_{>0}$. For brevity we call them fat sequences in $\mathbb{R}_{>0}$.

The group G acts on fat points by composition: If $x: (-1,1) \to \mathbb{R}_{>0}$ is a fat point with core \dot{x} , then $g.x \coloneqq g \circ x$ is a fat point with core $g.\dot{x}$. Since the action of G on $\mathbb{R}_{>0}$ is orientation-preserving, the action on sequences of fat points preserves the conditions defining \mathcal{F} . This makes \mathcal{F} into a G-poset of sequences. We now have to verify Items 2 and 3 from Theorem 2.2:

Lemma 3.2. For every fat sequence $\mathbf{x} \in \mathcal{F}$, the stabilizer of \mathbf{x} in G is isomorphic to the power $\text{Diff}_c^r(\mathbb{R})^{\mathbb{N}}$ and thus it is boundedly acyclic. In case r = 0, it is also acyclic.

Proof. Let $\mathbf{x} = (x_i)_{i \in \mathbb{N}^*}$. An element $g \in G$ is in the stabilizer of \mathbf{x} if and only if it fixes a neighbourhood of \dot{x}_i for each $i \in \mathbb{N}^*$. Therefore the stabilizer is isomorphic to the product

$$\operatorname{Diff}_{c}^{r}((0,\dot{x}_{1})) \times \prod_{i \geq 2} \operatorname{Diff}_{c}^{r}((\dot{x}_{i-1},\dot{x}_{i}))$$

which is isomorphic to $\text{Diff}_{c}^{r}(\mathbb{R})^{\mathbb{N}}$. The bounded acyclicity now follows from Theorem 2.4. In case r = 0, the acyclicity follows from Proposition 2.11, Lemma 2.12 and Theorem 2.9.

Lemma 3.3. The action of G on \mathcal{F} is transitive.

Proof. Let $\mathbf{x} = (x_i)_{i \in \mathbb{N}^*}$ be the fat sequence whose *i*-th fat point is the germ of the embedding $(-1,1) \to (i-1,i+1) \subset \mathbb{R}_{>0}$ defined as translation by *i*, so $\dot{x}_i = i$. Let $\mathbf{y} = (y_i)_{i \in \mathbb{N}^*}$ be any other fat sequence. For each *i*, choose $\varepsilon_i \in (0, \frac{1}{2})$ to be such that the closed intervals $y_i([-\varepsilon_i, \varepsilon_i]) \subset \mathbb{R}_{>0}$ are pairwise disjoint. Then the composition

$$g_i \colon (i - \varepsilon_i, i + \varepsilon_i) \xrightarrow{x_i^{-1}} (-\varepsilon_i, \varepsilon_i) \xrightarrow{y_i} \mathbb{R}_{>0}$$

sends the fat point x_i to the fat point y_i . Interpolating via cutoff functions matching the regularity of G, we can construct $g \in G$ such that $g|_{(i-\varepsilon_i,i+\varepsilon_i)} = g_i$. Then $g.\mathbf{x} = \mathbf{y}$. \Box

The transitivity is particularly easy to show in this case, but will require some extra work in higher dimensions.

We have thus proved:

Theorem 3.4. The group G satisfies the criterion from Theorem 2.2, and thus is boundedly acyclic. In case r = 0, the group G also satisfies the criterion from Theorem 2.3, and thus is acyclic.

This allows to conclude the proof of Theorems A and B for the case of the line:

Proof of Theorems A and B for n = 1. Let \mathcal{G}_{\pm} be the group of germs at $\pm \infty$ of $\text{Diff}_{+}^{r}(\mathbb{R})$, which gives the short exact sequence

$$1 \to \operatorname{Diff}_{c}^{r}(\mathbb{R}) \to \operatorname{Diff}_{+}^{r}(\mathbb{R}) \to \mathcal{G}_{-} \times \mathcal{G}_{+} \to 1.$$

Since $\operatorname{Diff}_{c}^{r}(\mathbb{R})$ is boundedly acyclic by Theorem 2.4, the quotient induces an isomorphism in bounded cohomology, by Proposition 2.7. Since moreover $\mathcal{G}_{+} \cong \mathcal{G}_{-}$, it suffices to show that \mathcal{G}_{+} is boundedly acyclic. For this, we consider the short exact sequence

$$1 \to G \cap \operatorname{Diff}_{c}^{r}(\mathbb{R}) \to G \to \mathcal{G}_{+} \to 1.$$

Now $G \cap \operatorname{Diff}_{c}^{r}(\mathbb{R}) \cong \operatorname{Diff}_{c}^{r}(\mathbb{R})$ is boundedly acyclic, so once again the quotient induces an isomorphism in bounded cohomology. We have shown that G is boundedly acyclic, so \mathcal{G}_{+} is also boundedly acyclic, which concludes the proof.

We similarly obtain Theorem C.

Proof of Theorem C. As in the previous proof, we pass from the acyclicity of G and $\text{Homeo}_c(\mathbb{R})$ to the acyclicity of $\text{Homeo}_+(\mathbb{R})$ using the analogue of Proposition 2.7 in homology; namely, if $1 \to N \to G \to Q \to 1$ is a short exact sequence and N is acyclic, then the quotient $G \to Q$ induces an isomorphism $H_n(G;\mathbb{Z}) \cong H_n(Q,\mathbb{Z})$ for all $n \in \mathbb{N}$ (this follows from an application of the Lyndon–Hochschild–Serre spectral sequence [Bro94, Therem VII.6.3]).

3.2 Transformation groups of Euclidean space

Let $r \in \mathbb{N} \cup \{\infty\}$ and $n \in \mathbb{N}_{\geq 2}$. In this section, we prove Theorems A and B for the group $G := \text{Diff}_+^r(\mathbb{R}^n)$; this includes also $\text{Homeo}_+(\mathbb{R}^n)$, which corresponds to the case of r = 0. Our goal is to construct a poset of sequences of *fat spheres* (replacing fat points in higher dimensions), to which we can apply Theorem 2.2. The construction is close to the one for \mathbb{R} , however, it presents some additional difficulties.

The main thing that needs addressing is transitivity. In the case of \mathbb{R} , we clearly have transitivity on fat points, a basic starting point for transitivity on fat sequences. In our setting, we will need transitivity on germs of embeddings of (locally flat) spheres in \mathbb{R}^n . The fact that this action is transitive is now non-trivial, and would have to rely on the annulus Theorem [Kir69], which in C^r -category for r > 0 is not known in dimension 4. To amend this, we work with a poset that only allows for embeddings of a certain type:

Definition 3.5. The model fat sphere in \mathbb{R}^n is the germ at $S^{n-1} \times \{0\}$ of the (orientationpreserving, C^r) embedding $\Sigma: S^{n-1} \times (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}^n$ defined by $(x, \rho) \mapsto (1 + \rho)x$.

The core of Σ , denoted by $\dot{\Sigma}$, is the embedding $S^{n-1} \to \mathbb{R}^n$ obtained by restricting Σ to $S^{n-1} \times \{0\}$. We denote by $\Sigma_b \coloneqq B_1^n(0)$ and by $\Sigma_u \coloneqq \mathbb{R}^n \setminus \overline{B}_1^n(0)$; these are respectively the bounded and unbounded components of $\mathbb{R}^n \setminus \operatorname{im} \dot{\Sigma}$.

By abuse of notation, we will often identify an embedding with its germ. Given $g \in G$, we define $g.\Sigma := g \circ \Sigma$, which is an orientation-preserving C^r -embedding $S^{n-1} \times (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}^n$, and again we use $g.\Sigma$ to also denote the germ at $S^{n-1} \times \{0\}$ of this embedding. We will consider all such possibilities.

Definition 3.6. A fat sphere in \mathbb{R}^n is the germ of an orientation-preserving, C^r -embedding $S: S^{n-1} \times (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}^n$ that can be obtained as $g.\Sigma$ for some $g \in G$.

The core of S, denoted by \dot{S} , is the embedding $S^{n-1} \to \mathbb{R}^n$ obtained by restricting S to $S^{n-1} \times \{0\}$. Note that if $S = g.\Sigma$ then $\dot{S} = g.\dot{\Sigma}$.

We denote by S_b and S_u the bounded and unbounded components of $\mathbb{R}^n \setminus \text{im } \dot{S}$. If $S = g.\Sigma$, then $S_b = g.\Sigma_b$ and $S_u = g.\Sigma_u$.

Note that since we defined fat spheres starting from a model fat sphere, everything is welldefined directly, without any need to appeal to the Jordan–Brower or Schoenflies Theorems. Moreover, the set of fat spheres comes endowed with a natural action of G, which is transitive by definition.

The poset we will be working on will consist of certain sequences of fat spheres. Having a well-defined notion of bounded components, we will be able to formalize the idea of going to infinity.

Definition 3.7. We say that a fat sphere S in \mathbb{R}^n englobes a set $B \subset \mathbb{R}^n$ if B is contained in the bounded component S_b . We say that a sequence of fat spheres $(S_i)_{i \in \mathbb{N}^*}$ is concentric going to infinity if S_i englobes im \dot{S}_{i-1} , and every compact subset of \mathbb{R}^n is englobed by S_i for large enough i.

A fat sequence in \mathbb{R}^n is a collection $\mathbf{S} = (S_i)_{i \in \mathbb{N}^*}$ of fat spheres that are concentric going to infinity. We write $\mathbf{S} \preceq \mathbf{T}$ if \mathbf{S} is a subsequence of \mathbf{T} , and denote by \mathcal{F} the corresponding poset.

As in the case of the line, we can now define relations which allow us to deduce from Proposition 2.14 that the poset of fat sequences has the **W** property. We define $S \simeq T \Leftrightarrow \dot{S} = \dot{T}$ and $S \sqsubseteq T$ if either $\dot{S} = \dot{T}$ or T englobes \dot{S} .

As we did with fat spheres, it will be useful to define a basepoint in the set of fat sequences.

Definition 3.8. Set $\Sigma_1 := \Sigma$ to be the model fat sphere. Define Σ_i to be the image of Σ under the homothety with ratio *i*; note that the image of the core of Σ_i in \mathbb{R}^n is the sphere of radius *i* around the origin. The sequence $(\Sigma_i)_{i \in \mathbb{N}^*}$ is concentric going to infinity, and so it defines a fat sequence Σ , which we call the *model fat sequence*.

We now verify the Items 2 and 3 from Theorem 2.2.

Lemma 3.9. The stabilizer of the model fat sequence Σ in G is isomomorphic to $\text{Diff}_{c}^{r}(\mathbb{R}^{n}) \times \text{Diff}_{c}^{r}(S^{n-1} \times \mathbb{R})^{\mathbb{N}}$, and thus is boundedly acyclic.

Proof. An element $g \in G$ is in the stabilizer of Σ if and only if it fixes a neighborhood of $\operatorname{im} \dot{\Sigma}_i$ for each $i \in \mathbb{N}^*$. Denote by B_i the open ball of radius *i* around the origin, that is the bounded component defined by Σ_i . Then the stabilizer is isomorphic to the product of

$$\operatorname{Diff}_{\operatorname{c}}^{r}(B_{1}) \times \prod_{i \geq 2} \operatorname{Diff}_{\operatorname{c}}^{r}(B_{i} \setminus \overline{B}_{i-1}).$$

Since B_1 is C^r -diffeomorphic to \mathbb{R}^n , and $B_{i+1} \setminus B_i$ is C^r -diffeomorphic to $S^{n-1} \times \mathbb{R}$, we obtain the result. The last statement follows from Theorem 2.4.

The last (and hardest) part is transitivity.

Lemma 3.10. The action of G on \mathcal{F} is transitive.

Proof. Let $\Sigma = (\Sigma_i)_{i \in \mathbb{N}^*}$ be the model fat sequence. Let $\mathbf{S} = (S_i)_{i \in \mathbb{N}^*}$ be any other fat sequence. Since the action on the set of fat spheres is transitive by construction, for each *i* there exists $g_i \in G$ such that $g_i \cdot \Sigma_i = S_i$, and $\varepsilon_i \in (0, \frac{1}{4})$ such that the compact sets $\Sigma_i(S^{n-1} \times [-\varepsilon_i, \varepsilon_i])$ are pairwise disjoint, and the compact sets $g_i \circ \Sigma_i(S^{n-1} \times [-\varepsilon_i, \varepsilon_i])$ are also pairwise disjoint. We want to show that there exists $g \in G$ such that the restriction of g on $\Sigma_i(S^{n-1} \times [-\varepsilon_i, \varepsilon_i])$ coincides with the restriction of g_i . Then $g_i \cdot \Sigma = \mathbf{S}$.

Given that fat spheres, by definition, are in the orbit of a model fat sphere, each fat sphere comes with data of a global C^r -diffeomorphism in G. This makes arguing for transitivity easier, otherwise we would have to use the annulus theorem in the C^r category. To find gsuch that $g.\Sigma = \mathbf{S}$, we inductively construct $h_k \in G$ such that

- $h_k(\Sigma_i) = S_i$ for $i \le k$.
- $h_{k+1}^{-1} \circ h_k$ is the identity in an open neighborhood of the standard ball that bounds Σ_k .

Then g is the composition of all h_i 's. This is well defined since each $x \in \mathbb{R}^n$ eventually lies in the interior of $(\Sigma_k)_b$ for some big k and the second condition assures that $h_j(x) = h_k(x)$ for $j \ge k$.

For k = 1, we let $h_1 = g_1$. Now suppose that h_k is given. To construct h_{k+1} , we shall modify h_k by an element in G whose support does not intersect $(S_k)_b$ as follows. We have two different embeddings of the thin annulus $\sum_{k+1}(S^{n-1} \times [-\varepsilon_{k+1}, \varepsilon_{k+1}])$ into \mathbb{R}^n ; one is given by the global C^r -diffeomorphism g_{k+1} which gives the fat sphere S_{k+1} , and the other is given the global C^r -diffeomorphism h_k . Recall from Remark (ii) in the introduction that $\operatorname{Diff}_+^r(\mathbb{R}^n) = \operatorname{Diff}_\circ^r(\mathbb{R}^n)$ for all r. Therefore h_k and g_{k+1} are isotopic to the identity, and so these two embeddings are isotopic embeddings in \mathbb{R}^n . But we need to make sure that these two embeddings are isotopic away from Σ_k . To show this we need a version of the annulus theorem in our setting that follows from the isotopy extension theorem (for r > 0 see [Hir76, Section 8] and [Pal60] and for r = 0 see [EK71]).

Claim 3.11. Let W_i be the region between $\dot{S}_i = g_i \circ \Sigma_i(S^{n-1} \times \{0\})$ and $\dot{S}_{i+1} = g_{i+1} \circ \Sigma_{i+1}(S^{n-1} \times \{0\})$. The manifold W_i is C^r -diffeomorphic to $S^{n-1} \times [0,1]$.

