On and Around the Bounded Cohomology of SL_2

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Abstract In order to illustrate some of the machinery of continuous bounded cohomology, we work out a couple of concrete questions in the particular case of SL_2. First we compute, in degree two, the continuous bounded cohomology of $SL_2(\mathbb{R})$ with unitary irreducible coefficients. Then we explore the connections between dilogarithm functions and the continuous bounded cohomology of $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$. In particular, we obtain that Rogers' dilogarithm is uniquely determined by the Spence–Abel functional equation.

1 Introduction

Although the theory of bounded cohomology has recently found many applications in various fields (see for instance [3] or [13]), for discrete groups it remains scarcely accessible to computation. As a matter of fact, almost all known results assert either a complete vanishing or yield intractable infinite dimensional spaces. On the other hand, the low degree continuous bounded cohomology $\mathrm{H}^\bullet_{\mathrm{cb}}$ of a Lie group (with unitary coefficients) can be described by means of the rich structure theory of the latter.

Our first result in this paper derives from the investigation, in a particular case, of the interplay between the infinite dimensionality of the bounded cohomology groups of surface groups and a concrete description of the bounded cohomology groups for SL_2.

Spectral Distribution

Let Σ be a compact orientable surface different from the sphere and the torus. The fundamental group $\Gamma = \pi_1\Sigma$ is Gromov hyperbolic and, as such, $\mathrm{H}^2_{\mathrm{b}}(\Gamma)$ is infinite dimensional, [7].

Any hyperbolization $\Gamma \hookrightarrow PSL_2(\mathbb{R})$ of Σ induces an injection

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma) \longrightarrow \mathrm{H}^2_{\mathrm{cb}}\big(PSL_2(\mathbb{R}), L^2(PSL_2(\mathbb{R})/\Gamma)\big)\,, \tag{1}$$

see [13, 11.1.5]. On the other hand, the $PSL_2(\mathbb{R})$-representation on the space $L^2(PSL_2(\mathbb{R})/\Gamma)$ decomposes into a direct sum of irreducible representations in a way dictated by the topology and spectral theory of the surface Σ with the chosen hyperbolic structure.

In this situation, it is natural to ask how the infinite dimensional space $\mathrm{H}^2_{\mathrm{b}}(\Gamma)$ gets distributed over the spectral decomposition. This is a difficult question. The first step is of course to understand which irreducible representations of $\mathrm{PSL}_2(\mathbb{R})$ carry bounded cohomology in degree two. Observe that, since $\mathrm{H}^2_{\mathrm{cb}}\big(\mathrm{PSL}_2(\mathbb{R}), L^2(\mathrm{PSL}_2(\mathbb{R})/\Gamma)\big)$ is infinite dimensional (see (1)), the spectral distribution is bound to be very different from the one concerning ordinary continuous cohomology, in which case only a finite number of unitary irreducible coefficients result in non-zero, finite dimensional cohomology groups (see the Table below). In contrast, we have:

Theorem 1.1. *(a) Let* $(\pi, \mathfrak{H})$ *be an irreducible unitary representation of* $\mathrm{SL}_2(\mathbb{R})$*:*

(i) $\dim \mathrm{H}^2_{\mathrm{cb}}(\mathrm{SL}_2(\mathbb{R}), \mathfrak{H}) = 1$ *if the representation* $(\pi, \mathfrak{H})$ *is spherical;*

(ii) $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{SL}_2(\mathbb{R}), \mathfrak{H}) = 0$ *in all other cases;*

(b) If $L^p\mathfrak{C}$ *denotes the series of complementary* L^p *representations, we have* $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{SL}_2(\mathbb{R}), L^p\mathfrak{C}) \neq 0$ *for all* $1 < p < \infty$.

Except for the Euler class, which corresponds to the trivial representation, the cohomology classes above are all new – they vanish in usual continuous cohomology. It is therefore worth mentioning that we can define them with a very explicit formula (Proposition 4.1).

We summarize below the present state of our knowledge for the list of all irreducible unitary $\mathrm{SL}_2(\mathbb{R})$-representations factoring through $\mathrm{PSL}_2(\mathbb{R})$; for the reader's convenience, we have recalled on the left hand side the well known situation in ordinary continuous cohomology. We denote the trivial representation by $\mathbf{1}$; the discrete series representations $\mathfrak{H}(n)$ are indexed by the minimal or maximal weight $n \in 2\mathbb{Z}$.

	$\mathrm{H}^0_{\mathrm{c}}$	$\mathrm{H}^1_{\mathrm{c}}$	$\mathrm{H}^2_{\mathrm{c}}$	$\mathrm{H}^{\geq 2}_{\mathrm{c}}$	$\mathrm{H}^0_{\mathrm{cb}}$	$\mathrm{H}^1_{\mathrm{cb}}$	$\mathrm{H}^2_{\mathrm{cb}}$	$\mathrm{H}^3_{\mathrm{cb}}$	$\mathrm{H}^{\geq 3}_{\mathrm{cb}}$
$\mathbf{1}$	$\mathbb{R}$	0	$\mathbb{R}$	0	$\mathbb{R}$	0	$\mathbb{R}$	0	?
Spherical $\neq \mathbf{1}$	0	0	0	0	0	0	$\mathbb{R}$	?	?
$\mathfrak{H}(2)$, $\mathfrak{H}(-2)$	0	$\mathbb{R}$	0	0	0	0	0	?	?
$\mathfrak{H}(n)$, $n \neq \pm 2$	0	0	0	0	0	0	0	?	?

It would be interesting to investigate the spectral distribution of the bounded cohomology for rank one locally symmetric spaces of higher dimension. On the other hand, if Σ is a compact (or finite volume) locally symmetric irreducible space of rank at least two, we have shown ([2], [3]) that $\mathrm{H}^2_{\mathrm{b}}(\Gamma)$, $\Gamma = \pi_1(\Sigma)$, injects into the ordinary cohomology by proving precisely that there are no new classes in the continuous bounded cohomology, with unitary coefficients, of the corresponding Lie groups.

* * *

We turn now to bounded cohomology in degree three. We observe first the connection between this cohomology group for $\mathrm{SL}_2(\mathbb{C})$ and the Bloch–Wigner dilogarithm. For $\mathrm{SL}_2(\mathbb{R})$, we show vanishing of this cohomology group and relate it to Roger's dilogarithm.

The Dilogarithm and $\mathbf{SL_2}$

Recall that, modulo its finite centre, $\mathrm{SL}_2(\mathbb{C})$ is (the connected component of) the group of isometries of Lobachevskiĭ's space $\mathcal{H}^3$. It follows, via Dupont's isomorphism [6], that the continuous cohomology group $\mathrm{H}^3_{\mathrm{c}}(\mathrm{SL}_2(\mathbb{C}))$ is generated by the volume form of $\mathcal{H}^3$. Since there is an upper bound to the volume of all geodesic simplices in this space, the volume form defines actually a class in $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_2(\mathbb{C}))$. The latter cohomology space can be computed (see Theorem 2.1) using measurable bounded cocycles on the space of *ideal* simplices, i.e. on the space of geodesic simplices with all four vertices on the sphere at infinity $\partial\mathcal{H}^3 \cong \widehat{\mathbb{C}}$.

It is well known that the volume of such a simplex is essentially given by the Bloch–Wigner dilogarithm of the crossratio of the four points in $\widehat{\mathbb{C}}$. In this realisation, the Spence–Abel functional equation for the dilogarithm corresponds simply to the cocycle equation for volume. S. Bloch has shown [1] that the functional equation essentially determines the dilogarithm among measurable functions; we shall rephrase his result as:

Theorem 1.2. *There is a natural isomorphism* $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_2(\mathbb{C})) \cong \mathrm{H}^3_{\mathrm{c}}(\mathrm{SL}_2(\mathbb{C}))$.

Remark 1.3. It is essential for this reformulation that Bloch's result is valid in the generality of measurable functions.

