ERRATUM AND ADDENDA TO "ISOMETRY GROUPS OF NON-POSITIVELY CURVED SPACES: DISCRETE SUBGROUPS"

PIERRE-EMMANUEL CAPRACE* AND NICOLAS MONOD

Abstract. We amend the statement of point (i) in Theorem 1.3 in [CM09b] and supply the additional arguments and minor changes for the results that depend on it. We also seize the occasion and generalize to non-finitely generated lattices.

1. The Euclidean Factor Theorem for CAT(0) lattices

I'm fixing a hole where the rain gets in
And stops my mind from wandering
Lennon–McCartney, *Fixing a Hole*, 1967

Let X be a proper CAT(0) space and let $n \geq 0$ be the dimension of the maximal Euclidean factor of X. Let $G < \mathrm{Is}(X)$ be a closed subgroup acting minimally and cocompactly on X, and let $\Gamma < G$ be a finitely generated lattice. Theorem 1.3(i) in [CM09b] states that Γ has a finite index subgroup Γ_0 that splits as $\Gamma_0 \cong \mathbf{Z}^n \times \Gamma'$, and that n coincides with the maximal rank of a free abelian normal subgroup of Γ.

This result is incorrect, as shown by the example below, which is a simple case of a beautiful general construction of Leary–Minasyan [LM] for which they prove a number of deep results. Our mistake lies in the erroneous assertion from the proof of Proposition 3.6 loc. cit. that "the commensurator of any lattice in A is virtually abelian".

However, the *normaliser* of any lattice in the Euclidean motion group $A = \mathbf{R}^k \rtimes O(k)$ is indeed virtually abelian (see Lemma 5 below). Relying on this, we state and prove an amended version of Theorem 1.3(i), namely Theorem 2 below. That result ensures similarly that n coincides with the maximal rank of a free abelian *commensurated* subgroup of Γ.

Concerning the virtual splitting of Γ as $\mathbf{Z}^n \times \Gamma'$, it does nonetheless hold provided Γ is residually finite, see Theorem 2(iv).

Example 1 (See [LM]). The construction starts from a Pythagorean triple, say $(3, 4, 5)$. The matrix $\alpha = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}$ is in $\mathrm{SL}_2(\mathbf{Z}[1/5]) \cap SO(2)$ and represents an irrational rotation of $\mathbf{R}^2$. Therefore the corresponding semi-direct product $\Lambda = (\mathbf{Z}[1/5])^2 \rtimes_\alpha \mathbf{Z}$ embeds as a dense subgroup in $A = \mathbf{R}^2 \rtimes SO(2)$. Moreover, Λ is finitely generated (by α together with a suitable pair of independent elements of $(\mathbf{Z}[1/5])^2$). Let now $D' = (\mathbf{Q}_5)^2 \rtimes_\alpha \mathbf{Z}$ be the semi-direct product of the additive group of the 2-dimensional vector space over the 5-adic rationals, by the cyclic automorphism group generated by α. The diagonal embedding of $(\mathbf{Z}[1/5])^2$ in $\mathbf{R}^2 \times (\mathbf{Q}_5)^2$ is a cocompact lattice embedding. It follows that Λ embeds diagonally as a cocompact lattice in $A \times D'$, with dense projection on both factors. Since D' is compactly generated (because it contains a dense copy of Λ) and does not have non-trivial compact normal subgroup, it has a continuous faithful

Date: August 2019.

* F.R.S.-FNRS senior research associate.

vertex-transitive action on a locally finite graph $\mathfrak{g}$ (a "Cayley–Abels graph", see e.g. the end of 5.2 in [Abe74]). Let D be the extension of D' by the fundamental group of $\mathfrak{g}$. Then D acts vertex-transitively on universal cover of $\mathfrak{g}$, which is a regular locally finite tree T. Moreover the cocompact irreducible lattice $\Lambda < A \times D'$ lifts to a cocompact irreducible lattice Γ in $G = A \times D$. Notice that G acts cocompactly and minimally on the proper CAT(0) space X defined by $X = \mathbf{R}^2 \times T$. In particular Γ is a CAT(0) group.