As we shall see in the proof of the claim, the reason that this statement is easier than the annulus theorem is that the two spheres come with data of embeddings that are (ambiently) isotopic.

Proof of the claim. Since by radial dilation, we can send $\Sigma_i(S^{n-1} \times \{0\})$ to $\Sigma_{i+1}(S^{n-1} \times \{0\})$, we can assume that there exists $f \in G$ such that $f(\dot{S}_{i+1}) = \dot{S}_i$. To simplify the notation, we shall consider the following situation. Suppose S(r) is the standard sphere of radius raround the origin and for $f \in G$ we know that the sphere S' = f(S(1)) is in the interior of the unit ball. We want to prove that the region between S' and S(1) is C^r -diffeomorphic to $S^{n-1} \times [0, 1]$. Since f is isotopic to the identity, the sphere S' is isotopic to S(1/2). Since spheres are compact, the support of this isotopy can be contained inside S(k) for some big k. The isotopy extension theorem says that the isotopy between S' and S(1/2) can be extended to a path of C^r -diffeomorphisms of D(k), the ball of radius k whose boundary is S(k), that is compactly supported (i.e. is the identity near S(k)). So there is g in $\text{Diff}_c^r(\text{int}(D(k)))$ such that g(S') = S(1/2). Note that the region between S' and S(1) is C^r -diffeomorphic to the region between S' and S(k) since we just added the standard annulus between S(1) and S(k) to one of the sphere boundaries S(1). So it is enough to show that the region between S' and S(k) is C^r -diffeomorphic to the region between S(1/2) and S(k). Note that g maps the region between S' and S(k) to the region between S(1/2) and S(k) which is what we wanted. \Box

So by the claim, the region W_{k+1} between $g_{k+1} \circ \Sigma_{k+1}(S^{n-1} \times \{0\})$ and $g_k \circ \Sigma_k(S^{n-1} \times \{0\})$ and the region W'_{k+1} between $h_k \circ \Sigma_{k+1}(S^{n-1} \times \{0\})$ and $g_k \circ \Sigma_k(S^{n-1} \times \{0\})$ are C^r -diffeomorphic to an annulus $S^{n-1} \times [0, 1]$.

So we can find an isotopy from $g_{k+1} \circ \Sigma_{k+1}(S^{n-1} \times \{0\})$ to a sphere close to $g_k \circ \Sigma_k(S^{n-1} \times \{0\})$ by pushing along the annulus W_{k+1} . Similarly we can isotope $h_k \circ \Sigma_{k+1}(S^{n-1} \times \{0\})$ to the sphere close to $g_k \circ \Sigma_k(S^{n-1} \times \{0\})$ by pushing along W'_{k+1} . Therefore, $g_{k+1} \circ \Sigma_{k+1}(S^{n-1} \times [-\varepsilon_{k+1}, \varepsilon_{k+1}])$ and $h_k \circ \Sigma_{k+1}(S^{n-1} \times [-\varepsilon_{k+1}, \varepsilon_{k+1}])$ are isotopic via an isotopy that does not intersect the sphere $g_k \circ \Sigma_k(S^{n-1} \times \{0\})$.

Since the thin annuli $g_{k+1} \circ \Sigma_{k+1}(S^{n-1} \times [-\varepsilon_{k+1}, \varepsilon_{k+1}])$ and $h_k \circ \Sigma_{k+1}(S^{n-1} \times [-\varepsilon_{k+1}, \varepsilon_{k+1}])$ are compact subsets of \mathbb{R}^n , the isotopy between them is compactly supported. Therefore, by the isotopy extension theorem, there exists $g \in \text{Diff}_c^r(\text{int}(D(k)))$ for a large enough k such that

- the support of g does not intersect the disc that bounds the sphere $g_k \circ \Sigma_k(S^{n-1} \times \{0\})$.
- g sends $h_k \circ \Sigma_{k+1}(S^{n-1} \times [-\varepsilon_{k+1}, \varepsilon_{k+1}])$ to $g_{k+1} \circ \Sigma_{k+1}(S^{n-1} \times [-\varepsilon_{k+1}, \varepsilon_{k+1}])$.

So now we define h_{k+1} to be $g \circ h_k$ which completes the induction step and the proof of Lemma 3.10.

We can now conclude the proof of Theorems A and B:

Theorem 3.12. The group $\text{Diff}^r_+(\mathbb{R}^n)$ satisfies the criterion from Theorem 2.2, and thus is boundedly acyclic.

Remark 3.13. A similar strategy might work to study the bounded cohomology of $\operatorname{Diff}_{\circ}^{r}(M \times \mathbb{R}^{n})$ for a closed manifold M (note that we used the isotopy extension theorem for the proof of Lemma 3.10, which is why we need to explicitly require that diffeomorphisms are isotopic to the identity to hope for a generalization). Since we do not have any application in mind for the possible bounded acyclicity of $\operatorname{Diff}_{\circ}^{r}(M \times \mathbb{R}^{n})$, we do not pursue it in this paper.

3.3 Homeomorphism group of the disc

In this section, we tackle discs D^n , which we model as the unit disc in \mathbb{R}^n . In case n = 1, there is no difference between homeomorphisms of the disc and of the line, therefore we assume $n \ge 2$. Our goal is to show Theorem F: The group $G \coloneqq \text{Homeo}(D^n, \partial)$ of homeomorphisms of the disc fixing the boundary pointwise is boundedly acyclic (note that since the boundary is fixed, the orientation is automatically preserved). We will construct a poset analogous to the case of \mathbb{R}^n , to which Theorem 2.2 can be applied; here controlling the behaviour at the boundary will present additional difficulties, and is the reason we only consider homeomorphisms.

Definition 3.14. The model fat sphere in D^n is the germ at $S^{n-1} \times \{0\}$ of the orientationpreserving C^0 -embedding $\Sigma: S^{n-1} \times (-\frac{1}{4}, \frac{1}{4}) \to D^n$ defined by $(x, \rho) \mapsto (\frac{1}{2} + \rho)x$. The core of Σ , denoted by Σ is the embedding $S^{n-1} \to D^n$ obtained by restricting Σ to $S^{n-1} \times \{0\}$.

We denote by $\Sigma_{in} \coloneqq B^n_{\frac{1}{2}}(0)$ and $\Sigma_{out} \coloneqq D^n \setminus \overline{B}^n_{\frac{1}{2}}(0)$, these are respectively the *inner* and *outer* components of $D^n \setminus \operatorname{im} \dot{\Sigma}$.

Once again we use the same notation for embeddings and their germs, and denote $g.\Sigma \coloneqq g \circ \Sigma$.

Definition 3.15. A fat sphere in D^n is the germ of an orientation-preserving C^0 -embedding $S: S^{n-1} \times (-\frac{1}{4}, \frac{1}{4}) \to D^n$ that can be obtained as $g.\Sigma$ for some $g \in G$.

The core of S, denoted by \dot{S} , is the embedding $S^{n-1} \to D^n$ obtained by restricting S to $S^{n-1} \times \{0\}$. Note that if $S = g \cdot \Sigma$ then $\dot{S} = g \circ \dot{\Sigma}$.

We denote by S_{in} and S_{out} the *inner* and *outer* components of $D^n \setminus \operatorname{im} \dot{S}$.

As in the case of \mathbb{R}^n we considered sequences of fat spheres going to infinity, here we will consider sequences of fat spheres going to the boundary. The definition of \simeq and \sqsubseteq is exactly the same; note that the cofinality amounts to asking that every compact subset of the interior of D^n is englobed by S_i for large enough *i*.

In the case of \mathbb{R}^n we simply defined fat sequences, as sequences of fat spheres that are concentric going to infinity. In the case of discs, there is a further condition that will be needed in order to have continuity at the boundary.

Definition 3.16. Let $(S_i)_{i \in \mathbb{N}^*}$ be a cofinal increasing sequence of pairwise inequivalent fat spheres. We say that $(S_i)_{i \in \mathbb{N}^*}$ has trivial germ at the boundary if for every $x \in S^{n-1}$ we have

$$\lim_{i \to \infty} \dot{S}_i(x) = \partial(x);$$

where $\partial: S^{n-1} \to D^n$ is the canonical embedding of the boundary.

Since G is continuous at the boundary, and restricts to the identity, the action of G on sequences of fat spheres preserves the properties of being concentric going to the boundary, and having trivial germ at the boundary. We now have our suitable definition of fat sequences:

Definition 3.17. A fat sequence in D^n is a cofinal increasing sequence of pairwise inequivalent fat spheres with trivial germ at the boundary. We denote by \mathcal{F} the corresponding poset.

Set $\Sigma_1 := \Sigma$ to be the model fat sphere. Define Σ_i to be the image of Σ under a homeomorphism of D^n fixing the boundary that restricts to homothety by $2(1 - 2^{-i})$ on a neighbourhood of $\min \dot{\Sigma}_1$; note that the image of the core of Σ_i in D^n is the sphere of radius $1 - 2^{-i}$ around the origin. The sequence $(\Sigma_i)_{i \in \mathbb{N}^*}$ is concentric going to the boundary and has trivial germ at the boundary, so it defines a fat sequence Σ , which we call the *model fat sequence*.

We now have a G-poset of sequences \mathcal{F} , and we need to verify the three conditions from Theorem 2.2. The first one, the W property, follows from Proposition 2.14 in view of Remark 2.15. The stabilizer of the model fat sequence does not split as a direct product to which Theorem 2.4 applies. Still, we will be able to apply Theorem 2.6 to obtain its bounded acyclicity.

Lemma 3.18. The stabilizer of the model fat sequence Σ in G has commuting \mathbb{Z} -conjugates, and thus is boundedly acyclic.

Proof. An element $g \in G$ belongs to the stabilizer G_{Σ} if and only if it fixes a neighbourhood of $\dot{\Sigma}_i$ for each $i \in \mathbb{N}^*$. Denote by B_i the open ball of radius $(1 - 2^{-i})$ around the origin, that is the inner component defined by Σ_i . Then G_{Σ} is a subgroup of the product

Homeo_c(
$$B_1$$
) × $\prod_{i\geq 2}$ Homeo_c($B_i \setminus \overline{B}_{i-1}$),

which in turn can be identified with a subgroup of homeomorphisms of the interior of D^n . However, unlike the case of \mathbb{R}^n , this time G_{Σ} is a proper subgroup, since given a sequence of elements in $B_i \setminus \overline{B}_{i-1}$, their image under every element $g \in G$ must have the same limit in the boundary.

Let $H \leq G_{\Sigma}$ be a finitely generated subgroup. For $i \geq 2$, denote by H_i the projection of H onto $\operatorname{Homeo}_{c}(B_i \setminus \overline{B}_{i-1})$. This is a finitely generated subgroup of a group of compactly supported homeomorphisms of a non-compact space, so all of H_i is supported on some compact set. We can choose this to be a compact annulus $A_i \subset B_i \setminus \overline{B}_{i-1}$. More explicitly, we can find a compact interval $I_i \subset (1 - 2^{1-i}, 1 - 2^{-i})$ such that A_i is the set of all elements of norm contained in I_i , and every element of H_i fixes $(B_i \setminus \overline{B}_{i-1}) \setminus A_i$. Similarly, we denote by H_1 the projection of H on $\operatorname{Homeo}_{c}(B_1)$, which is supported on some compact set $A_1 \subset B_1$.

We will construct an element $t \in G_{\Sigma}$ with the property that $t^{p}(A_{i}) \cap A_{i} = \emptyset$ for all $p, i \in \mathbb{N}^{*}$. Let us conclude the proof assuming the existence of such an element. For every $p \in \mathbb{N}^{*}$, the sets $\cup_{i}A_{i}$ and $\cup_{i}t^{p}(A_{i})$ are disjoint. Indeed, by construction A_{i} is disjoint from $t^{p}(A_{i})$, and moreover $A_{i} \subset B_{i} \setminus \overline{B}_{i-1}$ is disjoint from $t^{p}(A_{j}) \subset B_{j} \setminus \overline{B}_{j-1}$ for all $j \neq i$, because $t \in G_{\Sigma}$ preserves the sets $B_{i} \setminus \overline{B}_{i-1}$. It follows that the group H, supported on $\cup_{i}A_{i}$, and the group $t^{p}Ht^{-p}$, supported on $\cup_{i}t^{p}(A_{i})$, commute. This concludes the proof that G_{Σ} has commuting \mathbb{Z} -conjugates; the last statement follows from Theorem 2.6.

It remains to construct t. For $i \geq 2$, let $I_i \coloneqq (a_i, b_i)$, and choose an element $f_i \in \text{Homeo}_c(1-2^{1-i}, 1-2^{-i})$ with the property that $f_i(a_i) > b_i$. Since f_i is orientation-preserving, it follows that $f_i^p(I_i) \cap I_i = \emptyset$ for all $p \in \mathbb{N}^*$. We then define an element $t_i \in \text{Homeo}_c(B_i \setminus \overline{B}_{i-1})$ in cylindrical coordinates by

$$S^{n-1} \times (1 - 2^{1-i}, 1 - 2^{-i}) \to S^{n-1} \times (1 - 2^{1-i}, 1 - 2^{-i}) \colon (x, \rho) \mapsto (x, f_i(\rho)).$$

The choice of f_i then implies that $t_i^p(A_i) \cap A_i = \emptyset$ for all $p \in \mathbb{N}^*$. We also choose an element $t_1 \in \operatorname{Homeo}_{c}(B_1)$ such that $t_1^p(A_1) \cap A_1 = \emptyset$ for all $p \in \mathbb{N}^*$.

We define t on the interior of D^n so that $t|_{B_1} = t_1$ and $t|_{B_i \setminus \overline{B}_{i-1}} = t_i$ for all $i \geq 2$, and set $t|_{\partial D^n} = \operatorname{id}|_{\partial D^n}$. Clearly t is a homeomorphism when restricted to the interior of D^n , which restricts to the identity on a neighbourhood of $\operatorname{im} \dot{S}_i$ for all $i \in \mathbb{N}^*$, and it satisfies $t^p(A_i) \cap A_i = \emptyset$ for all $p, i \in \mathbb{N}^*$. So it only remains to show that $t \in G$, i.e. t is continuous at the boundary. For this, let $(y_j)_{j \in \mathbb{N}^*}$ be a sequence of points in D^n such that $y_j \to y \in \partial D^n$. Since t restricts to the identity on the boundary, we may assume without loss of generality that y_j belongs to the interior of D^n , for all j. Up to removing finitely many terms, we may assume that y_j has norm at least $\frac{1}{2}$, which allows to use cylindrical coordinates to check continuity: Say $y_j = (x_j, \rho_j), y = (x, 1)$ and so $x_j \to x \in S^{n-1}$ and $\rho_j \to 1 \in [\frac{1}{2}, 1]$. Let $i_j \in \mathbb{N}^*$ be such that $\rho_j \in [1 - 2^{1-i_j}, 1 - 2^{-i_j}]$. Then $t(y_j) = t(x_j, \rho_j) = (x_j, f(\rho_j)) \in \{x_j\} \times [1 - 2^{1-i_j}, 1 - 2^{-i_j}]$. In particular $\rho_j \to 1$ implies $f(\rho_j) \to 1$, and so $t(y_j) \to y$, which concludes the proof.