Rogers' dilogarithm is another relative of the classical Euler dilogarithm (see Sect. 5.1). It appears that Rogers' dilogarithm is connected to $\mathrm{SL}_2(\mathbb{R})$, but in a slightly different way. Using the corresponding version of the Spence–Abel functional equation and, denoting by $\wedge$ to the natural cup product in continuous bounded cohomology, we show:

Proposition 1.4. $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{SL}_2(\mathbb{R})) \wedge \mathrm{H}^2_{\mathrm{cb}}(\mathrm{SL}_2(\mathbb{R})) = 0$.

Further, the methods that we introduce for the spectral distribution allow us to show:

Theorem 1.5. $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_2(\mathbb{R})) = 0$.

This statement contains a uniqueness statement similar to Bloch's; indeed, our proof yields as a by-product:

Proposition 1.6. *Rogers' dilogarithm is the only integrable function* $L :]0,1[\to \mathbb{C}$ *satisfying both the Spence–Abel functional equation*

$$L\left(\frac{y-x}{1-x}\right) - L(y) + L(x) - L\left(\frac{x}{y}\right) + L\left(\frac{x(1-y)}{y(1-x)}\right) = 0 \tag{2}$$

and the symmetry $L(1-x) = \zeta(2) - L(x)$ *for all* $0 < x < y < 1$*, where* ζ *denotes the Riemann* ζ *function.*

(According to Gelfand and MacPherson [8, 4.1.2], there is no available reference for the uniqueness of Rogers' dilogarithm.)

2 Notations and Conventions

Throughout the paper, we write $G = \mathrm{SL}_2(\mathbb{R})$. We consider the action of G by fractional linear transformations on the upper half plane $\mathcal{H}^2 \subseteq \mathbb{C}$ and denote by K the stabilizer $\mathrm{SO}(2)$ of the point i. This action factors through the double covering $G \to \mathrm{PSL}_2(\mathbb{R})$ and extends to the geometric boundary $\partial\mathcal{H}^2 = \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. We denote by P the stabilizer of ∞, which is the subgroup of upper triangular matrices in G; one has $G = KP$. We shall repeteadly use that, up to null-sets, the diagonal G-action on the cartesian product $\widehat{\mathbb{R}}^3$ has exactly two orbits.

The matrix $\left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$ conjugates G to $\mathrm{SU}(1,1)$ within $\mathrm{SL}_2(\mathbb{C})$. Under this conjugation, the inverse of the stereographic projection

$$\rho : \ \mathbf{S}^1 \subseteq \mathbb{C} \longrightarrow \widehat{\mathbb{R}}, \quad \rho(z) = \frac{z+i}{iz+1}$$

intertwines the G-action on $\widehat{\mathbb{R}} \cong G/P$ with the homographic $\mathrm{SU}(1,1)$-action on the unit circle in $\mathbb{C}$. To avoid confusion, we use the notation $g_\star s$ for the action of $g \in G$ on $s \in \mathbf{S}^1$ obtained in this way. Occasionally, it will be convenient to use for $\mathbf{S}^1$ the additive parametrisation $\mathbb{R}/2\pi\mathbb{Z}$. With this notation, one has $(k_u)_\star s = s + 2u$ for $k_u = \left(\begin{smallmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{smallmatrix}\right)$, $u \in \mathbb{R}/2\pi\mathbb{Z}$.

We shall assume all continuous unitary representations to have separable range. For the complete classification of the *irreducible* continuous unitary representations of G, we refer to [10] or [11] (see also Sect. 3.2). A representation is called *spherical* if it has a (non-zero) K-invariant vector. As far as bounded cohomology is concerned, it is enough – as we shall recall below – to consider the representations which factor through the projection $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$. Irreducible representations of this kind are either spherical or belong to the discrete series. In Sect. 4.1, we shall recall the construction of the spherical representations.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}_\mathbb{C}$ its complexification. If $(\pi, \mathfrak{H})$ is a continuous unitary G-representation, we denote by $\mathfrak{H}_K$ the space of K-finite vectors. Besides the K-action, $\mathfrak{H}_K$ has also a structure of $\mathfrak{g}_\mathbb{C}$-module (if $\mathfrak{H}_K$ is irreducible or more generally admissible, the $(\mathfrak{g}_\mathbb{C}, K)$-structure turns it into a Harish-Chandra module, see [14]).

If S is a standard measure space and $\mathfrak{H}$ a separable Hilbert space, $L^\infty\big(S, \mathfrak{H}\big)$ denotes the space of measurable essentially bounded $\mathfrak{H}$-valued function classes;

observe that weak and strong measurability coincide here. Given a regular action of a locally compact group H on S (so that H-action on $L^\infty(S)$ is weak-$*$-continuous, [3])and a continuous unitary H-representation on $\mathfrak{H}$, we consider the corresponding representation on $L^\infty\big(S, \mathfrak{H}\big)$. We borrow Zimmer's notion of *amenability* for H-actions on S, see [15].

For the general theory of continuous bounded cohomology, we refer to [3] and [13]. We recall the following

Theorem 2.1 ([3], [13]). *Let H be a locally compact second countable group, $(\pi, \mathfrak{H})$ a continuous unitary H-representation and S an amenable regular H-space. Then the cohomology of the complex*

$$0 \longrightarrow L^\infty(S, \mathfrak{H})^H \xrightarrow{d^1} L^\infty\big(S^2, \mathfrak{H}\big)^H \xrightarrow{d^2} L^\infty\big(S^3, \mathfrak{H}\big)^H \xrightarrow{d^3} \cdots$$

is canonically isometrically isomorphic to the continuous bounded cohomology $\mathrm{H}^\bullet_{\mathrm{cb}}(H, \mathfrak{H})$. The same is true for the subcomplex of alternating cochains. $\square$

The above maps $d^n : L^\infty\big(S^n, \mathfrak{H}\big) \to L^\infty\big(S^{n+1}, \mathfrak{H}\big)$ are the usual Alexander–Spanier coboundaries $d^n = \sum_{j=0}^n (-1)^j d^n_j$, wherein d^n_j omits the j^{th} variable.

We shall mostly use the following particular case of the theorem:

Corollary 2.2. *For every continuous unitary G-representation $(\pi, \mathfrak{H})$, the cohomology of the complex*

$$0 \longrightarrow L^\infty(\mathbf{S}^1, \mathfrak{H})^G \xrightarrow{d^1} L^\infty\big((\mathbf{S}^1)^2, \mathfrak{H}\big)^G \xrightarrow{d^2} L^\infty\big((\mathbf{S}^1)^3, \mathfrak{H}\big)^G \xrightarrow{d^3} \cdots$$

is canonically isometrically isomorphic to $\mathrm{H}^\bullet_{\mathrm{cb}}(G, \mathfrak{H})$. The same is true for the subcomplex of alternating cochains. $\square$

3 A Differential Group

In this section, we introduce a graded differential group $(\mathcal{A}^\bullet_{\mathfrak{M}}, \partial)$ into which we shall translate questions about measurable cocycles on the circle by means of the Fourier transformation. The structure of $\mathcal{A}^\bullet_{\mathfrak{M}}$ will be well suited to find obstructions to the existence of cocycles.

We use the multiplicative parametrisation $\mathbf{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, so that $\mathbb{C}[z]$ is identified with the algebra of trigonometric polynomials. For $n \geq 0$, we denote by μ_n the normalized Haar measure on $(\mathbf{S}^1)^{n+1}$. If $\nu \in \mathbb{Z}^{n+1}$, we denote by χ_ν the character $\chi_\nu(z) = z_0^{\nu_0} \cdots z_n^{\nu_n}$ for $z \in (\mathbf{S}^1)^{n+1}$.