On the other hand, no finite index subgroup of Γ has a finitely generated free abelian normal subgroup. Indeed, we first observe that the projection $p(N)$ to A of such a subgroup $N < \Gamma$ must be trivial. To prove this claim, notice that by construction, the projection of Γ to A is the metabelian group Λ. The intersection of $p(N)$ with the translation subgroup $V = \mathbf{Z}[1/5]^2$ of Λ is a finitely generated free abelian normal subgroup. Any finitely generated subgroup of V is generated by at most two elements, and is thus a discrete subgroup of $\mathbf{R}^2$. Since $p(N) \cap V$ is normalised by the irrational rotation α, it follows that $p(N) \cap V$ is trivial. Therefore $p(N)$ commutes with V. But V is its own centraliser in Λ, hence $p(N) < V$ and the claim follows since $p(N) \cap V$ is trivial. It follows that every finitely generated free Abelian subgroup of Γ would be an abelian normal subgroup of D, and hence it is necessarily trivial by [CM09a, Theorem 1.10].

Observe that G is a compactly generated locally compact group whose amenable radical coincides with A. The example shows moreover that the image of a cocompact lattice in G need not project to a lattice in the quotient of G by its amenable radical. This lies in sharp contrast with the case of connected Lie groups, where the image of a lattice modulo the amenable radical is always a lattice, by Auslander's theorem [Rag72, Theorem 23.9.3].

Theorem 1.3(i) from [CM09b] should be replaced by the following; notice that no finite generation assumption is made in (i).

Theorem 2. *Let X be a proper* CAT(0) *space, $G < \mathrm{Is}(X)$ a closed subgroup acting minimally and cocompactly, and $\Gamma < G$ any lattice. Let $E \cong \mathbf{R}^n$ be the maximal Euclidean factor of X, where $n \geq 0$.*

(i) Γ commensurates a free abelian subgroup $\Gamma_A \cong \mathbf{Z}^n$, and n is the largest such rank. Moreover, any commensurated abelian subgroup of Γ acts properly on E.

We now assume Γ finitely generated.

(ii) If Γ virtually normalises a free abelian subgroup of rank k, then Γ virtually splits as $\mathbf{Z}^k \times \Gamma'$. Moreover there is a corresponding invariant decomposition $X \cong \mathbf{R}^k \times X'$, and the projection of $\mathbf{Z}^k$ (resp. Γ') to $\mathrm{Is}(X')$ is trivial (resp. discrete).

(iii) If the projection of Γ to $\mathrm{Is}(E)$ is virtually abelian, then Γ virtually splits as $\mathbf{Z}^n \times \Gamma'$.

(iv) If Γ is residually finite, then again Γ virtually splits as $\mathbf{Z}^n \times \Gamma'$.

(v) If Γ is cocompact, then Γ virtually splits as $\mathbf{Z}^k \times \Gamma'$ and there is a corresponding invariant decomposition $X \cong \mathbf{R}^k \times X'$ for some $k \geq 0$, satisfying the same properties as in (ii), and such that moreover every Γ'-orbit in $\partial X'$ is infinite.

We recall that the *virtual splittings* above mean by definition that a finite index subgroup splits as specified. Item (v) implies that if Γ fixes a point in ∂X, then Γ virtually splits as $\mathbf{Z} \times \Gamma'$.

The condition that Γ normalizes a free abelian subgroup in (ii) is natural, since the center of Γ, and more generally the amenable radical of Γ, is virtually free abelian. In particular, we obtain the following.