We are left to show transitivity, which follows from the results in the appendix, in particular a controlled version of the annulus theorem. **Lemma 3.19.** The action of G on \mathcal{F} is transitive.

Proof. Combine Theorem A.6 with Corollary A.10.

We can now conclude the proof of Theorem F:

Theorem 3.20. The group Homeo (D^n, ∂) satisfies the criterion from Theorem 2.2, and thus is boundedly acyclic.

We deduce from this Corollary G:

Proof of Corollary G. We have a short exact sequence

 $1 \to \operatorname{Homeo}(D^n, \partial) \to \operatorname{Homeo}_+(D^n) \to \operatorname{Homeo}_+(S^{n-1}) \to 1,$

where the quotient is given by the restriction. It is indeed surjective, as can be seen by a cone construction. By Theorem F the kernel is boundedly acyclic, and so the conclusion follows from Proposition 2.7. \Box

The bounded cohomology of $\text{Homeo}_+(S^1)$, and thus of $\text{Homeo}_+(D^2)$, was fully computed in [MN23, Theorems 1.1 and 1.3]. In higher dimensions, $\text{Homeo}_+(S^{n-1})$ has been computed in low degrees [MN23, Theorems 1.8 and 1.10], and using Corollary G we obtain:

Corollary 3.21. For all $n \ge 3$, the bounded cohomology of Homeo₊ (D^n) vanishes in degrees 2 and 3. Moreover, the bounded cohomology of Homeo₊ (D^4) vanishes in degree 4.

3.4 Homeomorphism group of the non-compact Cantor set

The non-compact Cantor set **K** is uniquely characterized as being a space that is non-compact, locally compact, totally disconnected, metrizable and without isolated points. Concretely, it can be modelled as $K \times \mathbb{N}$, where \mathbb{N} is the set of natural numbers with the discrete topology, and K is a Cantor set, for example in the dyadic model $2^{\mathbb{N}}$. We fix a basepoint o of K, and throughout we will use this specific model for **K**.

Following [And22], we say that a *fat point* in **K** is the germ at *o* of an embedding $x: K \to \mathbf{K}$ with clopen image. The *core* of the embedding is $\dot{x} \coloneqq x(o)$. We are again in the abstract setting of Section 2.2 if we define $x \simeq y \Leftrightarrow \dot{x} = \dot{y}$ and this time take for \sqsubseteq the trivial relation $(x \sqsubseteq y \forall x, y)$. We consider the subposet of the resulting poset \mathcal{F} of sequences of pairwise inequivalent sequences defined by requiring that the sequence of cores be discrete. We call these sequences simply *fat sequences*. This subposet satisfies the requirements of Remark 2.15 and therefore \mathcal{F} satisfies the **W** property.

Lemma 3.22. The action of G on \mathcal{F} is transitive.

Proof. Let \mathbf{x} and \mathbf{y} be two fat sequences. Let U_n, V_n be compact open subsets of \mathbf{K} , such that $\dot{x}_i \in U_n$ if and only if i = n, and similarly $\dot{y}_i \in V_n$ if and only if i = n. By the topological characterization of the Cantor set and the non-compact Cantor set, each U_i, V_i is homeomorphic to K, and each of their complements is homeomorphic to \mathbf{K} . Thus, up to choosing larger U_i, V_i , we may assume that $(U_n)_{n \in \mathbb{N}^*}$ and $(V_i)_{i \in \mathbb{N}^*}$ form partitions of \mathbf{K} by countably many compact open subsets.

Now, by [And22, Lemma 3.3.16], for each *n* there exists a homeomorphism $h_n: U_n \to V_n$ that sends x_n to y_n . The unique map $h: \mathbf{K} \to \mathbf{K}$ that agrees with h_n on U_n is then a homeomorphism that sends \mathbf{x} to \mathbf{y} .

Lemma 3.23. The stabilizers for the action of G on \mathcal{F} are binate, therefore they are acyclic and boundedly acyclic.

Proof. We consider the simplest fat sequence, namely $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ where $x_i \colon K \to \mathbf{K} = K \times \mathbb{N}$ is defined by $x_n(p) = (p, n)$. Let H be a finitely generated subgroup of the stabilizer $G_{\mathbf{x}}$. Then for each n there exists a clopen subset U_n of $o \in K$ such that H fixes pointwise $U_n \times \{n\}$, which is a compact open subset of \mathbf{K} . Let S_n denote the complement of U_n in K, and let V_n denote a proper compact open subset of U_n that contains o. Finally, we let $W_n \coloneqq U_n \setminus V_n$, which is again a compact open subset of K. Now S_n, U_n, V_n are all homeomorphic to K, and so we have a decomposition:

$$\mathbf{K} = \left(\bigcup_{n} V_{n} \times \{n\}\right) \bigsqcup \left(\bigcup_{n} W_{n} \times \{n\}\right) \bigsqcup \left(\bigcup_{n} S_{n} \times \{n\}\right) \eqqcolon \mathbf{K}_{V} \sqcup \mathbf{K}_{W} \sqcup \mathbf{K}_{S}$$

Each of $\mathbf{K}_V, \mathbf{K}_W, \mathbf{K}_S$ is homeomorphic to \mathbf{K} , the group H is supported on \mathbf{K}_S , and the fat sequence \mathbf{x} is contained in \mathbf{K}_V .

By the topological characterization, a countably infinite disjoint union of non-compact Cantor sets is homeomorphic to a non-compact Cantor set. Therefore there is a homeomorphism $\mathbf{K}_W \cong \mathbf{K}_0 \times (\mathbb{Z} \setminus \{0\})$, for some non-compact Cantor set \mathbf{K}_0 . This allows to define a homeomorphism $\mathbf{K}_W \sqcup \mathbf{K}_S \to \mathbf{K}_0 \times \mathbb{Z}$, that restricts to a homeomorphism $\mathbf{K}_S \to \mathbf{K}_0 \times \{0\}$, which gives a new decomposition

$$\mathbf{K} = \mathbf{K}_V \sqcup (\mathbf{K}_\mathbf{0} \times \mathbb{Z}) \,.$$

The group H is supported on $\mathbf{K}_0 \times \{0\}$.

We now define a homeomorphism $t: \mathbf{K} \to \mathbf{K}$ as follows:

$$t(x) \coloneqq \begin{cases} x \text{ if } x \in \mathbf{K}_V; \\ (x_0, i+1) \text{ if } x = (x_0, i) \in \mathbf{K}_0 \times \{i\} \subset \mathbf{K}_0 \times \mathbb{Z}. \end{cases}$$

Then $t \in G$, and since t restricts to the identity on \mathbf{K}_V , it belongs to the stabilizer $G_{\mathbf{x}}$.

It remains to define the homomorphism $\psi \colon H \to G$. We set

$$\psi(h)(x) \coloneqq \begin{cases} x \text{ if } x \in \mathbf{K}_V \sqcup (\mathbf{K}_0 \times \mathbb{Z}_{\leq 0}) \\ t^i h t^{-i}(x) \text{ if } x \in \mathbf{K}_0 \times \{i\} \subset \mathbf{K}_0 \times \mathbb{Z}_{> 0}. \end{cases}$$

Since *H* is supported on $\mathbf{K}_0 \times \{0\}$ and $\psi(H)$ is supported on $\mathbf{K}_0 \times \mathbb{Z}_{>0}$, we have $[H, \psi(H)] = 1$. Moreover, $t^{-1}\psi(h)t = h\psi(h)$ for all $h \in H$, which follows by a direct computation. This proves that $G_{\mathbf{x}}$ is binate, and we conclude using Theorems 2.9 and 2.10.

This concludes the proof of Theorem E:

Theorem 3.24. The group Homeo(\mathbf{K}) satisfies the criteria from Theorems 2.2 and 2.3, and thus is acyclic and boundedly acyclic.

3.5 Countably supported bijections of an uncountable set

Our last example is of a different flavour from the previous ones, but it will be crucial in the proofs of Theorems 2.2 and 2.3.

Definition 3.25. Let X be an uncountable set. We let G be the group of bijections of X that have countable support.

In contrast to the previous results in this section, we will not use the criteria from Theorems 2.2 and 2.3 to prove that G is [boundedly] acyclic; this is because the [bounded] acyclicity of G will be used in the proof. Instead, we establish the [bounded] acyclicity G directly:

Proposition 3.26. The group G is binate, therefore it is acyclic and boundedly acyclic.

Proof. Let $H \leq G$ be a finitely generated subgroup. Because each element of H has countable support, there exists a countably infinite subset $Y_0 \subset X$ such that H is supported on Y_0 . Now let $Y_i, i \in \mathbb{Z} \setminus \{0\}$ be countably infinite subsets of X that are disjoint from Y_0 and from each other. Fix a bijection $\pi_i \colon Y_{i-1} \to Y_i$ for all $i \in \mathbb{Z}$. We define

$$t(x) \coloneqq \begin{cases} \pi_i(x) \text{ if } x \in Y_{i-1}; \\ x \text{ otherwise.} \end{cases}$$

Then t is supported on the union of the Y_i , which is countable, so $t \in G$. Next, for each $i \ge 1$ let $\Pi_i := \pi_i \circ \cdots \circ \pi_1 \colon Y_0 \to Y_i$ and define

$$\psi(h)(x) \coloneqq \begin{cases} \Pi_i h \Pi_i^{-1}(x) \text{ if } x \in Y_i; \\ x \text{ otherwise.} \end{cases}$$

Then $\psi(h)$ is supported on $\bigcup_{i\geq 1}Y_i$, so it belongs to G. It is easy to see that $[H, \psi(H)] = 1$ and that $t^{-1}\psi(h)t = h\psi(h)$. This shows that G is binate, the last statements then follow from Theorems 2.9 and 2.10.

We now show that G satisfies the conditions for our two criteria.

Proposition 3.27. The group G satisfies the criteria from Theorems 2.2 and 2.3.

Proof. Let \mathcal{X} be the set of sequences of pairwise distinct elements in X. Then \mathcal{X} is a G-poset of sequences, which satisfies the \mathbf{W} property by Proposition 2.14 by taking \simeq to be equality and \sqsubseteq to be the trivial relation. Because sequences are countable, the action is transitive. The stabilizer $G_{\mathbf{x}}$ of a sequence \mathbf{x} with image Y is the subgroup of G supported on $X \setminus Y$; since this set has the same cardinality as X, it follows that $G_{\mathbf{x}}$ is isomorphic to G, and thus is acyclic and boundedly acyclic by Proposition 3.26.

4 Proof of the criterion

In this section, we prove the criteria from Theorems 2.2 and 2.3. Most of the section will be devoted to the proof of Theorem 2.2, which yields the bounded acyclicity results that are the main subject of this paper, we will show how to adapt the arguments to prove Theorem 2.3 in Subsection 4.5.

We start in Subsection 4.1 with some background on the bounded cohomology of semisimplicial sets. This will include a result from [MN23] which allows to compute the bounded cohomology of a group via an action on a boundedly acyclic semisimplicial set with boundedly acyclic stabilizers. In Subsection 4.2 we examine in detail the case of posets, and see how the W property implies bounded acyclicity; this will require the proof of several basic results about the cohomology of posets to be made uniform, in the spirit of [MM85, KS23].

In Subsection 4.3 we examine the orbit complex for the action of a group as in Theorem 2.2. This is quite straightforward, but it is perhaps the most surprising part of the argument: It will imply that a group as in Theorem 2.2 has the same bounded cohomology as the monoid $\text{Emb}_{<}(\mathbb{N}^*)$ of order-preserving embeddings of \mathbb{N}^* . Finally, in Subsection 4.4 we show that $\text{Emb}_{<}(\mathbb{N}^*)$ is boundedly acyclic, this will be proved indirectly, by appealing to the bounded acyclicity of the group of countably supported bijections of an uncountable set, which we treated in Subsection 3.5.

4.1 Bounded cohomology of semisimplicial sets

Let X_{\bullet} be a semisimplicial set. Its *bounded cohomology* $H^n_b(X_{\bullet})$ is the cohomology of the complex

$$0 \to \ell^{\infty}(X_0) \to \ell^{\infty}(X_1) \to \ell^{\infty}(X_2) \to \cdots$$

where the coboundary operators are defined as the duals of the face maps of X_{\bullet} . The coefficients are always understood to be \mathbb{R} unless specified otherwise. We say that X_{\bullet} is boundedly acyclic if $H_b^n(X_{\bullet}) = 0$ for all n > 0.

If G is a discrete group and X_{\bullet} is the nerve of G, this defines the *bounded cohomology* of G, denoted by $H_b^n(G)$. More explicitly, we have $X_{\bullet} = G^{\bullet}$, with face maps defined as

$$d_i: X_n \to X_{n-1}$$

$$d_0(g_1, \dots, g_n) = (g_2, \dots, g_n);$$

$$d_i(g_1, \dots, g_n) = (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n), \quad i \in \{1, \dots, n-1\};$$

$$d_n(g_1, \dots, g_n) = (g_1, \dots, g_{n-1}).$$

This definition can be directly generalized to define the bounded cohomology of a discrete semigroup (or monoid), which will appear in Section 4.4. In both cases, we say that the (semi)group G is boundedly acyclic if $H_b^n(G) = 0$ for all n > 0.

In the case of groups, the homogeneous resolution identifies the bounded cohomology of G with the cohomology of the subcomplex of invariants:

$$0 \to \ell^{\infty}(G)^G \to \ell^{\infty}(G^2)^G \to \ell^{\infty}(G^3)^G \to \cdots$$

In this context, we have a semisimplicial set X_{\bullet} , which is nothing but the full simplex on G with the usual face maps, endowed with a simplicial action of G; and the subcomplex of invariants is the complex computing the bounded cohomology of the semisimplicial set of orbits $G \setminus X_{\bullet}$. Simplicial actions of G can calculate its bounded cohomology in much greater generality:

Theorem 4.1 ([MN23, Theorem 3.3], see also [MR23]). Let G be a group acting on a boundedly acyclic and connected semisimplicial set X_{\bullet} . Suppose that, for all $p \in \mathbb{N}$:

- 1. The stabilizer of each point in X_p is boundedly acyclic;
- 2. There are only finitely many isomorphism types of such stabilizers.