3.1 Fourier Transformation with Coefficients

Let $(\pi, \mathfrak{H})$ be a continuous unitary G-representation. Since the inclusion

$$\mathbb{C}[z] \otimes \cdots \otimes \mathbb{C}[z] \otimes \mathfrak{H}_K \longrightarrow L^1(\mathbf{S}^1) \widehat{\otimes} \cdots \widehat{\otimes} L^1(\mathbf{S}^1) \widehat{\otimes} \mathfrak{H}$$

is dense and the dual of the right hand side is $L^\infty\big((\mathbf{S}^1)^{n+1},\mathfrak{H}\big)$ by the Dunford–Pettis theorem [5, VI.8], we deduce by duality that the canonical map to the *algebraic* dual of the left hand side is injective. Denoting by $\mathfrak{H}_K'$ the algebraic dual of $\mathfrak{H}_K$ and by $\mathcal{F}(\mathbb{Z}^{n+1},\mathfrak{H}_K')$ the space of all maps $\mathbb{Z}^{n+1}\to\mathfrak{H}_K'$ we conclude that the Fourier transformation

$$L^\infty\big((\mathbf{S}^1)^{n+1},\mathfrak{H}\big) \xrightarrow{\ \widehat{\ }\ } \mathcal{F}(\mathbb{Z}^{n+1},\mathfrak{H}_K')$$
$$\widehat{F}(\nu)(v) = \int_{(\mathbf{S}^1)^{n+1}} \langle F(z)|v\rangle \chi_\nu(z)\,d\mu_n(z)$$

(where $F\in L^\infty\big((\mathbf{S}^1)^{n+1},\mathfrak{H}\big)$, $\nu\in\mathbb{Z}^{n+1}$, and $v\in\mathfrak{H}_K$) is injective.

We define the operators $S_\pm$ on $\mathcal{F}(\mathbb{Z}^{n+1},\mathfrak{H}_K')$ by

$$(S_\pm\psi)(\nu) = \sum_{j=0}^{n}(\nu_j\pm 1)\psi(\nu\pm\epsilon_j) \qquad (\nu\in\mathbb{Z}^{n+1}), \tag{3}$$

where $(\epsilon_j)_{j=0}^n$ is the canonical basis of $\mathbb{Z}^{n+1}$. Further, define

$$\partial=\partial^n : \mathcal{F}(\mathbb{Z}^n,\mathfrak{H}_K') \longrightarrow \mathcal{F}(\mathbb{Z}^{n+1},\mathfrak{H}_K')$$

by $\partial^n=\sum_{j=0}^n(-1)^j\partial_j^n$ and

$$\partial_j^n\psi(\nu) = \begin{cases}\psi(\nu_0,\ldots,\widehat{\nu_j},\ldots,\nu_n) & \text{if } \nu_j=0,\\ 0 & \text{otherwise}.\end{cases} \tag{4}$$

One checks readily the

Lemma 3.1. *For all $n\geq 0$, the ∂^{n+1}-cocycles, i.e. functions belonging to* $\mathrm{Ker}\partial^{n+1}$*, are supported on the union $\Delta^{(n)}=\{\nu:\prod_{j=0}^n\nu_j=0\}$ of the canonical hyperplanes in $\mathbb{Z}^{n+1}$.* □

One can also show that the resulting complex is acyclic, but we shall not need this information.

The dual $\mathfrak{g}_\mathbb{C}$-structure on $\mathcal{F}(\mathbb{Z}^{n+1},\mathfrak{H}_K')$ is given by

$$(X^*\psi)(\nu)(v)=\psi(\nu)(-d\pi(X)v) \qquad (\psi\in\mathcal{F}(\mathbb{Z}^{n+1},\mathfrak{H}_K'), X\in\mathfrak{g}_\mathbb{C}),$$

where $d\pi$ is the differential of π which is well defined since $v\in\mathfrak{H}_K$ is smooth. In order to state the following proposition, we define $E_\pm\in\mathfrak{g}_\mathbb{C}$ by $E_\pm=\left(\begin{smallmatrix}1 & \pm i\\ \pm i & -1\end{smallmatrix}\right)$ and denote by T the map $T:\mathbb{Z}^{n+1}\to\mathbb{Z}$, defined by $T(\nu_0,\ldots,\nu_n)=\sum_{j=0}^n\nu_j$. We recall that $v\in\mathfrak{H}_K$ is of *weight* $\ell\in\mathbb{Z}$ if $\pi(k_t)v=e^{i\ell t}v$. The following proposition will motivate the introduction of our group $\mathcal{A}_\mathfrak{M}^\bullet$:

Proposition 3.2. *The Fourier transformation $L^\infty\big((\mathbf{S}^1)^\bullet,\mathfrak{H}\big)\hookrightarrow\mathcal{F}(\mathbb{Z}^\bullet,\mathfrak{H}_K')$ has the following properties:*

(a) If $F \in L^\infty\big((\mathbf{S}^1)^\bullet, \mathfrak{H}\big)$ *is G-equivariant, then* $E_\pm^* \widehat{F} = 2iS_\mp \widehat{F}$*;*
(b) If $F \in L^\infty\big((\mathbf{S}^1)^\bullet, \mathfrak{H}\big)$ *is G-equivariant and* $v \in \mathfrak{H}_K$ *is of weight* ℓ*, then* $\widehat{F}(\cdot)(v)$ *is supported on the hyperplane* $\{\nu : T(\nu) = -\ell/2\}$*. In particular,* $\widehat{F}(\cdot)(v)$ *vanishes if* ℓ *is odd;*
(c) The Fourier transformation preserves alternation and intertwines the coboundary d *with* ∂*.*

Proof. (a) For $X \in \mathfrak{g}_\mathbb{C}$ and $g_t = \exp(tX)$,

$$\widehat{g_t F}(\nu)(v) = \int_{(\mathbf{S}^1)^{n+1}} \langle F((g_t^{-1})_\star z) | \pi(g_t^{-1}) v\rangle \chi_\nu(z)\, d\mu_n(z)\,,$$

and thus the equivariance of F implies by product differentiation

$$X^* \widehat{F}(\nu)(v) = D \int_{(\mathbf{S}^1)^{n+1}} \langle F((g_t^{-1})_\star z) | v\rangle \chi_\nu(z)\, d\mu_n(z)\,,$$

where D is a shorthand for $\frac{\partial}{\partial t}\big|_{t=0}$. This becomes further

$$X^* \widehat{F}(\nu)(v) = \int_{(\mathbf{S}^1)^{n+1}} \langle F(z) | v\rangle D\Big(\chi_\nu((g_t)_\star z) \frac{d(g_t)_\star \mu_n}{d\mu_n}(z)\Big)\, d\mu_n(z)\,.$$

Using now

$$D\Big(\chi_\nu((g_t)_\star z)\Big) = \sum_{j=0}^n \nu_j \chi_{\nu-\epsilon_j}(z) D((g_t)_\star z_j)$$

and

$$D\Big(\frac{d(g_t)_\star \mu_n}{d\mu_n}(z)\Big) = \sum_{j=0}^n D\Big(\frac{d(g_t)_\star \mu_0}{d\mu_0}(z_j)\Big)\,,$$

we have

$$\begin{aligned} X^* \widehat{F}(\nu)(v) = \quad & \sum_{j=0}^n \nu_j \int_{(\mathbf{S}^1)^{n+1}} \langle F(z) | v\rangle D((g_t)_\star z_j) \chi_{\nu-\epsilon_j}(z)\, d\mu_n(z) & (5)\\ & + \sum_{j=0}^n \int_{(\mathbf{S}^1)^{n+1}} \langle F(z) | v\rangle \chi_\nu(z) D\Big(\frac{d(g_t)_\star \mu_0}{d\mu_0}(z_j)\Big)\, d\mu_n(z)\,. & (6)\end{aligned}$$

Now write $E_\pm = H \pm iV$ with $H = \big(\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\big)$ and $V = \big(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\big)$. The images of $a_t = \exp(tH)$ and $u_t = \exp(tV)$ in $\mathrm{SU}(1,1)$ under the conjugation by $\big(\begin{smallmatrix} 1 & i \\ i & 1\end{smallmatrix}\big)$ introduced in Sect. 4 are respectively $\big(\begin{smallmatrix} \cosh t & i\sinh t \\ -i\sinh t & \cosh t\end{smallmatrix}\big)$ and $\big(\begin{smallmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t\end{smallmatrix}\big)$. Using this, one computes

$$D\big((a_t)_\star z_j\big) = i(1 + z_j^2)\,, \quad \text{and} \quad D\big((u_t)_\star z_j\big) = 1 - z_j^2\,.$$