Corollary 3. *Let X be a proper* CAT(0) *space whose maximal Euclidean factor has dimension n. Let $G < \mathrm{Is}(X)$ be a closed subgroup acting minimally and cocompactly, and $\Gamma < G$ any lattice.*

Then the amenable radical of Γ is virtually isomorphic to $\mathbf{Z}^k$ for some $k \leq n$. If in addition Γ is finitely generated, then Γ has a finite index subgroup Γ_0 that splits as $\Gamma_0 \cong \mathbf{Z}^k \times \Gamma'$, where $k \leq n$ and Γ' has trivial amenable radical (hence a trivial centre).

Moreover, there is a corresponding Γ_0-invariant decomposition $X \cong \mathbf{R}^k \times X'$ and the projection of Γ_0 to $\mathrm{Is}(X')$ is discrete.

The proof of Theorem 2(i) requires the following fact, which supplements a result on normal subgroups proved in [CM09a, Theorem 1.10].

Proposition 4. *Let $X \neq \mathbf{R}$ be an irreducible proper $\mathrm{CAT}(0)$ space with finite-dimensional Tits boundary and $G < \mathrm{Is}(X)$ any subgroup whose action is minimal and does not have a global fixed point in ∂X.*

Then any commensurated subgroup $H < G$ either fixes a point in X, or still acts minimally.

Proof. Let ΔH be the **convex limit set** of H, defined as the visual boundary ∂Y of the closed convex hull Y of some H-orbit. By [CM09a, Lemma 4.2], this set is well-defined, i.e. it does not depend of the choice of a specific H-orbit. If $H_0 < H$ is a finite index subgroup, then each H_0-orbit is cobounded in the corresponding H-orbit and it follows that its convex hull is cobounded in the convex hull of the H-orbit. Thus, these two convex sets have the same visual boundary. It follows that if H' is commensurable to H in G, then $\Delta H' = \Delta H$. Since H is commensurated by G by hypothesis, the closed convex subset $\Delta H \subseteq \partial X$ is G-invariant. If ΔH is empty, then H has bounded orbits and fixes points in X. If ΔH is non-empty, then it must have circumradius $> \pi/2$ since otherwise G fixes a point in ∂X by [BL05, Proposition 1.4]. By [CM09a, Proposition 3.6] the union of all minimal closed convex subspaces $Z \subseteq X$ with $\partial Z = \Delta H$ is a non-empty closed convex subspace which splits as a product $Z_0 \cong Z \times Z'$. Since ΔH is G-invariant, so is Z_0. By hypothesis X is irreducible and the G-action is minimal. It follows that $X = Z_0 = Z$ and that $\Delta H = \partial X$. In particular for every closed convex subset $X' \subsetneq X$, we have $\partial X' \subsetneq \partial X$, and we infer that the convex hull of every H-orbit is dense in X. In other words H acts minimally on X. □

As mentioned above, we shall also need the following.

Lemma 5. *Let $G = \mathbf{R}^k \rtimes O(k)$ and $\Gamma \leq G$ be a lattice. Then the normaliser $N_G(\Gamma)$ has a finite index subgroup contained in the translation group $\mathbf{R}^k$. In particular $N_G(\Gamma)$ is virtually abelian.*

Proof. Let $V = \mathbf{R}^k$ be the translation subgroup of G and set $V_\Gamma = V \cap \Gamma$. Since V is normal in G, we have $N_G(\Gamma) \leq N_G(V_\Gamma) =: H$. The group V_Γ is a discrete normal subgroup of H. Therefore every element of V_Γ has a discrete conjugacy class in H, so that the centraliser $C_H(v)$ is open in H for all $v \in V_\Gamma$. By the Bieberbach Theorem, V_Γ is a lattice in V, hence it is finitely generated. It follows that $C_H(V_\Gamma)$ is open in H. Clearly $V \leq C_H(V_\Gamma) \leq C_G(V_\Gamma) \leq N_G(V_\Gamma) = H$. On the other hand the centraliser in G of the lattice V_Γ must act trivially at infinity of the Euclidean space, so that $C_G(V_\Gamma) \leq V$. Therefore $V = C_H(V_\Gamma) = C_G(V_\Gamma)$, and the natural image of H in $G/V \cong O(k)$ is closed (because $V \leq H$) and at most countable (because V is open in H). Since $O(k)$ is compact, the natural image of H in $O(k)$ is thus finite by Baire's theorem. In particular, so is the image of $N_G(\Gamma)$, as required. □