Then there is an isomorphism $H_b^n(G) \cong H_b^n(G \setminus X_{\bullet})$.

In order to prove bounded acyclicity of the relevant semisimplicial sets, we will use a framework, introduced by Matsumoto and Morita [MM85], and recently developed further by Kastenholz and Sroka [KS23]. Given a semisimplicial set X_{\bullet} , we let $\tilde{C}(X_{\bullet})$ denote the corresponding augmented simplicial chain complex, endowed with the ℓ^1 -norm with respect to the standard basis.

Definition 4.2. Let $f, g: X_{\bullet} \to Y_{\bullet}$ be two simplicial maps between semisimplicial sets. A bounded homotopy between f and g is a sequence of bounded linear maps $\{h_p: \widetilde{C}_p(X_{\bullet}) \to \widetilde{C}_{p+1}(Y_{\bullet})\}_{p\geq 0}$ such that

$$d_1 h_0 = f_0 - g_0$$

and for all $p \in \mathbb{N}^*$:

$$d_{p+1}h_p + h_{p-1}d_p = f_p - g_p.$$

If a bounded homotopy exists, we say that f and g are boundedly homotopic.

Two semisimplicial sets are boundedly homotopy equivalent if there exist simplicial maps $f: X_{\bullet} \to Y_{\bullet}$ and $g: Y_{\bullet} \to X_{\bullet}$ such that fg and gf are both boundedly homotopic to the identity.

Now we consider the notion of uniform acyclicity for semisimplicial sets, which is what we will use to prove [bounded] acyclicity in the sequel.

Definition 4.3. We say that a semisimplicial set X_{\bullet} is uniformly acyclic if the following holds. For every $p \ge 0$ there exists a constant $K_p^{X_{\bullet}} > 0$ such that for every cycle $z \in \widetilde{C}_p(X_{\bullet})$ there exists $c \in \widetilde{C}_{p+1}(X_{\bullet})$ such that $d_{p+1}(c) = z$ and $||c|| \le K_p^{X_{\bullet}} \cdot ||z||$.

This is relevant to us because of the following result, already implicit in [MM85]:

Lemma 4.4 ([KS23, Corollary 7.4]). If X_{\bullet} is uniformly acyclic, then it is acyclic and boundedly acyclic.

In order to check uniform acyclicity of a semisimplicial set, it can be useful to restrict to its finite sub-semisimplicial sets.

Definition 4.5. Let $f: X_{\bullet} \to Y_{\bullet}$ be a simplicial map. We say that f is uniformly acyclic if the following holds. For every $p \ge 0$ there exists a constant $K_p^f > 0$ such that for every cycle $z \in \widetilde{C}_p(X_{\bullet})$ there exists $c \in \widetilde{C}_{p+1}(Y_{\bullet})$ such that $d_{p+1}(c) = f_p(z)$ and $||c|| \le K_p^f \cdot ||z||$.

Lemma 4.6. Let Y_{\bullet} be a semisimplicial set. Suppose that there exist constants $K_p > 0$ such that every inclusion of a finite sub-semisimplicial set $\iota: X_{\bullet} \hookrightarrow Y_{\bullet}$ is uniformly acyclic with constants $K_p^{\iota} \leq K_p$. Then Y_{\bullet} is uniformly acyclic, with constants bounded by K_p .

Proof. This is a direct consequence of the fact that uniform acyclicity is defined in terms of simplicial cycles, which are supported on finitely many simplices. \Box

The easiest case of a uniformly acyclic semisimplicial set is that of a cone.

Lemma 4.7 ([KS23, Corollary 7.4]). A simplicial cone CX_{\bullet} over the semisimplicial set X_{\bullet} is uniformly acyclic, and the constant $K_p^{CX_{\bullet}}$ can be chosen to be equal to 1.

We rephrase this in the relative setting of uniformly acyclic maps:

Lemma 4.8. Let $f: X_{\bullet} \to Y_{\bullet}$ be a simplicial map that factors through the simplicial cone CX_{\bullet} . Then f is uniformly acyclic, and the constant K_p^f can be chosen to be equal to 1.

These notions are compatible with bounded homotopy equivalences.

Lemma 4.9. Let $f, g: X_{\bullet} \to Y_{\bullet}$ be two simplicial maps that are boundedly homotopic via the bounded homotopy $\{h_p\}_{p\geq 0}$. If f is uniformly acyclic, then so is g, with constants $K_p^g \leq K_p^f + ||h_p||$.

Proof. Let $z \in \widetilde{C}_p(X_{\bullet})$ be a reduced cycle. Since f is uniformly acyclic, there exists $c \in \widetilde{C}_{p+1}(Y_{\bullet})$ such that $d_{p+1}(c) = f_p(z)$ and $||c|| \leq K_p^f \cdot ||z||$. Now

$$g_p(z) = d_{p+1}(h_p(z)) + h_{p-1}(d_p(z)) - f_p(z) = d_{p+1}(h_p(z) - c),$$

where we used that z is a reduced cycle and the definition of c. Moreover

$$|h_p(z) - c|| \le ||h_p|| \cdot ||z|| + K_p^f \cdot ||z||.$$

4.2 Bounded cohomology of posets

Every poset (\mathcal{P}, \preceq) has an associated semisimplicial set: Its *nerve*, where a *p*-simplex is a chain $\{x_0 \preceq x_1 \preceq \cdots \preceq x_p\}$, and the face maps correspond to deleting an element from the chain. Let us stress that this definition allows for degenerate simplices, which is why we are working with semisimplicial sets instead of simplicial complexes.

We will denote posets by \mathcal{P} , and the corresponding semisimplicial sets by \mathcal{P}_{\bullet} . Posets are more amenable to proofs of acylicity than general semisimplicial sets [Bjö95], and some of those techniques can be adapted to bounded cohomology [KS23]. In this section, we prove bounded analogues of some basic lemmas in this context, which will lead to a useful combinatorial criterion for uniform acyclicity of a poset. We start with the *acyclic carrier lemma* [Bjö95, Lemma 10.1].

Definition 4.10. Let X_{\bullet}, Y_{\bullet} be semisimplicial sets. A *carrier* is a map Φ which to each simplex $\sigma \in X_p$ associates a sub-semisimplicial set $\Phi(\sigma)$ of Y_{\bullet} , and such that $\Phi(\sigma) \subseteq \Phi(\tau)$ whenever $\sigma \subseteq \tau$. A carrier Φ is said to be *uniformly acyclic* if for all $\sigma \in X_{\bullet}$, the semisimplicial set $\Phi(\sigma)$ is uniformly acyclic, and the corresponding constants $K_p^{\Phi(\sigma)}$ in degree p are bounded by constants K_p independent of σ .

A carrier for a simplicial map $f: X_{\bullet} \to Y_{\bullet}$ is a carrier Φ such that $f(\sigma) \in \Phi(\sigma)$ for all $\sigma \in X_{\bullet}$.

Lemma 4.11. Let $f, g: X_{\bullet} \to Y_{\bullet}$ be two simplicial maps. Suppose that f, g admit a common uniformly acyclic carrier Φ . Then f and g are boundedly homotopic. More precisely, if the constants K_p witness the uniform acyclicity of Φ , and satisfy $1 \leq K_p \leq K_{p+1}$, then the bounded homotopy $\{h_p\}_{p\geq 0}$ satisfies $||h_p|| \leq 2(p+1)(K_p)^p$.

The proof will show that a sharper bound is possible, even unconditionally, but this is all we will need for the proofs.

Proof. We construct the bounded homotopy $\{h_p\}_{p\geq 0}$ by induction. For $\sigma \in X_0$, both $f_0(\sigma)$ and $g_0(\sigma)$ represent an element in $\Phi(\sigma)_0$, and so $z := (f_0 - g_0)(\sigma) \in \widetilde{C}_0(\Phi(\sigma))$ is a reduced cycle. By the uniform acyclicity assumption, there exists a chain $c \in \widetilde{C}_1(\Phi(\sigma))$ such that $d_1(c) = z$ and $||c|| \leq K_0 \cdot ||z|| \leq 2K_0$. We define $h_0(\sigma) \coloneqq c$, extend it linearly to a map $h_0 \colon \widetilde{C}_0(X_{\bullet}) \to \widetilde{C}_1(Y_{\bullet})$ and notice that $d_1h_0 = f_0 - g_0$ and $||h_0|| \leq 2K_0$.

Now let $p \in \mathbb{N}^*$ and suppose by induction that for all $0 \leq i < p$ bounded maps $h_i \colon \widehat{C}_i(X_{\bullet}) \to \widetilde{C}(Y_{\bullet})$ have been defined so that the homotopy identity holds, and moreover $h_i(\sigma)$ is supported on $\Phi(\sigma)$ for all $\sigma \in X_i$. Fix $\sigma \in X_p$, and set $z \coloneqq (f_p - g_p - h_{p-1}d_p)(\sigma)$. Then the induction hypothesis shows that $d_p(z) = 0$ and z is supported on $\Phi(\sigma)$, so $z \in \widetilde{C}_p(\Phi(\sigma))$ is a cycle. By the uniform acyclicity assumption, there exists $c \in \widetilde{C}_p(\Phi(\sigma))$ such that $d_{p+1}(c) = z$ and $\|c\| \leq K_p \cdot \|z\| \leq K_p \cdot (2 + \|h_p\|)$. We define $h_p(\sigma) \coloneqq c$, extend it linearly to a map $h_p \colon \widetilde{C}_p(X_{\bullet}) \to \widetilde{C}_{p+1}(Y_{\bullet})$, and notice that $d_{p+1}h_p - h_{p-1}d_p = f_p - g_p$, and

$$||h_p|| \le K_p(2 + ||h_{p-1}||) \le 2K_p + K_p 2p(K^{p-1})^{p-1} \le 2(p+1)(K_p)^p.$$

This can be used to prove a bounded version of Quillen's Order Homotopy Theorem [Bjö95, Theorem 10.11]:

Theorem 4.12. Let X_{\bullet} be a semisimplicial set, and let \mathcal{P} be a poset. Let $f, g: X_{\bullet} \to \mathcal{P}_{\bullet}$ be simplicial maps. If $f(x) \leq g(x)$ for every $x \in X_0$, then f and g are boundedly homotopic via a bounded homotopy $\{h_p\}_{p\geq 0}$ with $||h_p|| \leq 2(p+1)$.

Proof. For each simplex $\sigma \in X_{\bullet}$ define $\Phi(\sigma)$ to be the sub-semisimplicial set of \mathcal{P}_{\bullet} spanned by $f(\sigma) \cup g(\sigma)$. Every vertex of $\Phi(\sigma)$ dominates the minimal element of $f(\sigma)$, so the corresponding semisimplicial set is a cone, which is uniformly acyclic and whose constant can be chosen to be equal to 1 in all degrees (Lemma 4.7). The result then follows from Lemma 4.11. \Box

We will only use the following consequence of Theorem 4.12 (the \mathbf{W} property is given in Definition 2.1).

Corollary 4.13. Let \mathcal{P} be a poset with the W property. Then \mathcal{P}_{\bullet} is uniformly acyclic.

Proof. We will show that \mathcal{P}_{\bullet} satisfies the criterion of Lemma 4.6. That is, for every finite subposet $\mathcal{Q} \subset \mathcal{P}$, we will show that the inclusion $\iota: \mathcal{Q}_{\bullet} \to \mathcal{P}_{\bullet}$ is uniformly acyclic with constants independent of \mathcal{Q} . Let x_1, \ldots, x_k be the minimal elements of \mathcal{Q} , and let $y_I, I \subseteq$ $\{1, \ldots, k\}$ be the elements of \mathcal{P} whose existence is ensured by the **W** property. Define the map $f: \mathcal{Q} \to \mathcal{P}$ by sending $x \in \mathcal{Q}$ to y_{I_x} , where $I_x = \{i \in \{1, \ldots, k\} \mid x_i \leq x\}$. Item 1 of Definition 2.1 states that $y_I \leq y_J$ if $I \subseteq J$, and this implies that f is order-preserving. Item 2 states that $y_{I_x} \leq x$, and so $f(x) \leq x$, for all $x \in \mathcal{Q}$. Therefore Theorem 4.12 applies and shows that $f: \mathcal{Q}_{\bullet} \to \mathcal{P}_{\bullet}$ is boundedly homotopic to ι , via a bounded homotopy $\{h_p\}_{p\geq 0}$ with $\|h_p\| \leq 2(p+1)$. However, every element of $f(\mathcal{Q})$ is dominated by $y_{\{1,\ldots,k\}}$, so $f(\mathcal{Q}_{\bullet})$ is contained in a cone. Lemma 4.8 then shows that f is uniformly acyclic with $K_p^f \leq 1$ for all p. Finally, we apply Lemma 4.9 to deduce that ι is uniformly acyclic with $K_p^f \leq K_p^f + \|h_p\| \leq 1 + 2(p+1)$. Since this bound is independent of \mathcal{Q} , we conclude from Lemma 4.6 that \mathcal{P} is uniformly acyclic.

4.3 The orbit complex

Let us now begin the proof of Theorem 2.2. Recall the setup: The group G is acting on a set X, and there is a G-invariant poset of sequences in X such that the action of G on \mathcal{X} is transitive, and has boundedly acyclic stabilizers. Moreover, \mathcal{X} satisfies the \mathbf{W} property, which implies that the semisimplicial set \mathcal{X}_{\bullet} is boundedly acyclic, by Corollary 4.13. Therefore the

action of G on \mathcal{X}_{\bullet} satisfies the hypotheses of Theorem 4.1. It remains to show that the orbit complex $G \setminus \mathcal{X}_{\bullet}$ is boundedly acyclic.

Consider a simplex $\{\mathbf{x}_0 \leq \mathbf{x}_1 \leq \cdots \leq \mathbf{x}_p\}$ for $p \geq 0$. Define $\iota_i \colon \mathbf{x}_i \to \mathbb{N}^*$ to be the index map, which associates to a fat point in \mathbf{x}_i its index in the sequence. Now \mathbf{x}_{p-1} is uniquely determined by the indices that describe it as a subsequence of \mathbf{x}_p . Therefore, once \mathbf{x}_p has been given, \mathbf{x}_{p-1} can be described in terms of an *order-preserving embedding* $\eta_p \colon \mathbb{N}^* \hookrightarrow \mathbb{N}^*$, such that the following diagram commutes:



Proceeding inductively in the same way, we define order-preserving embeddings $\eta_i \colon \mathbb{N}^* \to \mathbb{N}^*$ that intertwine the inclusion of \mathbf{x}_{i-1} into \mathbf{x}_i with the order isomorphisms ι_{i-1}, ι_i , so that the following diagram commutes:



We denote by $\operatorname{Emb}_{<}(\mathbb{N}^*)$ the monoid of order-preserving embeddings of \mathbb{N}^* . We write multiplication in $\operatorname{Emb}_{<}(\mathbb{N}^*)$ in terms of the right action of $\operatorname{Emb}_{<}(\mathbb{N}^*)$ on \mathbb{N}^* , so given $\eta, \zeta \in \operatorname{Emb}_{<}(\mathbb{N}^*)$, their product $\eta\zeta$ is the embedding given by applying η , then ζ . Let $I_p: \mathcal{X}_p \to \operatorname{Emb}_{<}(\mathbb{N}^*)^p$ be the map that associates to a *p*-simplex $\{\mathbf{x}_0 \preceq \cdots \preceq \mathbf{x}_p\}$ the *p*-tuple (η_1, \ldots, η_p) .