Computing the Radon–Nikodým derivatives yields

$$\frac{d(a_t)_\star \mu_0}{d\mu_0}(z_j) = \left(\cosh 2t + i\frac{z_j^{-1} - z_j}{2} \sinh 2t\right)^{-1}$$

and

$$\frac{d(u_t)_\star\mu_0}{d\mu_0}(z_j) = \left(\cosh 2t + \frac{z_j^{-1}+z_j}{2}\sinh 2t\right)^{-1},$$

thus one checks that

$$D\left(\frac{d(a_t)_\star\mu_0}{d\mu_0}(z_j)\right) = i(z_j - z_j^{-1}), \quad \text{and} \quad D\left(\frac{d(u_t)_\star\mu_0}{d\mu_0}(z_j)\right) = -(z_j + z_j^{-1}).$$

Replacing all this in (5), we find

$$\begin{aligned} H^*\widehat{F}(\nu)(v) &= iS_+\widehat{F}(\nu)(v) + iS_-\widehat{F}(\nu)(v) \\ V^*\widehat{F}(\nu)(v) &= -S_+\widehat{F}(\nu)(v) + S_-\widehat{F}(\nu)(v)\,, \end{aligned}$$

whence the claim.
(b) is a simpler form of this argument since the Radon–Nikodým derivatives for K are trivial.
(c) The orthogonality relations imply that $\widehat{d_j^n F} = \partial_j^n \widehat{F}$. □

3.2 The Differential Group $\mathcal{A}_{\mathfrak{M}}^{\bullet}$.

A classification of general $(\mathfrak{g}_{\mathbb{C}}, K)$-modules can be found in [9]. We shall however only need modules of the form $\mathfrak{M} = \mathfrak{H}_K$ where $\mathfrak{H}$ is an irreducible unitary representation of G factoring through $\mathrm{PSL}_2(\mathbb{R})$. They yield the following four types of irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-modules:

(a) *Spherical*: there is an element $v \in \mathfrak{M}$ such that $\mathfrak{M}$ is spanned by $(E_\pm^k v)_{k\geq 0}$. Moreover, there is $\lambda \in \mathbb{C}$ with $E_-E_+v' = \lambda v'$ for all v' of weight zero;
(b) *Positive minimal weight*: there is $v \in \mathfrak{M}$ such that $\mathfrak{M}$ is spanned by $(E_+^k v)_{k\geq 0}$ (in fact, since $\mathfrak{H}$ factors through $\mathrm{PSL}_2(\mathbb{R})$, only even weights occur);
(b') *Negative maximal weight*: there is $v \in \mathfrak{M}$ such that $\mathfrak{M}$ is spanned by $(E_-^k v)_{k\geq 0}$ (again, only even weights occur);
(c) The *trivial* module $\mathfrak{M} = \mathbb{C}$ is a particular case of (a).

Definition 3.3. Let $\mathfrak{M}$ be a $(\mathfrak{g}_{\mathbb{C}}, K)$-module and let $n \geq 0$. We define $\mathcal{A}_{\mathfrak{M}}^n$ to be the space of all maps $\psi : \mathbb{Z}^{n+1} \to \mathfrak{M}'$ satisfying:

$\mathrm{A_I}$ ψ is alternating;
$\mathrm{A_{II}}$ $E_\pm^*\psi = 2iS_\mp\psi$, where $E_\pm^*$ act by the natural dual $\mathfrak{g}_{\mathbb{C}}$-structure and the operators $S_\pm$ are defined as in (3);
$\mathrm{A_{III}}$ If $v \in \mathfrak{M}$ is of weight ℓ, then $\psi(\cdot)(v)$ is supported on the hyperplane $\{\nu : T(\nu) = -\ell/2\}$.

The differential $\partial_{\mathcal{A}}^{n+1} : \mathcal{A}_{\mathfrak{M}}^n \to \mathcal{A}_{\mathfrak{M}}^{n+1}$ is defined as above in (4), and thus again one checks:

A_{IV} ∂^{n+1}-cocycles are supported on the union $\Delta^{(n)}$ of the canonical hyperplanes.

Proposition 3.2, together with the injectivity of the Fourier transformation, implies that the complex of equivariant bounded alternating measurable $\mathfrak{H}$-valued cochains on the circle can be viewed as a subcomplex of $\mathcal{A}^{\bullet}_{\mathfrak{M}}$ for $\mathfrak{M} = \mathfrak{H}_K$.

The following finiteness result for general irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-modules, specialized to $\mathfrak{H}_K$, will be the final ingredient in the proof of Theorem 1.1, Theorem 1.5 and Proposition 1.6.

Proposition 3.4. *Let $\mathfrak{M}$ be an irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-module.*

(a) If $\mathfrak{M}$ is of spherical type, then $\dim \operatorname{Ker} \partial^3_{\mathcal{A}} \leq 1$;
(b) If $\mathfrak{M}$ is of positive minimal weight or negative maximal weight, then non-zero elements of $\operatorname{Ker} \partial^3_{\mathcal{A}}$ cannot vanish at infinity;
(c) If $\mathfrak{M}$ is the trivial $(\mathfrak{g}_{\mathbb{C}}, K)$-module $\mathbb{C}$, then $\operatorname{Ker} \partial^4_{\mathcal{A}} = 0$.

Proof. (a) Since $\mathfrak{M}$ is spherical and irreducible, there is a K-invariant element $v \in \mathfrak{M}^K$ such that $\mathfrak{M}$ is spanned over $\mathbb{C}$ by $(E^k_{\pm} v)_{k \geq 0}$. Moreover, there is $\lambda \in \mathbb{C}$ with $E_- E_+ v' = \lambda v'$ for all v' of weight zero. Fixing $\omega \in \operatorname{Ker} \partial^3_{\mathcal{A}}$, we shall show that $\omega(\cdot)(v)$ vanishes if $\omega(1, -1, 0)(v) = 0$; this will prove the claim (a) because of A_{II} and the structure of $\mathfrak{M}$ just described.

We define the map $\alpha : \mathbb{Z} \to \mathbb{C}$ by

$$\alpha(x) = \omega(x, -x, 0)(v)\,,$$

so that $\alpha(1) = 0$ by the assumption on ω and $\alpha(0) = 0$ by A_{I}. The properties A_{I}, $\mathrm{A}_{\mathrm{III}}$ and A_{IV} imply that α determines linearly $\omega(\cdot)(v)$, so that it is sufficient to show the vanishing of α. Implementing the definition of α in $S_+ S_- \omega(x, -x, 0)(v) = \lambda' \omega(x, -x, 0)(v)$, where $\lambda' = -\lambda/4$ (see A_{II}), we find

$$\begin{aligned}
&(x+1)^2 \alpha(x+1) + (x-1)^2 \alpha(x-1) \\
&\quad = (x-1)\big(\omega(x-1, -x, 1) + \omega(x, -x+1, -1)\big) + (2x^2 - \lambda')\alpha(x) \\
&\qquad - (x+1)\big(\omega(x+1, -x, -1) + \omega(x, -x-1, 1)\big)\,.
\end{aligned}$$

For $x = 1$, this reduces (by A_{I} and A_{IV}) to $4\alpha(2) = (2 - \lambda')\alpha(1)$, hence $\alpha(2) = 0$. For $x \geq 2$, the ω terms vanish because of A_{IV}, so that we have

$$(x+1)^2 \alpha(x+1) = (2x^2 - \lambda')\alpha(x) - (x-1)^2 \alpha(x-1)\,.$$

This propagates by induction the vanishing from $x = 1, 2$ to all $x \geq 3$. The negative values are handled with the formula $\alpha(-x) = -\alpha(x)$, which follows from A_{I}.

(b) We give the proof for an even positive minimal weight 2ℓ; the case of negative weights is analogous and the case of odd weights is trivial because

of A_{III}. Let ω be a cocycle $\omega \in \operatorname{Ker} \partial_{\mathcal{A}}^3$. We fix a $v \in \mathfrak{M}$ such that $\mathfrak{M}$ is spanned by $(E_+^k v)_{k \geq 0}$ (thus v is of minimal weight) and define the map $\beta : \mathbb{Z} \to \mathbb{C}$ by

$$\beta(x) = \omega(x, -\ell - x, 0)(v) .$$

We suppose that ω vanishes at infinity; then so does β. On the other hand, as for point (a), it is enough to show $\beta = 0$.