Lemma 6. *Let $A = \mathbf{R}^n \rtimes O(n)$, S be a semi-simple Lie group with trivial centre and no compact factor, D be a totally disconnected locally compact group, and $G = S \times A \times D$.*

Then any lattice $\Gamma < G$ commensurates a free abelian subgroup $\Gamma_A < \Gamma$ of rank n. Moreover the projection of Γ_A to A (resp. to S) is a lattice (resp. is trivial).

Proof. Let $Q < D$ be a compact open subgroup. By [CM09b, Lemma 3.2], the intersection $\Gamma^* = \Gamma \cap (S \times A \times Q)$ is a lattice in $S \times A \times Q$, which is commensurated by Γ. Since Q is compact, the projection of Γ^* to $S \times A$ is a lattice. Upon replacing Γ^* by a finite index subgroup, we may then assume by Lemmas 3.4 and 3.5 from [CM09b] that Γ^* possesses two normal subgroups Γ^*_S and Γ^*_A, both commensurated by Γ, such that $\Gamma^* = \Gamma^*_S \cdot \Gamma^*_A$ and $\Gamma^*_S \cap \Gamma^*_A \subseteq Q$, where $\Gamma^*_A = \Gamma^* \cap (1 \times A \times Q)$ is a finitely generated free abelian group whose projection to A is a lattice. In particular Γ^*_A is a free abelian group of rank n which is commensurated by Γ and we can define $\Gamma_A = \Gamma^*_A$. □

Proof of Theorem 2. By [CM13, Theorem M], the only points of ∂X that are possibly fixed by G lie in the visual boundary of the maximal Euclidean factor of X. In particular the full isometry group $\mathrm{Is}(X)$ has no fixed point in ∂X. Theorem 1.6 and Addendum 1.8 from [CM09a] then provide a canonical $\mathrm{Is}(X)$-invariant decomposition $X \cong M \times \mathbf{R}^n \times Y$, where $\mathbf{R}^n$ is the maximal Euclidean factor of X, $S = \mathrm{Is}(M)$ is a semi-simple Lie group with trivial center and no compact factor, and $D = \mathrm{Is}(Y)$ is totally disconnected. In particular Γ is a lattice in $\mathrm{Is}(X) = S \times A \times D$, where $A \cong \mathbf{R}^n \rtimes O(n)$.

The fact that Γ contains a commensurated free abelian subgroup of rank n now follows from Lemma 6. Assume conversely that Γ commensurates some abelian subgroup H; we only need to consider the case where H is infinite. By Borel density, the projection of H to S is finite; we may thus assume that it is trivial upon replacing H by a finite index subgroup. Recall that an abelian group cannot act minimally on a non-trivial proper CAT(0) space without Euclidean factor (indeed, since the displacement function of an element g is invariant under the centraliser $\mathscr{Z}_H(g)$, the minimality of H implies that g has constant displacement function, and is thus a Clifford isometry). Therefore, Proposition 4 implies that H fixes a point in each irreducible factor of Y, hence also in Y itself. In other words the closure of the projection of H to $S \times D$ is compact. It follows that the H-action on the maximal Euclidean factor $\mathbf{R}^n$ is indeed proper. In particular, if H is free abelian of rank m, then $m \leq n$. This proves (i).

For the rest of the proof, we suppose that Γ is finitely generated.