Lemma 4.14. The map $I_p: \mathcal{F}_p \to \operatorname{Emb}_{<}(\mathbb{N}^*)^p$ induces a bijection $G \setminus \mathcal{X}_p \to \operatorname{Emb}_{<}(\mathbb{N}^*)^p$, which we also denote by I_p .

Proof. Given a *p*-simplex $\{\mathbf{x}_0 \leq \cdots \leq \mathbf{x}_p\}$, the action of *G* intertwines the maps ι_i , i.e. the following diagram commutes:



Since this is true for each i, we see that I_p is invariant under the action of G.

Now let $\{\mathbf{x}_0 \leq \cdots \leq \mathbf{x}_p\}$ and $\{\mathbf{y}_0 \leq \cdots \leq \mathbf{y}_p\}$ be two *p*-simplices with the same image under I_p . Let $g \in G$ be an element such that $g.\mathbf{x}_p = \mathbf{y}_p$; this exists by Lemma 3.3. Since the embedding $\eta_p \colon \mathbb{N}^* \to \mathbb{N}^*$ is valid for both $g.\mathbf{x}_{p-1} \to g.\mathbf{x}_p$ and $\mathbf{y}_{p-1} \to \mathbf{y}_p$, the indices that define \mathbf{y}_{p-1} as a subsequence of \mathbf{y}_p are the same that describe $g.\mathbf{x}_{p-1}$ as a subsequence of $g.\mathbf{x}_p = \mathbf{y}_p$. This shows that $g.\mathbf{x}_{p-1} = \mathbf{y}_{p-1}$, and repeating the same argument by induction we conclude that these two *p*-simplices are in the same *G*-orbit. This produces an identification of the simplices of the orbit complex. We now show that this extends to an isomorphism:

Lemma 4.15. The maps $I_p: G \setminus \mathcal{X}_p \to \operatorname{Emb}_{<}(\mathbb{N}^*)^p$ define an isomorphism of semisimplicial sets $G \setminus \mathcal{X}_{\bullet} \to \operatorname{Emb}_{<}(\mathbb{N})^{\bullet}$ from the orbit complex to the nerve of $\operatorname{Emb}_{<}(\mathbb{N})^{\bullet}$. In particular, there is an isomorphism in bounded cohomology $H_b^n(\operatorname{Emb}_{<}(\mathbb{N}^*)) \to H_b^n(G \setminus \mathcal{X}_{\bullet})$.

Proof. We already know that I_p is a bijective map sending simplices to simplices. We need to show that these maps intertwine the face maps. Let $\{\mathbf{x}_0 \leq \cdots \leq \mathbf{x}_p\}$ be a simplex, whose image under I_p is (η_1, \ldots, η_p) . Removing \mathbf{x}_i yields the following commutative diagrams. For 0 < i < p:

While for i = 0 and i = p we are simply removing the first and last columns, respectively. We now compute the effect on face maps, starting with i = 0:

$$I_{p-1}(d_p^0(\{\mathbf{x}_0 \leq \cdots \leq \mathbf{x}_p\})) = I_{p-1}(\{\mathbf{x}_1 \leq \cdots \leq \mathbf{x}_p\})$$
$$= (\eta_2, \dots, \eta_p)$$
$$= d_p^0(\eta_1, \dots, \eta_p) = d_p^0(I_p(\{\mathbf{x}_0 \leq \cdots \leq \mathbf{x}_p\})).$$

Analogously:

$$I_{p-1}(d_p^n(\{\mathbf{x}_0 \leq \cdots \leq \mathbf{x}_p\})) = d_p^n(I_p(\{\mathbf{x}_0 \leq \cdots \leq \mathbf{x}_p\})).$$

And for 0 < i < p:

$$I_{p-1}(d_p^i(\{\mathbf{x}_0 \leq \cdots \leq \mathbf{x}_p\})) = I_{p-1}(\{\mathbf{x}_1 \leq \cdots \leq \mathbf{x}_{i-1} \leq \mathbf{x}_{i+1} \leq \cdots \leq \mathbf{x}_p\})$$
$$= (\eta_2, \dots, \eta_{i-1}, \eta_i \eta_{i+1}, \eta_{i+2}, \dots, \eta_p)$$
$$= d_p^i(\eta_1, \dots, \eta_p) = d_p^i(I_p(\{\mathbf{x}_0 \leq \cdots \leq \mathbf{x}_p\})).$$

The discussion in this subsection is summarized in the following result:

Proposition 4.16. Let G be a groups satisfying the criterion from Theorem 2.2. Then there is an isomorphism in bounded cohomology $H_b^n(\text{Emb}_{<}(\mathbb{N}^*)) \cong H_b^n(G)$.

Proof. Consider the action of G on \mathcal{X}_{\bullet} . This semisimplicial set is uniformly acyclic by Corollary 4.13, so it is acyclic and boundedly acyclic by Lemma 4.4. The stabilizer of a simplex $\{\mathbf{x}_0 \leq \cdots \leq \mathbf{x}_p\}$ coincides with the stabilizer of \mathbf{x}_p , which is boundedly acyclic, and all such stabilizers are conjugate by transitivity. So Theorem 4.1 applies and gives an isomorphism $H_b^n(G) \cong H_b^n(G \setminus \mathcal{X}_{\bullet})$. By Lemma 4.14, the latter is in turn isomorphic to $H_b^n(\operatorname{Emb}_{<}(\mathbb{N}^*))$.

4.4 Bounded cohomology of the embedding monoid

It remains to show that the embedding monoid $\text{Emb}_{<}(\mathbb{N}^*)$ is boundedly acyclic. This will be proved indirectly, using the group of countably supported permutations on an uncountable set, from Subsection 3.5.

Theorem 4.17. The monoid $\text{Emb}_{<}(\mathbb{N}^*)$ is boundedly acyclic.

Proof. By Proposition 4.16, it suffices to exhibit a boundedly acyclic group that satisfies the criterion from Theorem 2.2. This is the group of countably supported permutations of an uncountable set, by Propositions 3.26 and 3.27. \Box

Proof of Theorem 2.2. Combine Proposition 4.16 and Theorem 4.17.

4.5 Acyclicity

A completely analogous proof holds in the case of acyclicity. As before, we split the proof into two statements.

Proposition 4.18. Let G be a groups satisfying the criterion from Theorem 2.3. Then there is an isomorphism in homology $H_n(\text{Emb}_{\leq}(\mathbb{N}^*);\mathbb{Z}) \cong H_n(G;\mathbb{Z})$.

Proof. The analogue of Theorem 4.1 holds for homology with integral coefficients, in our case. Indeed, by [Bro94, Chapter VII.7], there is a spectral sequence

$$E_{p,q}^{1} = \bigoplus_{\sigma \in \Sigma_{p}} H_{p}(G_{\sigma}; \mathbb{Z}_{\sigma}) \Rightarrow H_{p+q}(G; \mathbb{Z}),$$

where G_{σ} is the stabilizer of the simplex σ , and \mathbb{Z}_{σ} is the module \mathbb{Z} with the $\mathbb{Z}[G_{\sigma}]$ -module structure defined by the sign of the permutation that an element of G_{σ} induces on the set of vertices of σ . In our setting, the semisimplicial sets in questions are posets, so G_{σ} fixes the vertices of σ pointwise, and thus we recover the isomorphism $H_p(G;\mathbb{Z}) \cong H_p(G \setminus X_{\bullet})$. Since the orbit complex is isomorphic to the nerve of $\text{Emb}_{<}(\mathbb{N}^*)$ (Lemma 4.15), we conclude. \Box

Theorem 4.19. The monoid $\text{Emb}_{\leq}(\mathbb{N}^*)$ is acyclic.

Proof. As in the proof of Theorem 4.17, we appeal to Proposition 4.18 and the acyclicity of the group of countably supported bijections of an uncountable set (Proposition 3.26). \Box

Proof of Theorem 2.3. Combine Proposition 4.18 and Theorem 4.19.

Remark 4.20. As we have seen in the previous section, actions on posets of sequences with *boundedly acyclic* stabilizers are easier to construct than actions with *acyclic* stabilizers. However, let us point out that the isotropy spectral sequence that we used in the proof of Proposition 4.18 only requires the acyclicity of the semisimplicial set being acted on, which in our setting follows from Corollary 4.13. A more careful analysis of this spectral sequence for the groups in question, aided by the acyclicity of the orbit complex established via Theorem 4.19, could shed some light on their homology, even when we know that acyclicity does not hold.

4.6 An alternative approach

The final step in the proof of the criteria is the [bounded] acyclicity of the monoid $\text{Emb}_{<}(\mathbb{N}^*)$, which we achieved by appealing to the auxiliary group of countably supported permutations of an uncountable set. This is the most streamlined approach that we were able to find, which has the peculiarity of leaving the world of transformation groups. In this section, we sketch an

alternative approach that is more technically involved, but more thematically coherent: We show how to prove that the group $G \leq \text{Homeo}_+(\mathbb{R})$ of elements that fix a neighbourhood of $(-\infty, 0]$ pointwise, is [boundedly] acyclic directly. Then the same arguments as above can be applied to deduce the [bounded] acyclicity of $\text{Emb}_{<}(\mathbb{N}^*)$, concluding the proof of the criterion. We only deal with bounded acyclicity in this sketch, but the proof could be carried through analogously for acyclicity (alternatively, recall that the acyclicity of $\text{Homeo}_+(\mathbb{R})$ is already known [McD80], and the acyclicity of the group G follows from the same manipulations we did in the proof of Theorem C).

For the scope of this argument, we call a *fat sequence* in $\mathbb{R}_{>0}$ a collection $\mathbf{x} = (x_i)_{i \in \mathbb{N}^*}$ of fat points in $\mathbb{R}_{>0}$ such that the cores $\dot{\mathbf{x}} = (\dot{x}_i)_{i \in \mathbb{N}}$ are pairwse distinct and form a discrete subset of $\mathbb{R}_{>0}$. This is a similar definition as before, except that the sequence of cores does not need to be increasing, although the discreteness assumption implies that it is diverging.

The crucial difference is in the order relation. We denote $\mathbf{x} \leq \mathbf{y}$ if \mathbf{x} is a *subset* of \mathbf{y} : We do not require that the inclusion preserves the order given by the indices in the sequences. Now the relation \leq is not antisymmetric anymore, and so we are dealing with a preordered set (*proset* for short), and not a poset. We denote by \mathcal{F} this proset. The group G acts on \mathcal{F} by preserving the preorder, and thus it also acts on the nerve \mathcal{F}_{\bullet} , which is a semisimplicial set in the usual way.

The uniform acyclicity of \mathcal{F}_{\bullet} now requires modifications of the arguments in Subsection 4.2, where posets are replaced with prosets, cones are replaced with nerves of prosets admitting a global maximum, and the **W** property has to be modified appropriately. By the discreteness and disjointness assumption, the stabilizer of a fat sequences is isomorphic to a product of copies of Homeo_c(\mathbb{R}), and thus it is boundedly acyclic.

It remains to study the orbit complex $G \setminus \mathcal{F}_{\bullet}$. Since G preserves the natural order on $\mathbb{R}_{>0}$, but fat sequences are not required to follow this order, the action of G on \mathcal{F} is not transitive and accordingly the orbit complex will be larger. We fix a basepoint in \mathcal{F} , for example the fat sequence with cores the natural numbers and trivial germs. Every other fat sequence with the same cores is in a different G-orbit, and is parametrized by a permutation of \mathbb{N}^* . This permutation is a complete invariant of the G-orbit, since G can send any fat sequence to a unique fat sequence with cores the natural numbers.

Now, to a *p*-simplex $\{\mathbf{x}_0 \leq \cdots \leq \mathbf{x}_p\}$ we can associate both the underlying permutations of \mathbb{N}^* associated to each \mathbf{x}_i , and also *p* embeddings $\mathbb{N}^* \to \mathbb{N}^*$ which describe the way \mathbf{x}_{i-1} sits inside \mathbf{x}_i . The key difference is that now these embeddings need not be order-preserving. Unraveling the definitions, and describing explicitly the face maps as we did in Subsection 4.3, we see that the orbit complex is isomorphic to the nerve of the category $\text{Emb}(\mathbb{N}^*) \times \Pi$. Here $\text{Emb}(\mathbb{N}^*)$ is the monoid of all self-embeddings of \mathbb{N}^* , seen as a category with one object; Π is a category with objects indexed by permutations of \mathbb{N}^* and a unique morphism between any two objects; and \times denotes the product of categories.

We can now study the bounded cohomology of this nerve via a spectral sequence, noticing that the complex $\ell^{\infty}(\text{Nerve}_{\bullet}(\text{Emb}(\mathbb{N}^*) \times \Pi))$ is the total complex of the double complex $\ell^{\infty}(\text{Nerve}_p(\text{Emb}(\mathbb{N}^*)); \ell^{\infty}(\text{Nerve}_q(\Pi)))$. The bounded acyclicity will then follow from the fact that both $\text{Nerve}_{\bullet}(\Pi)$ and $\text{Emb}(\mathbb{N}^*)$ are boundedly acyclic.

First, Nerve_•(Π) is boundedly acyclic by virtue of being the nerve of a proset with a global maximum, which as we saw before is a replacement for cones in the context of prosets.

As for the monoid $\text{Emb}(\mathbb{N}^*)$, one can prove this similarly to an argument sketched (for acyclicity) by de la Harpe and McDuff [dlHM83, Appendix 2], who in turn attribute it to

unpublished work of Quillen [Qui10]. The fundamental notion is that of *semiconjugacy*: Two endomorphisms $f, g: \operatorname{Emb}(\mathbb{N}^*) \to \operatorname{Emb}(\mathbb{N}^*)$ are semiconjugate if there exists $\mu \in \operatorname{Emb}(\mathbb{N}^*)$ such that $\mu f(x) = f(x)\mu$ for all $x \in \operatorname{Emb}(\mathbb{N}^*)$. The key property of semiconjugacy is that semiconjugate maps are homotopic. In fact they are even boundedly homotopic, and this can be proven analogously: See e.g. [Mon01, Lemma 8.2.7] for a proof in the case of groups, where semiconjugacy is the more familiar notion of conjugacy. Then one shows that there exists an endomorphism of $\operatorname{Emb}(\mathbb{N}^*)$ that is semiconjugate to both the identity endomorphism, and to the constant endomorphism at the identity embedding [dlHM83, Appendix 2]. This implies that the identity and a constant map induce the same map in bounded cohomology, which is only possible if $\operatorname{Emb}(\mathbb{N}^*)$ is boundedly acyclic.