Now we use $E_-(v) = 0$: writing out $S_+\omega(x, -\ell - x - 1, 0)(v) = 0$, we have

$$(x + 1)\beta(x + 1) = (x + \ell)\beta(x) - \omega(x, -\ell - x, 1)(v) .$$

Therefore, ω being supported on $\Delta^{(n)}$, the positivity of ℓ implies that for all $x \geq 1$ we have $x, -\ell - x \neq 0$ and hence $(x+1)\beta(x+1) = (x+\ell)\beta(x)$. Since $x + \ell \geq x + 1 > 0$, the vanishing at infinity implies $\beta(x) = 0$ for all $x \geq 1$. The alternation of ω implies $\beta(x) = -\beta(-\ell - x)$, so that we remain only with the case $-\ell < x \leq 0$, which we settle by descending induction starting from $\beta(0) = 0$ (by A_{I}). If $1 - \ell < x < 0$, then $S_+\omega(x - 1, -\ell - x, 0)(v) = 0$ reads

$$x\beta(x) + (1 - \ell - x)\beta(x - 1) + \omega(x - 1, -\ell - x, 1) = 0 .$$

The first term vanishes by the induction hypothesis and the third because of $x \neq 1, -\ell$ and A_{IV}. Therefore $\beta(x - 1) = 0$ since $x \neq 1 - \ell$. For the last step $x = 1 - \ell$, we have $\beta(x) = -\beta(-1)$, which is already done if $1 - \ell \neq -1$ and follows from this formula if $1 - \ell = -1$.

(c) Notice first that $\operatorname{Ker} \partial_{\mathcal{A}}^4$ is invariant under the linear map σ defined by $\sigma\omega(\nu) = \omega(-\nu)$. Therefore we have a decomposition $\operatorname{Ker} \partial_{\mathcal{A}}^4 = Z_+ \oplus Z_-$ according to the eigenvalues ± 1 of σ. Now, for $\omega \in Z_+ \cup Z_-$, we define $\psi : \mathbb{Z}^2 \to \mathbb{C}$ by

$$\psi(x, y) = \omega(x, y, -x - y, 0) .$$

We remark (by A_{I}) that ψ is alternating and vanishes if x or y is zero. As before, it is enough for point (c) to prove that ψ vanishes everywhere. For simplicity, we write the operators $S_\pm$ as S_ϵ for $\epsilon \in \{-1, 1\}$. Now if

$$x \neq 0, \ y \neq 0, \ x + y + \epsilon \neq 0 , \tag{7}$$

then the condition A_{IV} reduces $S_\epsilon\omega = 0$ to

$$(x + y)\psi(x, y) = (x + \epsilon)\psi(x + \epsilon, y) + (y + \epsilon)\psi(x, y + \epsilon) . \tag{8}$$

The simpler case is when x and y are of the same sign. Indeed, if $y < x < 0$, we start with $\psi(x, x) = \psi(0, y) = 0$ and check by descending induction on y that $\psi(x, y)$ is zero: if this is so for some $y \leq -1$, then $(\epsilon = 1)$

$$\psi(-1, y - 1) = \frac{y}{y - 2}\psi(-1, y) = 0 ,$$

and thus by a additional induction on $y < x < -1$

$$\psi(x, y - 1) = \frac{x + 1}{x + y - 1}\psi(x + 1, y - 1) + \frac{y}{x + y - 1}\psi(x, y) = 0 .$$

This completes the (main) induction step and thus $\psi(x, y) = 0$ for all $y < x < 0$. By alternation, the same holds for $x < y < 0$; recalling that $\omega \in Z_+ \cup Z_-$, we deduce also that $\psi(x, y) = 0$ for all $x, y > 0$.

The second case is when x and y are of opposite sign. Remark first that on the line $x + y = 0$, the condition (7) holds away from zero, and that ψ vanishes. Therefore (8) yields

$$(x+\epsilon)\psi(x+\epsilon, -x) = (x-\epsilon)\psi(x, -x+\epsilon) \qquad (\forall x \neq 0)\,.$$

On the line $x + y = \epsilon$, the condition (7) holds for $x \neq 0, \epsilon$. This implies first that $\psi(2\epsilon, \epsilon) = 0$ (set $x = \epsilon$) and then, by induction, that $\psi(x + \epsilon, -x) = 0$ for all x with sign ϵ. The x of opposite sign (with ϵ kept fixed) are obtained by alternation together with $\omega \in Z_+ \cup Z_-$.

We have shown that $\psi(x, y) = 0$ holds on the two lines $x+y = \epsilon$ ($\epsilon = \pm 1$). Now we may use (8) for every $x < 0, y > 0$ with $x + y = h \geq 2$ in order to deduce by induction on h that $\psi(x, y)$ vanishes. The remaining points in $x + y \leq -2$ are taken care of by $\psi(-x, -y) = \pm\psi(x, y)$. □

4 Constructing Cocycles

Apart from the trivial representation, there are two types of spherical irreducible continuous unitary representations of G factoring through $\mathrm{PSL}_2(\mathbb{R})$: the principal and complementary series [10,11]. They can be defined as follows.

4.1 Representation Spaces

Consider for $\sigma \in \mathbb{C}$ the character δ_σ of P defined by

$$\delta_\sigma(p) = |a|^\sigma\,, \qquad p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in P\,.$$

For every $\sigma \neq -1$ one introduces the space of continuous functions

$$E(\sigma) = \Big\{ F \in C(G) : F(px) = \delta_{\sigma+1}(p)F(x)\ \forall x \in G, p \in P \Big\}$$

and endows it with the right regular G–action. For σ pure imaginary, one obtains the *principal series* representations by taking the completion $\mathfrak{P}(\sigma)$ of $E(\sigma)$ with respect to the G–invariant pre-Hilbertian structure induced by the inclusion $E(\sigma) \to L^2(K)$ obtained by restricting functions from G to K.

For σ real with $0 < \sigma < 1$, one gets the *complementary series* representations by taking the completion $\mathfrak{C}(\sigma)$ with respect to the G–invariant pre-Hilbertian structure

$$\langle f, g\rangle = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{f(k_u)\overline{g(k_v)}}{|\sin(u-v)|^{1-\sigma}}\,du\,dv\,, \qquad f, g \in E(\sigma)\,.$$

As a small excursion away from unitary representations, we shall also consider the L^p *complementary series* for $1 < p < \infty$, which is the completion $L^p\mathfrak{C}$ of the space $E((2-p)/p)$ for the norm induced by the inclusion $E((2-p/)p) \to L^p(K)$ as above, see [4, Chap. 6].

4.2 Basic Construction

For all $a \in \mathbb{R}$ and $b \in \mathbb{C}^*$, we define the *odd exponential*

$$\{a\}_o^b = \begin{cases} a^b & \text{if } a \geq 0 \\ -(-a)^b & \text{if } a < 0\,, \end{cases}$$

and extend it by $\{\infty\}_o^b = \infty$. We aim at the following

Proposition 4.1. *For $\sigma \in \mathbb{C} \smallsetminus \{-1\}$, there is a G-equivariant cocycle*

$$\omega:\ \widehat{\mathbb{R}} \times \widehat{\mathbb{R}} \times \widehat{\mathbb{R}} \longrightarrow E(\sigma)$$

defined almost everywhere by

$$\omega(x,y,z)(g) = \{gx - gy\}_o^{(\sigma+1)/2} + \{gy - gz\}_o^{(\sigma+1)/2} + \{gz - gx\}_o^{(\sigma+1)/2}\ .$$

(a) If σ is pure imaginary, then the cocycle ω represents a non-trivial class in $\mathrm{H}^2_{\mathrm{cb}}(G, \mathfrak{P}(\sigma))$.
(b) If $0 < \sigma < 1$, then ω represents a non-trivial class in $\mathrm{H}^2_{\mathrm{cb}}(G, \mathfrak{C}(\sigma))$.
(c) If $1 < p < \infty$ and $\sigma = (2-p)/p$, then ω represents a non-trivial class in $\mathrm{H}^2_{\mathrm{cb}}(G, L^p\mathfrak{C})$.