Assume that Γ normalises a free abelian subgroup of rank k, say H. By (i), the group H acts properly on the maximal Euclidean factor $\mathbf{R}^n$. Since it is normal, it yields a Γ-invariant decomposition $\mathbf{R}^n = \mathbf{R}^k \times \mathbf{R}^{n-k}$, and the H-action on the $\mathbf{R}^{n-k}$-factor is trivial. Since the projection of Γ to $\mathrm{Is}(\mathbf{R}^k)$ normalises a lattice, it is virtually abelian by Lemma 5. Upon passing to a finite index subgroup, we may thus assume that the image of Γ in $\mathrm{Is}(\mathbf{R}^k)$ is free abelian of the form $\mathbf{Z}^k \oplus \mathbf{Z}^m$, where $\mathbf{Z}^k$ is the image of H. Let Λ be the preimage of the $\mathbf{Z}^m$-factor. Then $\Gamma = H \cdot \Lambda$. Moreover the intersection $H \cap \Lambda$ acts trivially on $\mathbf{R}^k$. By construction H acts trivially on the complementary Euclidean factor $\mathbf{R}^{n-k}$. Since moreover H is an abelian normal subgroup of a lattice, it acts trivially on $M \times Y$ by [CM09a, Theorem 1.10] and [CM09b, Theorem 1.1]. Therefore $H \cap \Lambda$ is trivial, whence $\Gamma \cong H \times \Lambda$. Moreover the projection of H to $\mathrm{Is}(\mathbf{R}^{n-k}) \times S \times D$ is trivial, hence the projection of Γ is discrete by [CM09b, Proposition 3.1]. This proves (ii).

Assume now that Γ projects to a virtually abelian subgroup of A. Since Γ is finitely generated, upon replacing Γ by a finite index subgroup, we may thus assume that the image of Γ is free abelian of the form $\mathbf{Z}^n \oplus \mathbf{Z}^m$, where $\mathbf{Z}^n$ is the image of the commensurated subgroup Γ_A provided by (i), which acts thus properly on $\mathbf{R}^n$ and fixes a point in $M \times Y$. Let Λ be the preimage of the $\mathbf{Z}^m$-factor in Γ. Thus Λ is normal in Γ and we have $\Gamma = \Gamma_A \cdot \Lambda$.

Since the closure of the projection of Γ_A to $S \times D$ is compact, it follows that the closure of the projection of Λ, which is denoted by L, is cocompact in the closure of the projection of Γ, which is denoted by J. Since J is compactly generated (because Γ is finitely generated and maps densely to J), the cocompact subgroup L is compactly generated as well. Moreover the derived group $[\Lambda, \Lambda]$ maps trivially to A and is thus a discrete normal subgroup of L. It follows that the quotient $L/[\Lambda, \Lambda]$ is a compactly generated abelian locally compact group. It is therefore of the form $K \times \mathbf{R}^p \times \mathbf{Z}^q$ for some $p, q \geq 0$, where K is a compact subgroup (see §29 in [Wei40] or 9.8 in [HR63]). The identity component of L maps trivially to D. It is thus a connected subgroup of S normalised by the projection of a lattice. By Borel density, it follows that L° is semi-simple, and can therefore not have any non-trivial connected abelian quotient. In view of Lemma 2.4 from [CCMT15], it follows that $p = 0$, i.e. $L/[\Lambda, \Lambda] \cong K \times \mathbf{Z}^q$. Now the preimage of $\mathbf{Z}^q$ in L is a cocompact discrete normal subgroup. It must therefore be finitely generated, and its centraliser is thus an open subgroup of L. By [CM09b, Theorem 1.1], the group L acts minimally without a fixed point at infinity on $M \times Y$, and any cocompact subgroup of L must therefore have trivial centraliser. It follows that L is discrete. Thus L is a discrete cocompact normal subgroup of J. By the same arguments we have just used, it follows that L is finitely generated and has an open centraliser in J, from which it follows that J is discrete as well.

This now implies that, upon replacing Γ_A by a finite index subgroup, the projection of Γ_A to $S \times D$ is trivial. It then follows that Γ_A is normal in Γ, and assertion (iii) therefore follows from (ii).