It is worth pointing out that, even in this approach, the bounded acyclicity of embedding monoids plays a crucial role in the proof.

5 Unboundedness of the topological Pontryagin classes

Recall from the introduction that the map

 $B\mathrm{Homeo}^{\delta}(M) \to B\mathrm{Homeo}(M),$

is an acyclic map when M is a compact manifold or the interior of a compact manifold with boundary ([McD80, Section 2, Theorem 2.5]). The same holds for Homeo₊(M). If we combine this result for $M = \mathbb{R}^n$ and the non-vanishing result of Galatius and Randal-Williams [GRW23] we obtain that

$$\mathbb{R}[e, p_1, p_2, \dots] \to H^*(B\mathrm{Homeo}^{\delta}_+(\mathbb{R}^{2n}); \mathbb{R}),$$

is injective for $2n \ge 6$, and in the odd-dimensional case, it follows from [GRW23, Corollary 1.2] that

$$\mathbb{R}[p_1, p_2, \dots] \to H^*(B\mathrm{Homeo}^{\delta}_+(\mathbb{R}^{2n+1}); \mathbb{R}),$$

is injective for $2n+1 \ge 7$. In dimension 2 and 3, we obtain that $\mathbb{R}[p_1] \to H^*(B\mathrm{Homeo}^{\delta}_+(\mathbb{R}^3);\mathbb{R})$ and $\mathbb{R}[e] \to H^*(B\mathrm{Homeo}^{\delta}_+(\mathbb{R}^2);\mathbb{R})$ are isomorphisms. Given that $\mathrm{Homeo}_+(\mathbb{R}^n)$ is a boundedly acyclic group by Theorem A, all the nontrivial classes in $H^*(B\mathrm{Homeo}^{\delta}_+(\mathbb{R}^n);\mathbb{R})$ are unbounded. This gives a proof of Corollary K. In general, for oriented C^r -flat \mathbb{R}^n -bundles for any regularity r, we have the following result.

Theorem 5.1. For oriented C^r -flat \mathbb{R}^n -bundles of any regularity $r \ge 0$, the polynomials in terms of the Euler class (when n is even) and Pontryagin classes p_i that are of degree less than n + 2 are all unbounded.

Proof. By a deep result of Segal [Seg78, Propositions 1.3 and 3.1], we know that there is a natural map

$$BDiff_{+}^{r,\delta}(\mathbb{R}^n) \to BS\Gamma_n^r,$$

which is a homology isomorphism, where $BS\Gamma_n^r$ is the classifying space of Haefliger structures for codimension n foliations that are transversely oriented, see [Seg78, Section 1] for more details. There is also a map

$$\nu \colon BS\Gamma_n^r \to BGL_n(\mathbb{R})_+,$$

which classifies oriented normal bundles to the codimension n foliations where $\operatorname{GL}_n(\mathbb{R})_+$ is the group invertible matrices with positive determinant. For all regularities, it is known that the map ν is at least (n + 1)-connected, see [Hae71, Remark 1, Section II.6]. Hence, in particular, the induced map

$$H^*(B\mathrm{GL}_n(\mathbb{R})_+) \to H^*(B\mathrm{Diff}_+^{r,\delta}(\mathbb{R}^n)),$$

is an isomorphism for $* \leq n$. So the polynomials of the Euler class, when n is even, and the Pontryagin classes p_i are nontrivial for all r when the degree of the polynomial is less than n+2. Given that $\text{Diff}_+^r(\mathbb{R}^n)$ is a bounded acyclic group by Theorem B, all these nontrivial classes in $H^*(B\text{Diff}_+^{r,\delta}(\mathbb{R}^n))$ are unbounded.

To prove the unboundedness of Pontryagin classes for C^0 -flat sphere bundles, we need the following lemma.

Lemma 5.2. Let st: Homeo(\mathbb{R}^n) \rightarrow Homeo(\mathbb{R}^{n+1}) be the stabilization map that sends a homeomorphism f to $f \times id$. Let α be the following composition

$$\operatorname{Homeo}(\mathbb{R}^n) \to \operatorname{Homeo}(S^n) \to \operatorname{Homeo}(D^{n+1}) \to \operatorname{Homeo}(\operatorname{int}(D^{n+1})) \xrightarrow{\cong} \operatorname{Homeo}(\mathbb{R}^{n+1})$$

where the first map is induced by the one-point compactification, the second map is induced by taking a cone over the origin, the third map is the restriction to the interior of the disc which is homeomorphic to \mathbb{R}^{n+1} . By choosing an orientation preserving homomeorphism between $\operatorname{int}(D^{n+1})$ and \mathbb{R}^{n+1} , we obtain a continuous homomorphism α : Homeo $(\mathbb{R}^n) \to \operatorname{Homeo}(\mathbb{R}^{n+1})$ which is independent of the choice up to homotopy. Then also the maps st and α are homotopic through homomorphisms.

Proof. First note that different identifications are induced by conjugation by an orientation preserving homomorphism of \mathbb{R}^n . Since conjugation by elements that are isotopic to the identity induces a homomorphism on Homeo(\mathbb{R}^n) that is homotopic to the identity, the homomorphism α is well defined up to homotopy.

We identify S^n as the one-point compactification of \mathbb{R}^n and choose ∞_n , the point at infinity, as the base point of S^n . Let CS^n be the cone $S^n \times [0,1]/S^n \times \{0\}$ and let $\dot{C}S^n$ be the open cone $S^n \times [0,1)/S^n \times \{0\}$. Let also $SS^n = CS^n/S^n \times \{1\}$ be the unreduced suspension of S^n . Let us denote the path in CS^n and SS^n which is the image of $\infty_n \times [0,1]$ by l. We identify S^{n+1} with SS^n and the point ∞_{n+1} with the image of $S^n \times \{1\}$.

For a closed subset Y of a space X, we write Homeo(X, Y) to denote those homomorphisms of X that restrict to the identity on Y. We can identify $\text{Homeo}(\mathbb{R}^n)$ with $\text{Homeo}(S^n, \infty_n)$. So we consider the following model of the stabilization map

 $\operatorname{Homeo}(S^n, \infty_n) \to \operatorname{Homeo}(S^n \times [0, 1], \infty_n \times [0, 1]) \to \operatorname{Homeo}(SS^n, l) \to \operatorname{Homeo}(S^{n+1}, \infty_{n+1}).$

On the other hand, we can identify $\operatorname{int}(D^{n+1})$ with $\dot{C}S^n$ such that the line l is identified with the line from the origin to $\infty_n \in S^n$. With this identification, we have a model of the homomorphism α as the following composition

$$\operatorname{Homeo}(S^n, \infty_n) \to \operatorname{Homeo}(\dot{C}S^n, l) \to \operatorname{Homeo}(SS^n, l) \to \operatorname{Homeo}(S^{n+1}, \infty_{n+1}),$$

which is the same homomorphism as the above model for the stabilization map.

Using this lemma we can prove the unboundedness of certain Pontryagin classes for flat sphere bundles as follows.

Proof of Theorem M. First we shall see the nonvanishing of topological Pontryagin classes in $H^*(B\text{Homeo}^{\delta}(S^n); \mathbb{R})$. By Lemma 5.2 and the Thurston–McDuff homology isomorphism [McD80], we know that the composition

$$H^*(B\mathrm{Homeo}^{\delta}(\mathbb{R}^{n+1});\mathbb{R}) \to H^*(B\mathrm{Homeo}^{\delta}(S^n);\mathbb{R}) \to H^*(B\mathrm{Homeo}^{\delta}(\mathbb{R}^n);\mathbb{R}),$$

is induced by the stabilization map. By Galatius–Randal-Williams [GRW23] we know that

$$\mathbb{R}[p_1, p_2, \dots] \hookrightarrow H^*(B\mathrm{Homeo}^{\delta}(\mathbb{R}^n); \mathbb{R}),$$

is injective for all $n \ge 6$. Therefore, we also have the injective map

$$\mathbb{R}[p_1, p_2, \dots] \hookrightarrow H^*(B\mathrm{Homeo}^{\delta}(S^n); \mathbb{R})$$

Now consider the following commutative diagram

$$\begin{array}{c} H^*(B\mathrm{Homeo}^{\delta}(S^{n+1});\mathbb{R}) \to H^*(B\mathrm{Homeo}^{\delta}(\mathbb{R}^{n+1});\mathbb{R}) \longrightarrow H^*(B\mathrm{Homeo}^{\delta}(S^n);\mathbb{R}) \\ \uparrow & \uparrow \\ H^*_b(B\mathrm{Homeo}^{\delta}(S^{n+1});\mathbb{R}) \to H^*_b(B\mathrm{Homeo}^{\delta}(\mathbb{R}^{n+1});\mathbb{R}) \longrightarrow H^*_b(B\mathrm{Homeo}^{\delta}(S^n);\mathbb{R}). \end{array}$$

By Theorem A, we know that $H_b^*(B\text{Homeo}^{\delta}(\mathbb{R}^{n+1});\mathbb{R}) = 0$ in positive degrees. A diagram chase implies that all the classes $\mathbb{R}[p_1, p_2, \dots]$ in $H^*(B\text{Homeo}^{\delta}(S^{n+1});\mathbb{R})$ are unbounded.

For
$$4 \leq n \leq 6$$
, we can use iterations of these composition of maps to get the diagram $H^*(B\operatorname{Homeo}^{\delta}(S^n);\mathbb{R}) \longrightarrow H^*(B\operatorname{Homeo}^{\delta}(\mathbb{R}^n);\mathbb{R}) \longrightarrow \cdots \longrightarrow H^*(B\operatorname{Homeo}^{\delta}(S^3);\mathbb{R})$
 $\uparrow \qquad \uparrow \qquad \uparrow$
 $H^*_b(B\operatorname{Homeo}^{\delta}(S^n);\mathbb{R}) \longrightarrow H^*_b(B\operatorname{Homeo}^{\delta}(\mathbb{R}^n);\mathbb{R}) \longrightarrow \cdots \longrightarrow H^*_b(B\operatorname{Homeo}^{\delta}(S^3);\mathbb{R}).$

Recall that $Homeo(S^3) \simeq O(4)$, so the Thurston–McDuff theorem implies that

$$\mathbb{R}[p_1, p_2] \to H^*(B\mathrm{Homeo}^{\delta}(S^3); \mathbb{R})$$

is an isomorphism. Therefore, the same argument as before implies that the classes $\mathbb{R}[p_1, p_2] \hookrightarrow H^*(B\mathrm{Homeo}^{\delta}(S^n); \mathbb{R})$ for $4 \leq n \leq 6$ are unbounded.

If we continue the same diagram up $H^*(B\text{Homeo}^{\delta}(S^1);\mathbb{R})$, we obtain that the classes $\mathbb{R}[p_1] \hookrightarrow H^*(B\text{Homeo}^{\delta}(S^n);\mathbb{R})$ for n = 2, 3 are also unbounded. \Box

6 Further results and questions

We conclude by mentioning other settings in which Theorem 2.2 applies, with minor variations of the arguments of Section 3. On the other hand, when trying to tackle other transformation groups of interest, we encounter technical difficulties, and we leave these as open questions. Let us stress that even though Theorem 2.2 may not apply for some of these cases, the general proof strategy seems fruitful beyond its specific scope. For example, our proof shows that the \mathbf{W} property is nothing but a simple combinatorial condition to ensure bounded acyclicity of posets of sequences, and more sophisticated criteria could work in more general contexts. Moreover, our statement of Theorem 4.1 made the convenient assumption that the stabilizers belong to finitely many isomorphism classes, which was enough for our purposes, but the principle works more generally, as long as the *vanishing modulus* of the stabilizers is controlled [MN23, Theorem 3.3].

6.1 Results

We start by considering other transformation groups of the line. Let $PL^{\omega}_{+}(\mathbb{R})$ denote the group of orientation preserving piecewise linear homeomorphisms of \mathbb{R} with a discrete set of breakpoints. Subgroups of this group are central in the theory of left orderable groups, and include lifts of piecewise linear groups of the circle [GS87], as well as the first examples of finitely generated simple left orderable groups [HL19, MBT20]. The bounded acyclicity of these simple groups was already proven in [CFFLM23, Example 5.41]; the proof relies on their quasi-periodic nature, and in particular it does not generalize to the whole of $PL^{\omega}_{+}(\mathbb{R})$. With a very minor variation on the arguments in Subsection 3.1, we obtain:

Theorem 6.1. The group $PL^{\omega}_{+}(\mathbb{R})$ is boundedly acyclic.

Proof. Any subgroup of Homeo_c(ℝ) acting without global fixed points has commuting Zconjugates [CFFLM23, Corollary 5.32]. Therefore, by the same argument as in the proof of Theorem A for Homeo₊(ℝ), it suffices to prove bounded acyclicity of the subgroup $G < PL_+^{\omega}(\mathbb{R})$ of elements that fix pointwise $(-\infty, 0]$. We consider the poset \mathcal{X} of strictly increasing diverging sequences of *points* in $\mathbb{R}_{\geq 0}$. The action of G is transitive, and this poset satisfies the \mathbb{W} property, by Proposition 2.14 (where $x \simeq y \Leftrightarrow x = y$ and $x \sqsubseteq y \Leftrightarrow x \leq y$). The stabilizer of a sequence is isomorphic to $PL_+(I)^{\mathbb{N}}$, where I = [0, 1] and $PL_+(I)$ denotes the group of piecewise linear homeomorphisms of I with a *finite* set of breakpoints. Because germs at fixed points are abelian, the commutator subgroup of $PL_+(I)^{\mathbb{N}}$ is a product of subgroups of Homeo_c(ℝ) acting without global fixed points, so it has commuting \mathbb{Z} -conjugates by [CFFLM23, Corollary 5.32] and [CFFLM23, Lemma 4.8]. It then follows from Theorem 2.6 and coamenability [Mon01, Proposition 8.6.6] that $PL_+(I)^{\mathbb{N}}$ is boundedly acyclic. Thus Theorem 2.2 applies, and we conclude. \square

The proof generalizes seamlessly to the piecewise projective case $PP^{\omega}_{+}(\mathbb{R})$; again, subgroups of this group have provided very important examples in group theory, particularly in the realm of non-amenable groups without free subgroups [Mon13, LM16]:

Theorem 6.2. The group $PP^{\omega}_{+}(\mathbb{R})$ is boundedly acyclic.