More formally, we define for every $\sigma \in \mathbb{C} \smallsetminus \{-1\}$ the function

$$F^{(\sigma)} : \mathbb{R} \times \mathbb{R} \longrightarrow \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

by

$$F^{(\sigma)}(s,t) = \left\{ \frac{2\sin(\frac{s-t}{2})}{\cos(\frac{s-t}{2}) - \sin(\frac{s+t}{2})} \right\}_o^{(\sigma+1)/2},$$

with the convention $F^{(\sigma)}(s,t) = 0$ if both s and t are in $\pi/2 + 2\pi\mathbb{Z}$. Henceforth, we freely view $F^{(\sigma)}$ as a function on $\mathbf{S}^1 \times \mathbf{S}^1$.

Lemma 4.2. *For all distinct $s, t \in \mathbf{S}^1 \smallsetminus \{\pi/2\}$*

(a) $F^{(\sigma)}(s,t) = \{x-y\}_o^{(\sigma+1)/2}$ for $x = \tan\frac{2s+\pi}{4}$, $y = \tan\frac{2t+\pi}{4}$;
(b) $F^{(\sigma)}(s,t) = -F^{(\sigma)}(t,s)$ (where $-\infty = \infty$);
(c) $F^{(\sigma)}(p_\star s, p_\star t) = \delta_{\sigma+1}(p) F^{(\sigma)}(s,t)$ for all $p \in P$.

Up to a multiple, the two properties (b) and (c) determine $F^{(\sigma)}$ entirely on $s, t \neq \pi/2$ (without further regularity assumptions).

Proof. The properties of $F^{(\sigma)}$ follow from trigonometric identities. The uniqueness statement is due to the transitivity of P on pairs of distinct points of $\mathbf{S}^1 \smallsetminus \{\pi/2\}$ in a given order with respect to $\pi/2$; the pairs with reversed order are taken care of by (b). $\square$

Now, for every $s \neq t$ we define

$$F^{(\sigma)}_{s,t} : \mathbf{S}^1 \longrightarrow \widehat{\mathbb{C}}$$
$$u \longmapsto F^{(\sigma)}_{s,t}(u) = F^{(\sigma)}(s+2u, t+2u)\,.$$

Notice that $F^{(\sigma)}_{s,t}$ is of period π.

Lemma 4.3. *For all distinct $s,t \in \mathbf{S}^1 \smallsetminus \{\pi/2\}$*

(a) If $\Re(\sigma) > -1$, then $F^{(\sigma)}_{s,t}$ is infinite on $\{\frac{\pi-2s}{4}, \frac{\pi-2t}{4}\} + \pi\mathbb{Z}$ (while it is finite and analytic outside this set);
(b) If σ is pure imaginary, then for $1 \leq p \leq \infty$ we have

$$F^{(\sigma)}_{s,t} \in L^p(\mathbf{S}^1) \iff p < 2\,;$$

(c) For $1 < p < \infty$ and $\sigma = (2-p)/p$, the function $F^{(\sigma)}_{s,t}$ does not belong to $L^p\mathfrak{C}$;
(d) For all distinct $s,t \in \mathbf{S}^1 \smallsetminus \{\pi/2\}$ and $0 < \sigma < 1$, the function $F^{(\sigma)}_{s,t}$ does not belong to $\mathfrak{C}(\sigma)$.

Proof. The first three points follow from elementary calculus. By the transitivity properties of P and Lemma 4.2, it is enough to show (d) for a particular pair of distinct s,t. Therefore we set, say, $f = F^{(\sigma)}_{0,\pi}$ so that $f(u) = \{2/\cos 2u\}_o^{(\sigma+1)/2}$. We need to show that $(u,v) \mapsto f(u)\overline{f(v)}|\sin(u-v)|^{\sigma-1}$ is not integrable in a neighborhood of the point $(\pi/4, \pi/4)$. Now since f behaves around $\pi/4$ as $X^{-(\sigma+1)/2}$ behaves around zero, this amounts to study the expression

$$\int_0^\epsilon \int_0^\epsilon \frac{X^{-(\sigma+1)/2}Y^{-(\sigma+1)/2}}{|X-Y|^{1-\sigma}}\,dX\,dY\,, \qquad (\epsilon > 0)\,.$$

If this were convergent, we could change to polar coordinates $X = r\cos\eta$, $Y = r\sin\eta$ and deduce the convergence of

$$\int_0^\epsilon \frac{1}{r}\int_0^{\pi/2} \frac{(\frac{1}{2}\sin 2\eta)^{-(\sigma+1)/2}}{|\cos\eta - \sin\eta|^{1-\sigma}}\,d\eta\,dr,$$

which is an *absurdum.* $\square$

Now we come to the major feature of the functions $F^{(\sigma)}_{s,t}$, namely that their singularities can be made to cancel each other in coboundary-like sums:

Lemma 4.4. *For all* $\sigma \in \mathbb{C} \smallsetminus \{-1\}$ *with* $\Re(\sigma) < 1$ *and all distinct* $s, t, u \in \mathbf{S}^1$, *there is a continuous function* $\mathbf{S}^1 \to \mathbb{C}$ *which coincides with* $F^{(\sigma)}_{s,t} + F^{(\sigma)}_{t,u} + F^{(\sigma)}_{u,s}$ *outside* $\big\{(\pi - 2s)/4,\ (\pi - 2t)/4,\ (\pi - 2u)/4\big\} + \pi\mathbb{Z}$.

Proof. We claim that for distinct $s, t \in \mathbf{S}^1 \smallsetminus \{\pi/2\}$ the function

$$F^{(\sigma)}_{s,\pi/2} + F^{(\sigma)}_{\pi/2,t}$$

can by continuously extended (by zero) at the point $0 \in \mathbf{S}^1$. This immediately implies the statement of the lemma because of the transitivity properties of P.

Applying the stereographic projection as in Lemma 4.2 (a) (which sends $\pi/2$ to ∞), we see that the claim follows from the fact that

$$\lim_{\substack{z \to \pm\infty, \\ x \to x_0, y \to y_0}} \Big(\{x - z\}_o^{(\sigma+1)/2} + \{z - y\}_o^{(\sigma+1)/2} \Big) = 0\,,$$

where $x_0 = \tan\big((2s + \pi)/4\big)$ and $y_0 = \tan\big((2t + \pi)/4\big)$. In fact, writing the expression in the limit as

$$\frac{\{z - y\}_o^{\sigma+1} - \{z - x\}_o^{\sigma+1}}{\{z - y\}_o^{(\sigma+1)/2} + \{z - x\}_o^{(\sigma+1)/2}}\,,$$

the above convergence statement follows from $\Re(\sigma) < 1$. $\square$

4.3 The Spectral Distribution

We have now collected all the ingredients to establish Proposition 4.1 and Theorem 1.1. We realize the bounded cohomology of G as in Corollary 2.2 (with alternating cochains).

Proof of Proposition 4.1. Under the stereographic projection $\rho : \mathbf{S}^1 \to \widehat{\mathbb{R}}$, the cocycle ω will be defined almost everywhere on $(\mathbf{S}^1)^3$ by

$$\rho^*\omega(s, t, u)(g) = F^{(\sigma)}_{g_\star s, g_\star t}(0) + F^{(\sigma)}_{g_\star t, g_\star u}(0) + F^{(\sigma)}_{g_\star u, g_\star s}(0)\,.$$

By Lemma 4.4 and Lemma 4.2 (c), $\rho^*\omega$ ranges indeed in $E(\sigma)$, so that it is bounded because the transitivity properties of G force it to have essentially constant norm.