We now turn to (iv). Let $\Lambda < \Gamma$ be the profinite closure of Γ_A in Γ, that is, the smallest separable subgroup of Γ containing Γ_A. Then Λ contains a finite index subgroup $\Lambda_0 < \Lambda$ that is normal in Γ. Indeed, this follows from the main result of [CKRW] since Γ is finitely generated (or it could be deduced from [CM11, Cor. 4.1]; see [CKRW, Rem. 5]).

Since we now assume that Γ is residually finite, its profinite topology is Hausdorff and hence Λ is commutative. In particular, Λ_0 acts properly on E by (i); it is thus finitely generated and hence, upon possibly replacing it by a further finite index subgroup that is normal in Γ, we have $\Lambda_0 \cong \mathbf{Z}^k$ with $k \leq n$. Recalling that Λ contains Γ_A, we have $k = n$ and now the conclusion follows from (ii).

It remains to prove (v). Let $\Gamma_1 \leq \Gamma$ be a finite index subgroup. By [CM09b, Th. 3.14], the group Γ_1 acts minimally on X. By [CM09b, Prop. 3.15], there is a Γ_1-invariant splitting $X \cong E \times X'$ such that E is flat, Γ_1 acts trivially on ∂E and has no fixed point in $\partial X'$. Among all such finite index subgroups, we now pick one, say Γ_0, for which the flat factor E is of maximal possible dimension. Thus we have a Γ_0-invariant splitting $X \cong \mathbf{R}^k \times X'$, and the choice of Γ_0 ensures that every Γ_0-orbit in $\partial X'$ is infinite.

Fix k pairs $(\xi_1, \xi_1'), \dots, (\xi_k, \xi_k')$ of antipodal pairs in the boundary of the flat factor $\mathbf{R}^k$, so that the Tits distance from ξ_i to ξ_j is $\pi/2$ for all $i \neq j$. Invoking [CM13, Prop. K] k times successively, we deduce that the kernel Γ' of the projection of Γ_0 to $\mathrm{Is}(\mathbf{R}^k)$ acts cocompactly on X'. Denoting by H the closure of the projection of Γ_0 to $\mathrm{Is}(X')$, we infer that Γ' is a normal cocompact lattice in H. Since Γ_0 is finitely generated, the group H is compactly generated, so that Γ' is finitely generated. The centralizer $\mathscr{Z}_H(\Gamma')$ is thus open in H, so that $\mathscr{Z}(\Gamma') = \Gamma' \cap \mathscr{Z}_H(\Gamma')$ is a cocompact lattice in $\mathscr{Z}_H(\Gamma')$. By [CM09a, Th. 1.10] and [CM09b, Cor. 2.7], the centralizer $\mathscr{Z}_H(\Gamma')$ acts properly on the maximal Euclidean flat factor of X'. Therefore the discrete group $\mathscr{Z}(\Gamma')$ also acts properly on a flat, and is thus finitely generated. If $\mathscr{Z}(\Gamma')$ were infinite, then Γ_0 would normalize a free abelian subgroup of positive rank contained in Γ'. In view of (ii), this would yield a finite Γ_0-orbit in $\partial X'$, which is impossible. Therefore $\mathscr{Z}(\Gamma')$ is finite, so that $\mathscr{Z}_H(\Gamma')$ is compact (because it contains $\mathscr{Z}(\Gamma')$ as a lattice). Since $\mathscr{Z}_H(\Gamma')$ is normal in H and H acts minimally on X', we deduce that H is discrete. This implies that the kernel, denoted by V, of the projection $\Gamma_0 \to \mathrm{Is}(X')$ is a lattice in $\mathrm{Is}(\mathbf{R}^k)$. Thus V is a normal subgroup of Γ_0 virtually isomorphic to $\mathbf{Z}^k$. The required conclusion now follows from (ii). $\square$