Proof. The proof of Theorem 6.1 applies, with the only modification that germs at fixed points are not abelian but metabelian. \Box

Next, we consider point stabilizers in transformation groups of Euclidean spaces. Let $\operatorname{Diff}_+^r(\mathbb{R}^n, 0)$ denote the stabilizer of 0 in $\operatorname{Diff}_+^r(\mathbb{R}^n)$. For $r \geq 1$, the derivative at 0 gives a surjective homomorphism $\operatorname{Diff}_+^r(\mathbb{R}^n, 0) \to \operatorname{GL}_n(\mathbb{R})_+$, which admits a section: The inclusion $\operatorname{GL}_n(\mathbb{R})_+ \to \operatorname{Diff}_+^r(\mathbb{R}^n, 0)$. This implies that the bounded cohomology of $\operatorname{GL}_n(\mathbb{R})_+$ embeds into the bounded cohomology of $\operatorname{Diff}_+^r(\mathbb{R}^n)$, which is therefore not boundedly acyclic [Gro82]. On the other hand, we prove:

Theorem 6.3. For all $n \in \mathbb{N}^*$, the group Homeo₊($\mathbb{R}^n, 0$) is boundedly acyclic.

Proof. We identify Homeo₊($\mathbb{R}^n, 0$) \cong Homeo₊($S^n, \{0, \infty\}$), which gives a short exact sequence

 $1 \to \operatorname{Homeo}(S^n, \operatorname{near}\{0, \infty\}) \to \operatorname{Homeo}_+(S^n, \{0, \infty\}) \to \mathcal{G}_0 \times \mathcal{G}_\infty \to 1.$

Here Homeo(S^n , near{ $(0, \infty)$ }) denotes the group of homeomorphisms of S^n that fix pointwise a neighbourhood of { $(0, \infty)$ }, and \mathcal{G}_0 and \mathcal{G}_∞ denote the germs at 0 and ∞ , respectively. By Theorem 2.4 the group Homeo $(S^n, \text{near}\{0, \infty\}) \cong \text{Homeo}_c(S^{n-1} \times \mathbb{R})$ is boundedly acyclic, so Proposition 2.7 reduces the statement to the bounded acyclicity of $\mathcal{G}_0 \times \mathcal{G}_\infty$. Since $\mathcal{G}_0 \cong \mathcal{G}_\infty$, it suffices to show that \mathcal{G}_∞ is boundedly acyclic. We also have a short exact sequence

$$1 \to \operatorname{Homeo}_{c}(\mathbb{R}^{n}) \to \operatorname{Homeo}_{+}(\mathbb{R}^{n}) \to \mathcal{G}_{\infty} \to 1,$$

so Theorem 2.4, Theorem A and Proposition 2.7 imply that \mathcal{G}_{∞} is boundedly acyclic. \Box

In a similar way, we can deal with compact annuli, which were mentioned in the introduction. We denote by Homeo_o $(S^n \times [0, 1], \partial)$ the identity component of the group of homeomorphisms of the annulus fixing both boundary components pointwise.

Proof of Theorem H. Consider the short exact sequence

$$1 \to \operatorname{Homeo}_{c,\circ}(S^n \times (0,1)) \to \operatorname{Homeo}_{\circ}(S^n \times [0,1],\partial) \to \mathcal{G}_0 \times \mathcal{G}_1 \to 1.$$

Here \mathcal{G}_0 and \mathcal{G}_1 denote the germs at the boundary component $S^n \times \{0\}$ and $S^n \times \{1\}$, respectively. As in the proof of Theorem 6.3, it suffices to show that $\mathcal{G}_0 \cong \mathcal{G}_1$ is boundedly acyclic. This follows from Theorem F, by considering the short exact sequence

$$1 \to \operatorname{Homeo}_{c}(B^{n+1}) \to \operatorname{Homeo}(D^{n}, \partial) \to \mathcal{G}_{1} \to 1.$$

Corollary 6.4. For all $n \in \mathbb{N}^*$, the restriction $\operatorname{Homeo}_{\circ}(S^n \times [0,1]) \to \operatorname{Homeo}_{\circ}(S^n) \times \operatorname{Homeo}_{\circ}(S^n)$ induces an isomorphism in bounded cohomology in all degrees.

Proof. Consider the short exact sequence

$$1 \to \operatorname{Homeo}_{\circ}(S^n \times [0,1], \partial) \to \operatorname{Homeo}_{\circ}(S^n \times [0,1]) \to \operatorname{Homeo}_{\circ}(S^n) \times \operatorname{Homeo}_{\circ}(S^n) \to 1$$

and apply Theorem H and Proposition 2.7.

As in Corollary 3.21, using [MN23, Theorems 1.8 and 1.10] we can now compute the bounded cohomology of Homeo_o($S^n \times [0,1]$) for $n \ge 2$, in low degrees (note that in this case we also need a degree-wise version of Proposition 2.7, see [MN23, Proposition 2.4]).

Corollary 6.5. For all $n \ge 2$, the bounded cohomology of Homeo_o $(S^n \times [0,1])$ vanishes in degrees 2 and 3. The bounded cohomology of Homeo_o $(S^3 \times [0,1])$ vanishes in degree 4.

In case n = 1 we can say more.

Proof of Corollary I. The first statement was already proved in Corollary 6.4 for r = 0. For $r \geq 1$, it follows from a similar argument, given the bounded acyclicity of $\operatorname{Diff}_{\circ}^{r}(D^{2}, \partial)$, which is part of the proof that the restriction $\operatorname{Diff}_{\circ}^{r}(D^{2}) \to \operatorname{Diff}_{\circ}^{r}(S^{1})$ induces an isomorphism in bounded cohomology [MN23, Theorem 1.4]. For the second statement, [MN23, Theorem 1.2] gives an isomorphism of graded \mathbb{R} -algebras $H_{b}^{*}(\operatorname{Diff}_{\circ}^{r}(S^{1})) \cong \mathbb{R}[\mathscr{E}]$. The computation for the product follows from a suitable Künneth argument: See [FFLM23, Proposition 6.13] for a proof of the analogous statement for the product of two copies of Thompson's group T (the statement is conditional on the bounded acyclicity of Thompson's group F, which was proven in [Mon22]). Alternatively, the desired statement can be recovered from [FFLM23, Proposition 6.13], by using the fact that there is an embedding $T \to \operatorname{Diff}_{\circ}^{r}(S^{1})$ which is semiconjugate to the standard embedding [GS87], and therefore induces an isomorphism in bounded cohomology in all degrees.

Using the computation in degree 2, we recover a theorem of Militon. For a group G, we denote by Q(G) the space of homogeneous quasimorphisms $G \to \mathbb{R}$, and by $H^1(G)$ the subspace of homomorphisms. There is a natural homogeneous quasimorphisms on $\text{Diff}^r_{\circ}(S^1 \times [0, 1])$, called the *torsion number* [Mil14]. Given $f \in \text{Diff}^r_{\circ}(S^1 \times [0, 1])$, let $F \in \text{Diff}^r_{\circ}(\mathbb{R} \times [0, 1])$ be a lift to the universal cover. The restriction to the two boundary components defines maps F_0, F_1 in the universal cover of $\text{Homeo}_{\circ}(S^1)$, to which we can associate the asymptotic translation number τ . We define the torsion number as

$$\rho(f) \coloneqq \tau(F_0) - \tau(F_1).$$

This is well-defined: If we replace F by another lift, then $\tau(F_0)$ and $\tau(F_1)$ are both shifted by an integer, so their difference is the same. Moreover, ρ is a homogeneous quasimorphisms, in fact, $\delta\rho$ is the difference of the canonical real bounded Euler cocycles of \mathscr{E}_1 and \mathscr{E}_0 , as follows from the definitions [Mat86] (see also [Fri17, Section 10.8]).

Proof of Corollary J. There is an exact sequence (see e.g. [Fri17, Proposition 2.8]):

$$0 \to H^1(G) \to Q(G) \to H^2_b(G) \to H^2(G).$$

We know that $H_b^2(G) = \mathbb{R}\mathscr{E}_0 \oplus \mathbb{R}\mathscr{E}_1$ by Corollary I. Both \mathscr{E}_0 and \mathscr{E}_1 are non-trivial in $H^2(G)$; this is because the restriction $G \to \operatorname{Diff}_{\circ}^r(S^1)$ to either boundary component admits a section. The fact that $\delta\rho$ is the difference of the canonical real bounded Euler cocycles of \mathscr{E}_1 and \mathscr{E}_0 implies at once that $\mathscr{E}_1 - \mathscr{E}_0$ generates the kernel of $H_b^2(G) \to H^2(G)$, and that $Q(G)/H^1(G)$ is one-dimensional, spanned by ρ .

The last statement is a consequence of the fact that, for $r \neq 2, 3$, the group G is perfect [Ryb98, Theorem 5] (see also [AF09]).

6.2 Questions

A very obvious question is whether Theorem F holds in higher regularity:

Question 6.6. Is $\text{Diff}^r(D^n, \partial)$ boundedly acyclic for $r \ge 1$ and $n \ge 3$?

The case of n = 2 is known [MN23, Theorems 1.3 and 1.4], by methods specific to dimension 2. Our method is not dependent on dimension, but dealing with differentiability at the boundary presents several complications.

The main application of Theorem F is the isomorphism in bounded cohomology between homeomorphism groups of discs and spheres (Corollary G). However, the bounded cohomology of Homeo₊(S^n), and more generally of Diff^r₊(S^n), is understood (beyond low degrees) only in dimension 1 [MN23, Theorems 1.1 and 1.2].

Question 6.7. Is $\text{Diff}_+^r(S^n)$ boundedly acyclic for $r \ge 0$ and $n \ge 2$?

See also [MN23, Questions 7.3 and 7.4] for related questions.

One interesting feature of the proof of bounded acyclicity of $Homeo(\mathbb{R}^n)$ was that we had to study the bounded cohomology of certain monoids of self-embeddings. Normally the classifying space of the monoid of self-embeddings is more flexible to study than that of their maximal subgroup. But we had to go through auxiliary groups to prove bounded acyclicity of $\operatorname{Emb}_{<}(\mathbb{N}^{*})$. On the other hand $\operatorname{Homeo}(\mathbb{R}^{n})$ is the maximal subgroup of the monoid $\operatorname{Emb}(\mathbb{R}^{n})$ of self-embeddings of \mathbb{R}^{n} . As the topological monoid, Kister [Kis64] showed that the inclusion

$$\operatorname{Homeo}(\mathbb{R}^n) \to \operatorname{Emb}(\mathbb{R}^n),$$

induces a weak homotopy equivalence. As a discrete monoid, Segal's results in [Seg78] imply that

BHomeo^{δ}(\mathbb{R}^n) $\rightarrow B$ Emb^{δ}(\mathbb{R}^n),

induces a homology isomorphism. Now that our main theorem proves the bounded acyclicity of Homeo(\mathbb{R}^n) as a discrete group, it seems natural to ask the following question.

Question 6.8. Is $\text{Emb}^{\delta}(\mathbb{R}^n)$ boundedly acyclic?

Hirsch–Thurston [HT75] proved that the Euler class for flat C^0 -sphere bundles over an amenable base vanishes. Moreover, Calegari [Cal04, Theorem D] showed that the Euler class for C^1 -flat \mathbb{R}^2 -bundles over a torus should vanish. Even though the Euler class vanishes for such bundles with amenable bases, we know that the Euler class is unbounded [MN23, Cal04]. This suggests the following question.

Question 6.9. Are the Euler classes in $H^{2n}(\text{Diff}^r(\mathbb{R}^{2n}, \text{vol}); \mathbb{R})$ and in $H^{2n}(\text{Diff}^r(S^{2n-1}, \text{vol}); \mathbb{R})$ bounded?

Let us point out that $\operatorname{Diff}_{c}^{r}(\mathbb{R}^{n}; \operatorname{vol})$ is boundedly acyclic [CFFLM23, Corollary 1.12], but our methods from Section 3.2 break down in several ways when trying to prove that $\operatorname{Diff}^{r}(\mathbb{R}^{n}; \operatorname{vol})$ is boundedly acyclic.

A Controlled annulus theorem (by Alexander Kupers)

In this appendix, we will prove that the action of $\text{Homeo}(D^n, \partial)$ of homeomorphisms of the disc fixing the boundary pointwise on the set \mathcal{F} of "fat sequences" is transitive: Combine Theorem A.6 with Corollary A.10.

A.1 Sequences of spheres in \mathbb{R}^n

As preparation, we will study the action of homeomorphisms of \mathbb{R}^n on sequences of spheres. A *parametrised sphere* in \mathbb{R}^n is a locally flat embedding $S: S^{n-1} \hookrightarrow \mathbb{R}^n$ so that the composition

 $S^{n-1} \xrightarrow{S} \mathbb{R}^n \setminus \{ \text{point in bounded component of } \mathbb{R}^n \setminus \text{im}(S) \} \xrightarrow{\pi} S^{n-1}$

is orientation-preserving (i.e. preserves the fundamental class). We say S_1 englobes S_0 if S_0 is contained in the bounded component of $\mathbb{R}^n \setminus S_1$.

Definition A.1. A sequence of parametrised spheres is a collection $\mathbf{S} = \{S_i\}_{i \in \mathbf{N}^*}$ of disjoint parametrised spheres such that S_i englobes S_{i-1} .

Example A.2. The standard sequence Σ of parametrised spheres is given by letting $\Sigma_i: S^{n-1} \hookrightarrow \mathbb{R}^n$ be given by $x \mapsto i \cdot x$.

The identity component of the group of homeomorphisms of \mathbb{R}^n acts on the collection \mathcal{S} of all sequences of parametrised spheres by post-composition. As a warm-up we will prove the following:

Theorem A.3. A sequence **S** lies in the orbit of the standard nested sequence Σ under Homeo_o(\mathbb{R}^n) if and only if every compact subset of \mathbb{R}^n intersects finitely many S_i .

We refer to the condition in the previous definition as **S** going to infinity. It is equivalent to \mathbb{R}^n being equal to union of the bounded components of $\mathbb{R}^n \setminus \operatorname{im}(S_i)$ for all $i \geq 1$: if a compact K intersects the image of infinitely many S_i , pick a sequence $k_j \in K$ with $k_j \in \operatorname{im}(S_{i_j})$ where $i_j \to \infty$ and let k be the limit of a convergent subsequence, then we must have that k must lie in the unbounded component of $\mathbb{R}^n \setminus \operatorname{im}(S_{i_j})$ for all j and hence the intersection of the unbounded components of all $\mathbb{R} \setminus \operatorname{im}(S_{i_j})$ is non-empty.

The proof will use as main input two slightly sharpened variants of the Schoenflies and annulus theorems.