The only point remaining to be justified is non-triviality. If in any of the three settings the class of $\rho^*\omega$ were trivial, we could find an alternating equivariant cochain α on $\mathbf{S}^1 \times \mathbf{S}^1$ with $\rho^*\omega = d\alpha$. But the uniqueness statement of Lemma 4.2 would then imply, via Fubini's theorem and $G = PK$, that $\alpha(s, t)(pk_u) = \delta_{\sigma+1}(p) F^{(\sigma)}_{s,t}(u)$ almost everywhere and up to a multiple. This would be incompatible with respectively Lemma 4.3 (b), (c) and Lemma 4.3 (d). $\square$

Proof of Theorem 1.1. *(ii)* Let $Z = \{\pm I\}$ be the kernel of $G \to H = \mathrm{PSL}_2(\mathbb{R})$. For every unitary representation $(\pi, \mathfrak{H})$ of G we have

$$\mathrm{H}^\bullet_{\mathrm{cb}}(G, \mathfrak{H}) \cong \mathrm{H}^\bullet_{\mathrm{cb}}(G, \mathfrak{H}^Z)\,,$$

see e.g. [13, 8.5.3]. If $\mathfrak{H}$ is irreducible, $\mathfrak{H}^Z$ can only be $\mathfrak{H}$ or zero and thus $\mathrm{H}^\bullet_{\mathrm{cb}}(G, \mathfrak{H})$ vanishes unless the representation π factors through $G \to H$. Therefore, for point (b) in the theorem, we have only to consider the discrete series. So assume that $\mathfrak{H}$ is such a representation; by Corollary 2.2, any class of $\mathrm{H}^2_{\mathrm{cb}}(G, \mathfrak{H})$ can be represented by a G-equivariant alternating bounded measurable cocycle

$$\omega :\ \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \longrightarrow \mathfrak{H}\,.$$

Applying Proposition 3.2, we get a cocycle for the corresponding differential group $\mathcal{A}^\bullet_{\mathfrak{H}_K}$ of Definition 3.3. By the Riemann–Lebesgue lemma, the corresponding function on $\mathbb{Z}^3$ must vanish at infinity. Therefore, the second point of Proposition 3.4 forces this function to be zero. By injectivity of the Fourier transformation, ω vanishes, too.

(i) Given Corollary 2.2, the case of the trivial representation $\mathfrak{H} = \mathbb{C}$ is just the following well known fact: up to scalar multiple, there is one and only one G-invariant alternating map $(\mathbf{S}^1)^3 \to \mathbb{C}$, and it is given by the cyclic orientation cocycle.

So let $\mathfrak{H}$ be a non-trivial irreducible unitary representation of spherical type. The conjunction of Proposition 3.2 with the first point of Proposition 3.4 gives that the dimension of $\mathrm{H}^2_{\mathrm{cb}}(G, \mathfrak{H})$ is at most one. Since we are left with representations of the principal and complementary series, we can apply the two first points in Proposition 4.1. This completes the proof of Theorem 1.1. □

5 Above Degree Two

We begin by collecting what we need from Sect. 3:

Proposition 5.1. *There is no non-zero alternating integrable G-invariant cocycle $(\mathbf{S}^1)^4 \to \mathbb{C}$.*

Proof. Suppose there were such a cocycle; then by Proposition 3.2 its Fourier transform would be a cocycle in the group $\mathcal{A}^3_{\mathbb{C}}$ as defined in Definition 3.3. But Proposition 3.4 (c) would then force it to vanish. □

Remark 5.2. In view of the precise statement of the Proposition 3.4 and of the proof of Proposition 3.2, we see that we have established the Proposition 5.1 not only for integrable functions, but for the whole algebraic dual of the space of trigonometric polynomials.

Now we have already:

Proof of Theorem 1.5. According to Corollary 2.2, every class of $\mathrm{H}^3_{\mathrm{cb}}(G)$ can be represented by an essentially bounded measurable alternating G-invariant cocycle $(\mathbf{S}^1)^4 \to \mathbb{C}$. Such a cocycle is integrable since the measure on $\mathbf{S}^1$ is finite, so by Proposition 5.1 the cocycle must be zero. □

We observe the immediate

Corollary 5.3. *The space* $\mathrm{H}^3_{\mathrm{cb}}(G')$ *is trivial for* $G' = \mathrm{PSL}_2(\mathbb{R})$, $\mathrm{GL}_2(\mathbb{R})$, $\mathrm{PGL}_2(\mathbb{R})$.

Proof. As recalled in the above proof of Theorem 1.1, $\mathrm{H}^\bullet_{\mathrm{cb}}(\mathrm{PSL}_2(\mathbb{R}))$ coincides with $\mathrm{H}^\bullet_{\mathrm{cb}}(\mathrm{SL}_2(\mathbb{R}))$. Since $\mathrm{PSL}_2(\mathbb{R})$ is a closed subgroup of finite index in $\mathrm{PGL}_2(\mathbb{R})$, the restriction map $\mathrm{H}^\bullet_{\mathrm{cb}}(\mathrm{PGL}_2(\mathbb{R})) \to \mathrm{H}^\bullet_{\mathrm{cb}}(\mathrm{PSL}_2(\mathbb{R}))$ is injective ([3]) and thus $\mathrm{H}^3_{\mathrm{cb}}$ vanishes also for the former. Finally, we have $\mathrm{H}^\bullet_{\mathrm{cb}}(\mathrm{PGL}_2(\mathbb{R})) = \mathrm{H}^\bullet_{\mathrm{cb}}(\mathrm{GL}_2(\mathbb{R}))$ since the canonical map $\mathrm{GL}_2(\mathbb{R}) \to \mathrm{PGL}_2(\mathbb{R})$ has amenable kernel, see e.g. [13, 8.5.3]. □

As for our interpretation of Bloch's result:

Proof of Theorem 1.2. Write $G_{\mathbb{C}} = \mathrm{SL}_2(\mathbb{C})$. In view of the discussion in the introduction, we have only to justify that $\mathrm{H}^3_{\mathrm{cb}}(G_{\mathbb{C}})$ contains no other class than the class determined by the volume form. Let us apply Theorem 2.1 to $H = G_{\mathbb{C}}$ and $S = \widehat{\mathbb{C}}$ with its H-action coming from the identification $\widehat{\mathbb{C}} \cong \partial\mathcal{H}^3$. This action is amenable since $\widehat{\mathbb{C}}$ is an homogeneous space with amenable isotropy [15, 4.3.2], the isotropy groups being minimal parabolic. Now Bloch's Theorem 7.4.4 in [1] states that there is only one measurable $G_{\mathbb{C}}$-invariant cocycle on $\widehat{\mathbb{C}}^4$ – and it is precisely given by the Bloch–Wigner dilogarithm of the crossratio (we do not need this information here). □

5.1 Rogers' Dilogarithm

Recall that the classical Euler dilogarithm Li_2 is defined by

$$Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad (|z| \leq 1)$$

and can be extended to $\mathbb{C} \smallsetminus [1, \infty[$ by

$$Li_2(z) = -\int_0^z \frac{\log(1-t)}{t}\, dt\,.$$

Rogers introduced for $0 < x < 1$ the following modification L_2 of the dilogarithm:

$$L_2(x) = -\frac{1}{2}\int_0^x \Big(\frac{\log t}{1-t} + \frac{\log(1-t)}{t}\Big)\, dt = \frac{Li_2(x) - Li_2(1-x) + Li_2(1)}{2}\,.$$

Since $L_2(1) = Li_2(1) = \zeta(2)$, there is the symmetry $L_2(1-x) = \zeta(2) - L_2(x)$. One verifies by differentiation that L_2 satisfies the functional equation (2) of Proposition 1.6; various forms of this equation can be found e.g. in [12].