Proof of Corollary 3. Retaining the notation of the proof of Theorem 2, we know from [CM09a, Theorem 1.10] and [CM09b, Theorem 1.1] that the amenable radical of Γ acts trivially on $M \times Y$. Therefore it is a discrete group acting properly on the flat factor $\mathbf{R}^n$, and is consequently virtually isomorphic to $\mathbf{Z}^k$. If Γ is finitely generated, we may invoke Theorem 2(ii) and the required conclusions follow. □

We end this section with an example showing that the finite generation hypothesis cannot be discarded in Theorem 2(ii).

Example 7. We construct a CAT(0) lattice Γ with a normal subgroup isomorphic to $\mathbf{Z}$, but no finite index subgroup splitting as a direct product $\mathbf{Z} \times \Gamma'$, as follows.

Let $A = \mathbf{Z}[\frac{1}{2}]$ and let $B = \mathbf{Z}[\frac{1}{2}]/\mathbf{Z}$ be the the Prüfer 2-group. Then A is a non-trivial central extension of B. Let $B^{*3} = B * B * B$ and consider the retraction $\pi \colon B^{*3} \to B$ onto the first factor. We define Γ to be the pull-back by π of the central extension A, thus containing a central subgroup $\mathbf{Z}$. We refer to e.g. [Bro94, p. 94] for the pull-back of an extension and highlight just two facts. First, Γ is a fibered product $A \times_B B^{*3}$ over B and hence, in particular, Γ is a subgroup of $A \times B^{*3}$. Secondly, the central extension Γ is non-trivial because it is classified by the inflation map $\pi^* \colon H^2(B, \mathbf{Z}) \to H^2(B^{*3}, \mathbf{Z})$ and the latter is injective since π admits a right inverse, the canonical inclusion.

We claim that Γ does not have any finite index subgroup Γ_0 that split as $\Gamma_0 = \mathbf{Z} \times \Gamma'$. Indeed, assume for a contradiction that such a finite index subgroup Γ_0 exists. We identify A with its image in Γ and set $A_0 = A \cap \Gamma_0$. Since the center of Γ is contained in A, we see that A_0 contains a finite index subgroup of the center of Γ. The group A is 2-divisible (i.e. for every n, every element admits a 2^n-th root), so every finite index subgroup is of odd index, and is thus itself 2-divisible. In particular, the restriction of the projection map $\Gamma_0 \to \mathbf{Z}$ to A_0 must be trivial. It follows that the restriction of the projection map $\Gamma_0 \to \Gamma'$ to A_0 is injective. Since B is both 2-divisible and 2-torsion, it does not admit any non-trivial finite quotient. Therefore the same holds for B^{*3}. Using the classification of commuting elements in free products (see e.g. [MKS66, 4.5]), we see that the restriction of the projection map $\Gamma \to B^{*3}$ to Γ_0 is trivial on the $\mathbf{Z}$ factor, and induces an isomorphism $\Gamma' \to B^{*3}$. From the discussion above, we infer that A_0, which contains an infinite cyclic central subgroup of Γ, injects in B^{*3} under the projection map $\Gamma \to B^{*3}$. This is absurd, since the center of B^{*3} is trivial. This contradiction confirms the claim.

Finally, we proceed to realize Γ as a lattice in the isometry group of the cocompact CAT(0) space $\mathbf{R} \times T$, where T is the trivalent tree. The group B^{*3} can be seen as the fundamental group of a graph of finite groups, consisting of three rays of groups emanating from a common origin. The vertex and edge groups attached to vertices and edges at distance n from the origin are cyclic groups of order 2^n. The fundamental group of that graph of groups indeed acts properly on T, and is a non-uniform lattice in $\mathrm{Aut}(T)$ by Serre's covolume formula. By construction, Γ is a subgroup of $A \times B^{*3} \leq \mathrm{Is}(\mathbf{R}) \times \mathrm{Aut}(T)$. Since the image of the projection of Γ to $\mathrm{Aut}(T)$ is a lattice in $\mathrm{Aut}(T)$ and the kernel of that projection is a lattice in $\mathrm{Is}(\mathbf{R})$ (namely $\mathbf{Z}$), it follows that Γ is indeed a CAT(0) lattice, as required.