Theorem A.4 (Schoenflies theorem). Let S be a parametrised sphere and W denote the closure of the bounded component of $\mathbb{R}^n \setminus \operatorname{im}(S)$. Then there is a homeomorphism $h: D^n \to W$ that agrees with S on S^{n-1} .

Proof. The usual Schoenflies theorem provides a homeomorphism $g: D^n \to W$ ([Bro60, Maz59, Mor60]). By the Alexander trick $(g|_{S^{n-1}})^{-1} \circ S$ extends to a homeomorphism of D^n and precompose g with this to get the desired homeomorphism.

Theorem A.5 (Annulus theorem). Let S_1, S_2 be parametrised spheres with S_1 englobed by S_2 , and let W denote the closure of the region between these. Then there is a homeomorphism $h: S^{n-1} \times [1,2] \to W$ that agrees with S_1 on $S^{n-1} \times \{1\}$ and S_2 on $S^{n-1} \times \{2\}$.

Proof. The usual annulus theorem provides a homeomorphism $h: S^{n-1} \times [1, 2] \to W$ [Moi52, Kir69, Qui82]. We may assume by precomposing with a reflection if necessary that h is orientation-preserving on both boundary components. Then $(h|_{S^{n-1}\times\{1\}})^{-1} \circ S_1$ is orientationpreserving and since Homeo₊ (S^{n-1}) is path-connected, there is an isotopy $b_t: S^{n-1} \times [0, 1] \to S^{n-1}$ from it to the identity. Similarly, there is an isotopy b'_t from the identity to $(h|_{S^{n-1}\times\{2\}})^{-1} \circ S_2$. Now define a homeomorphism of $S^{n-1} \times [1, 2]$ by

$$(x,t) \longmapsto \begin{cases} (b_{2t}(x),t) & \text{if } t \in [0,1/2], \\ (b'_{2t-1}(x),t) & \text{if } t \in [1/2,1] \end{cases}$$

and precompose h with it to get the desired homeomorphism.

Proof of Theorem A.3. The direction \Leftarrow is clear as the standard nested sequence has the property of going to infinity and this property is preserved by homeomorphisms. For the direction \Rightarrow , without any hypothesis on $\mathbf{S} = \{S_i\}_{i \in \mathbf{N}^*}$ we will inductively construct an embedding $h: \mathbb{R}^n \to \mathbb{R}^n$ so that $h\Sigma_i = S_i$. The condition that the sequence goes to infinity guarantees this is surjective, and any surjective embedding is a homeomorphism. It will lie in the identity component because by construction it will be orientation-preserving.

Let W_1 denote the closure of the bounded component of $\mathbb{R}^n \setminus \operatorname{im}(S_1)$ and $W_{i+1} \colon S_i \rightsquigarrow S_{i+1}$ for $i \geq 1$ denote the closure of the region between $\operatorname{im}(S_i)$ and $\operatorname{im}(S_{i+1})$. Apply Theorem A.4 to get a homeomorphism $h_1 \colon D^n \to W_1$ and apply Theorem A.5 to get homeomorphisms

 $h_{i+1}: S^{n-1} \times [i, i+1] \to W_{i+1}$, all agreeing with the appropriate S_i 's on their boundary components. Now we define h as

$$h(x) := \begin{cases} h_1(x) & \text{if } x \in D^n, \\ h_{i+1}(x) & \text{if } x \in S^{n-1} \times [i, i+1] \text{ for } i \ge 1, \end{cases}$$

where we identify $S^{n-1} \times [i, i+1]$ with a subset of \mathbb{R}^n using radial coordinates.

A.2 Sequences of spheres in D^n

Let $\operatorname{Homeo}(D^n, \partial)$ denote those homeomorphisms that are the identity on ∂D^n (not necessarily near the boundary). Identifying the interior of D^n with \mathbb{R}^n , these are those homeomorphisms h of \mathbb{R}^n such the function $\frac{h(tx)}{|h(tx)|} \colon S^{n-1} \to S^{n-1}$, defined for t sufficiently large, converges uniformly to the identity at $t \to \infty$. We may ask what the orbit of the standard nested sequence under $\operatorname{Homeo}(D^n, \partial) \subset \operatorname{Homeo}_+(\mathbb{R}^n)$ is.

To state the answer, we introduce a notion of "radial control" using reference maps to S^{n-1} , with metric d induced from the standard one on Euclidean space. Given reference maps $r_0: X_0 \to S^{n-1}$ and $r_1: X_1 \to S^{n-1}$, we say that a continuous map $f: X_0 \to X_1$ has diameter $\langle \epsilon \text{ if } \sup\{d(r_0(x), r_1(f(x))) \mid x \in X_0\} < \epsilon$. The map $x \mapsto \frac{x}{|x|}$ induces a map to S^{n-1} on any subset of \mathbb{R}^n avoiding the origin; unless we say otherwise, this is the reference map to S^{n-1} on such subsets. Let us make some elementary observations about this notion:

- If f, g are composable so that $\operatorname{diam}(f) < \epsilon$ and $\operatorname{diam}(g) < \delta$ then $\operatorname{diam}(gf) < \epsilon + \delta$.
- If h is a homeomorphism so that $\operatorname{diam}(h) < \epsilon$ then $\operatorname{diam}(h^{-1}) < \epsilon$.
- If $\{A_i\}$ is a cover then diam $(f) = \sup \operatorname{diam}(f|_{A_i})$.

Theorem A.6. A sequence of parametrised spheres $\mathbf{S} = \{S_i\}_{i \in \mathbb{N}^*} \in \mathcal{S}$ lies in the orbit of the standard sequence Σ under Homeo (D^n, ∂) if and only if diam $(S_i) \to 0$ as $i \to \infty$.

We refer to the condition in the theorem as \mathbf{S} having trivial germ at the boundary.

This theorem will follow by essentially the same proof as for Theorem A.3 once we establish the following "controlled annulus theorem". Such a result was surely known to the experts, but we could not find it in the literature.

Proposition A.7 (Controlled annulus theorem). Let S_1, S_2 be parametrised spheres with S_1 englobing the origin and S_2 englobing S_1 , and let W denote the closure of the region between these. For all $\epsilon > 0$ there exists a $\delta > 0$ so that if S_1, S_2 have diameter $< \delta$, then there is a homeomorphism $h: S^{n-1} \times [1,2] \to W$ of diameter $< \epsilon$ that agrees with S_1 on $S^{n-1} \times \{1\}$ and S_2 on $S^{n-2} \times \{2\}$.

The proof relies on local contractibility of spaces of locally flat embeddings, in the compactopen topology. If Y is a metric space and X any space, for any two maps $f, g: X \to Y$ we define $d(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\}$. The following is a consequence of [EK71], explicitly stated as [DV09, Theorem 7.3.1].

Theorem A.8. Let N be a compact PL-manifold with metric and $M \subset \operatorname{int} N$ be a two-sided closed PL-submanifold of codimension one. For all $\epsilon > 0$ there exists a $\delta > 0$ so that for any locally flat embedding $e: M \to \operatorname{int} N$ satisfying $d(e, \operatorname{inc}_M) < \delta$ there is a homeomorphism $h: N \to N$ fixing the boundary pointwise, satisfying $he = \operatorname{inc}_M$, and $d(h, \operatorname{id}_N) < \epsilon$. Proof of Proposition A.7. We suppose S_1, S_2 have diameter $< \delta$ for some small δ to be specified at the end of the proof.

Picking spheres of radii R_{\min} and R_{\max} such that S_1 lies between them, we can regard using radial coordinates S_1 as an embedding $S^{n-1} \to S^{n-1} \times [R_{\min}, R_{\max}]$. By picking a suitable self-homeomorphism λ of $[R_{\min}, R_{\max}]$ fixing the boundary, we can assume $d((\operatorname{id} \times \lambda)S_1, \operatorname{inc}_{S^{n-1} \times \{R_{\operatorname{ave}}\}}) < \delta$ where $R_{\operatorname{ave}} \coloneqq \frac{1}{2}(R_{\min} + R_{\max})$. For ϵ' to be picked later, by taking δ to be small we can use Theorem A.8 to find a homeomorphism \widetilde{g}_1 of $S^{n-1} \times [R_{\min}, R_{\max}]$ that fixes the boundary, satisfying $d(\widetilde{g}_1, \operatorname{id}_{S^{n-1} \times [R_{\min}, R_{\max}]}) < \epsilon'$ and $\widetilde{g}_1(\operatorname{id} \times \lambda)S_1 = \operatorname{inc}_{S^{n-1} \times \{R_{\operatorname{ave}}\}}$. Extending $\widetilde{g}_1(\operatorname{id} \times \lambda)$ by the identity and rescaling the radial coordinates by $\frac{1}{R_{\operatorname{ave}}}$, we get a homeomorphism g_1 of diameter $< \epsilon'$ so that $g_1S_1 = \operatorname{inc}_{S^{n-1} \times \{1\}}$.

Now pick a sphere of radius R'_{\max} so that g_1S_2 lies between it and of that of radius 1. Using that diam $(g_1S_2) < \epsilon' + \delta$, by making ϵ' and δ smaller we can use the same argument to construct a homeomorphism g_2 supported in $S^{n-1} \times [1, R'_{\max}]$ of diameter $< \epsilon''$ for $\epsilon' \le \epsilon''$ and satisfying $g_2g_1S_2 = \operatorname{inc}_{S^{n-1}\times\{2\}}$. Then $(g_2g_1)^{-1}$ restricts to a homeomorphism $S^{n-1} \times [1, 2] \to W$ of diameter $< \epsilon' + \epsilon''$ agreeing with S_i on $S^{n-1} \times \{i\}$ for i = 1, 2. By taking δ small enough we can make $\epsilon' + \epsilon'' \le \epsilon$.

Proof of Theorem A.6. The direction \Leftarrow is clear as the standard sequence satisfies the condition of having trivial germ at the boundary and it is preserved by homeomorphisms of D^n fixing ∂D^n pointwise. For the direction \Rightarrow , we will construct an embedding $h: \mathbb{R}^n \to \mathbb{R}^n$ so that $h\Sigma_i = S_i$ and $\operatorname{diam}(h|_{S^{n-1}\times t}) \to 0$ as $t \to \infty$ using the second condition on the nested sequence. As before, the first condition guarantees this is surjective so a homeomorphism and it will be orientation-preserving so in the identity component.

Let W_1 denote the closure of the bounded component of $\mathbb{R}^n \setminus \operatorname{im}(S_1)$ and $W_{i+1} \colon S_i \rightsquigarrow S_{i+1}$ denote the closure of the region between $\operatorname{im}(S_i)$ and $\operatorname{im}(S_{i+1})$. Pick now for each $k \geq 1$ a $\delta_k > 0$ so that Proposition A.7 applies with $\epsilon = 1/k$.

Apply the Schoenfies theorem (Theorem A.4) to get a homeomorphism $h_1: D^n \to W_1$ and apply the controlled annulus theorem (Proposition A.7) to get a homeomorphisms of $h_{i+1}: S^{n-1} \times [i, i+1] \to W_{i+1}$ of diameter < 1/k for k largest such that diam (S_i) , diam $(S_{i+1}) < \delta_k$ when possible. There will be finitely many *i* where this is not possible, and then apply the annulus theorem (Theorem A.5) to get homeomorphisms $h_{i+1}: S^{n-1} \times [i, i+1] \to W_{i+1}$. All of these agree with the appropriate S_i 's on the boundary. Now we can define *h* as

$$h(x) := \begin{cases} h_1(x) & \text{if } x \in D^n, \\ h_{i+1}(x) & \text{if } x \in S^{n-1} \times [i, i+1] \text{ for } i \ge 1, \end{cases}$$

where we identify $S^{n-1} \times [i, i+1]$ with a subset of \mathbb{R}^n using radial coordinates, and by construction diam $(h|_{S^{n-1} \times \{t\}}) \to 0$ as $t \to \infty$.

A.3 Sequences of germs in \mathbb{R}^n or D^n

We can also consider germs near $S^{n-1} \times \{0\}$ of orientation-preserving topological embeddings $\overline{S}: S^{n-1} \times (-\frac{1}{4}, \frac{1}{4}) \hookrightarrow \mathbb{R}^n$.

Definition A.9. A sequence of germs of parametrised spheres is a collection $\overline{\mathbf{S}} = \{\overline{S}_i\}_{i\geq 1}$ of disjoint germs of parametrised spheres such that \overline{S}_i englobes \overline{S}_{i-1} .

There is a standard such sequence $\overline{\Sigma}$ given by the germs of $\overline{\Sigma}_i: S^{n-1} \times (-\frac{1}{4}, \frac{1}{4}) \to \mathbb{R}^n$ given by $(x, s) \mapsto (i + s) \cdot x$. We will use the notation $\dot{S}_i \coloneqq \overline{S}_i|_{S^{n-1} \times \{0\}}$ for the cores and \dot{S} for the resulting sequence of parametrised spheres.

Corollary A.10. A sequence $\overline{\mathbf{S}}$ of germs of parametrised spheres lies in the orbit of the standard sequence $\overline{\boldsymbol{\Sigma}}$ under Homeo⁺(\mathbb{R}^n) or Homeo(D^n, ∂) if and only if its sequence of cores $\dot{\mathbf{S}}$ does.

Proof. The direction \Rightarrow is obvious. For \Leftarrow , given $\overline{\mathbf{S}} = \{\overline{S}_i\}_{i \in \mathbf{N}^*}$ we pick a homeomorphism sending $\dot{\mathbf{S}}$ to the standard sequence. It remains to adjust the germs near the standard spheres and to do so, we will use that for any germ of embedding $S^{n-1} \times [0, 1) \to S^{n-1} \times [0, 1)$ near $S^{n-1} \times \{0\}$ that is the identity on $S^{n-1} \times \{0\}$, there is a compactly-supported homeomorphism that is the identity on the boundary which sends it to the germ of the identity of arbitrarily small diameter. To find this, note that both the given germ and the identity germ can be represented by collars on the boundary and invoke the collaring uniqueness theorem [KS77, Theorem A.1, Essay I], which comes with control. We now pick for each germ \overline{S}_i extending Σ_i two homeomorphisms as above—one for each side, in $S^{n-1} \times (i-1/4, i]$ and $S^{n-1} \times [i, i+1/4)$ that map its germ to the standard germ, have diameter smaller than 1/i, and necessarily agree on $S^{n-1} \times \{i\}$ so may be combined into a single homeomorphisms. The resulting homeomorphisms, one for each $i \in \mathbf{N}^*$, by construction have disjoint compact support so may be combined into a single homeomorphism h so that $h\overline{S}_i = \overline{\Sigma}_i$ and $diam(h|_{S^{n-1} \times \{t\}}) \to 0$ as $t \to \infty$.

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