5.2 Crossratio

Let us introduce some notation. We denote by $\mathcal{S}_n$ the symmetric group on n elements considered with its action on $\widehat{\mathbb{R}}^n$ by permutation of the coordinates. Let $\mathcal{C}_n < \mathcal{S}_n$ be the subgroup of cyclic permutations. We denote by $c : \widehat{\mathbb{R}}^4 \to \widehat{\mathbb{R}}$ the crossratio defined almost everywhere as

$$c(x_1, x_2, x_3, x_4) = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)}\,.$$

With the convention $\infty/\infty = 1$, this definition makes sense for all quadruples of distinct points in $\widehat{\mathbb{R}}$. Recall that c is invariant under the diagonal action of $G = \mathrm{PSL}_2(\mathbb{R})$. Endow $\widehat{\mathbb{R}}$ with the orientation induced by the natural order on $\mathbb{R}$ and denote by $\mathcal{D}_n^+$ the set of n-tuples of cyclicly positively ordered distinct points in $\widehat{\mathbb{R}}$; then $\mathcal{C}_n$ preserves $\mathcal{D}_n^+$. We write Δ_n for the set of points with at least two identical coordinates, so that

$$\widehat{\mathbb{R}}^n = \Delta_n \sqcup \bigsqcup_{[\pi] \in \mathcal{S}_n/\mathcal{C}_n} \pi \mathcal{D}_n^+\,, \tag{9}$$

where π ranges over a set of coset representatives. We remark that the image $c(\mathcal{D}_4^+)$ of $\mathcal{D}_4^+$ under the crossratio is the open interval $]0,1[$. Indeed, since G is transitive on $\mathcal{D}_3^+$, it suffices to notice that for all $0 < x < 1$ one has $c(0, x, 1, \infty) = x$.

Let now $F :]0,1[\to \mathbb{C}$ be an integrable function and write $\tau F(x) = F(1-x)$. We define $\Omega_F : \widehat{\mathbb{R}}^4 \to \mathbb{C}$ as follows. Set first $\Omega_F(x) = F \circ c(x)$ for all $x \in \mathcal{D}_4^+$, and observe that the condition $\tau F = -F$ is actually equivalent to the $\mathcal{C}_4$-alternation of Ω_F. Indeed, denoting by σ any generator of $\mathcal{C}_4$, one checks that

$$c \circ \sigma = 1 - c\,.$$

Therefore, if $\tau F = -F$, there is a unique extension of the definition of Ω_F to an alternating map $\widehat{\mathbb{R}}^4 \to \mathbb{C}$ because of (9); Ω_F must be zero on Δ_4 by alternation. Moreover, Ω_F is G-invariant by definition, for the diagonal G-action commutes with $\mathcal{S}_n$. Writing out the crossratio, we have

$$\begin{aligned} &d\Omega_F(0, x, y, 1, \infty) \\ &\quad = F\left(\frac{y-x}{1-x}\right) - F(y) + F(x) - F\left(\frac{x}{y}\right) + F\left(\frac{x(1-y)}{y(1-x)}\right) \end{aligned} \tag{10}$$

for all $0 < x < y < 1$. Finally, since the projective measure on $\widehat{\mathbb{R}}$ is finite, the Fubini–Lebesgue theorem together with the integrability of F implies that Ω_F is integrable on $\widehat{\mathbb{R}}^4$.

Proof of Proposition 1.4. As we have already mentioned in the proof of Theorem 1.1, there is up to scalar multiple only one G-invariant alternating map $(\mathbf{S}^1)^3 \to \mathbb{C}$, and it is given by the cyclic orientation cocycle ω defined on $\widehat{\mathbb{R}}^3 \smallsetminus \Delta_3$ by $\omega(\pi x) = \mathrm{sign}(\pi)$ for $\pi \in \mathcal{S}_3$ and $x \in \mathcal{D}_3^+$. It is therefore enough in view of Corollary 2.2 to show that $\omega \wedge \omega$ is of the form $d\Omega$ for some Ω in $L^\infty(\widehat{\mathbb{R}}^4)$. Since $\omega \wedge \omega$ is locally constant on $\widehat{\mathbb{R}}^5 \smallsetminus \Delta_5$, it is uniquely determined by its value on a fixed $x = (x_0, \ldots, x_4) \in \mathcal{D}_5^+$. We contend that $\omega \wedge \omega(x) = 1/3$. To see this, we consider the $\mathcal{S}_5$-action on $\{0, \ldots, 4\}$ in order to define the subgroup $\mathcal{B}$ which permutes block-wise $\{0,1\}$ and $\{3,4\}$:

$$\mathcal{B} = \Big\{\pi \in \mathrm{Stab}_{\mathcal{S}_5}(2) : \pi(\{0,1\}) \in \big\{\{0,1\},\{3,4\}\big\}\Big\}.$$

Now the number

$$\mathrm{sign}(\pi)\omega \times \omega(\pi^{-1}x) := \mathrm{sign}(\pi)\omega(x_{\pi(0)}, x_{\pi(1)}, x_{\pi(2)})\omega(x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)})$$

depends only on the class of π in $\mathcal{B}\backslash\mathcal{S}_5/\mathcal{C}_5$. There are three such double cosets, and one checks that the above number is positive for two of them, negative for the third. Therefore,

$$\omega \wedge \omega(x) = \frac{1}{|\mathcal{S}_5|} \sum_{\pi \in \mathcal{S}_5} \mathrm{sign}(\pi)\, \omega \times \omega\, (\pi^{-1}x) = \frac{1}{3}\,,$$

as claimed. If we set now $F = (1 - 2L_2/\zeta(2))/3$, we have $\tau F = -F$ and the above construction yields an alternating integrable G-invariant function Ω_F. We claim that $d\Omega_F = \omega \wedge \omega$; indeed, by alternation we may restrict to $\mathcal{D}_5^+$, and by G-invariance even to the points $(0, x, y, 1, \infty)$ for $0 < x < y < \infty$. Now the Spence–Abel equation (2) applied to (10) yields

$$d\Omega_F(0, x, y, 1, \infty) = \frac{1}{3} = \omega \wedge \omega(0, x, y, 1, \infty)\,,$$

finishing the proof. □

Proof of Proposition 1.6. If a function L has the two properties assumed, then setting $F' = (1 - 2L/\zeta(2))/3$ we would as above get $d\Omega_{F'} = \omega \wedge \omega$, so that $d(\Omega_F - \Omega_{F'}) = 0$. Therefore, applying Proposition 5.1, we deduce $\Omega_F = \Omega_{F'}$. Since the crossratio sends $\mathcal{D}_4^+$ onto $]0,1[$, we conclude that $F = F'$ whence $L = L_2$. □

References

1. S. Bloch *Higher regulators, algebraic K-theory, and zeta functions of elliptic curves* manuscript (1978), published as CRM Monograph **11** AMS, Providence (2000).
2. M. Burger, N. Monod *Bounded cohomology of lattices in higher rank Lie groups* J. Eur. Math. Soc. **1** no. 2 (1999) 199–235.
3. M. Burger, N. Monod *Continuous bounded cohomology and applications to rigidity theory* preprint (2000) to appear in Geom. Funct. Anal.

4. M. G. Cowling *Unitary and uniformly bounded representations of some simple Lie groups* Harmonic analysis and group representations 49–128 Liguori editore, Napoli (1982).
5. N. Dunford, J. T. Schwartz *Linear operators, I: General theory* Pure and Appl. Math. Vol. VII Interscience Publishers, New York, London (1958).
6. J. L. Dupont, *Bounds for characteristic numbers of flat bundles* Algebraic topology, Aarus 1978, Lecture Notes in Mathematics **763** Springer Verlag (1979).
7. D. B. A. Epstein, K. Fujiwara *The second bounded cohomology of word-hyperbolic groups* Topology **36** no. 6 (1997) 1275–1289.
8. I. M. Gelfand, R. D. MacPherson *Geometry in Grassmannians and a generalization of the dilogarithm* Adv. in Math. **44** (1982) 279–312.
9. R. E. Howe, E.-Ch. Tan *Non-Abelian harmonic analysis* Universitext, Springer-Verlag (1992).
10. A. W. Knapp *Representation theory of semisimple groups* Princeton Math. Series **36** Princeton University Press, Princeton (1986).
11. S. Lang $SL_2(\mathbb{R})$ Graduate Texts in Math. **105** Springer-Verlag (1985).
12. L. Lewin *The evolution of the ladder concept* In: Structural properties of polylogarithms Mathematical Surveys and Monographs **37** (1991) 1–10.
13. N. Monod *Continuous Bounded Cohomology of Locally Compact Groups* Lecture Notes in Mathematics **1758** Springer-Verlag (2001).
14. D. A. Vogan *Unitary representations of reductive Lie groups* Annals of Mathematical Studies **118** Princeton University Press, Princeton, NJ (1987).
15. R. J. Zimmer *Ergodic theory and semisimple groups* Monographs in Math. **81** Birkhäuser, Basel (1984).