2. Further corrections

The main point of this erratum is that Theorem 1.3(i) in [CM09b] should be replaced by Theorem 2 above. All other results presented throuhout the Introduction of [CM09b] remain valid without change.

We now proceed to indicate all modifications in the body of [CM09b] that are required by this correction. We only discuss the results that rely directly or indirectly on Theorem 1.3(i) as all the other results are unaffected and hold without changes, with one exception: in Section 6.C of [CM09b], the construction of pathological examples is mistaken. Specifically, the properness asserted in Lemma 6.9 does not hold.

- Theorem 1.3(ii) remains valid, but it contains a typo: the statement should read "Is(X) has no fixed point at infinity". We remark that Theorem 1.3(ii) has been generalised to infinitely generated lattices in [CM13, Theorems L and M].
- Proposition 3.6 should be replaced by Lemma 6 above.
- Theorem 3.8, which is a reformulation of Theorem 1.3(i), should be replaced by Theorem 2 above.
- Corollary 3.10 should be replaced by Theorem 2(i) above. When the lattice is residually finite, then Corollary 3.10 holds as is by Theorem 2(iv).
- Theorem 4.2 remains valid as is, although its proof currently relies on Theorem 3.8. The proof can be corrected as follows. We use the notation from loc. cit. Let $X'_2 = \mathbf{R}^n \times X''_2$ be the canonical decomposition of X'_2, where $\mathbf{R}^n$ is the maximal Euclidean factor. Let G_2 be the closure of the projection of Γ_0 to $\mathrm{Is}(X'_2)$ and $\Gamma_2 < G_2$ be the kernel of the projection of Γ_0 to $\mathrm{Is}(X'_1)$. The current proof shows that Γ_2 is a normal, hence cocompact and finitely generated, lattice in G_2. Moreover the centraliser $\mathscr{Z}_{G_2}(\Gamma_2)$ is open in G_2. Therefore the intersection $\Gamma_2 \cap \mathscr{Z}_{G_2}(\Gamma_2)$ is a lattice in $\mathscr{Z}_{G_2}(\Gamma_2)$. Since $\Gamma_2 \cap \mathscr{Z}_{G_2}(\Gamma_2)$ is nothing but the center of Γ_2, it is an abelian normal subgroup of Γ_0. Since Γ_0 is irreducible by hypothesis, Theorem 2(ii) implies that $\Gamma_2 \cap \mathscr{Z}_{G_2}(\Gamma_2)$ is finite. Therefore $\mathscr{Z}_{G_2}(\Gamma_2)$ is compact, hence trivial since G_2 acts minimally on X'_2. Hence the group G_2 is discrete. We infer that the inclusion $\Gamma_0 < G_1 \times G_2$ is of finite index, which contradicts the irreducibility assumption on Γ_0. This finishes the first part of the proof when the G-action on X is minimal.

  The case when the G-action on X is not minimal reduces to the minimal case, since a Γ_0-invariant splitting $X = X_1 \times X_2$ induces a Γ_0-invariant splitting $X' = X'_1 \times X'_2$ of the canonical minimal Γ_0-invariant subspace X'. The argument given in [CM09b] relies on the fact that X' has no Euclidean factor; however the statement holds even in the presence of a Euclidean factor, since X' is co-bounded in X and thus has the same Tits boundary. The desired decomposition of X' can then be obtained as a consequence of Propositions 3.6 and 3.11 from [CM09a].

  Finally, it remains to prove that if G acts minimally on X and $\Gamma = \Gamma' \times \Gamma''$ splits non-trivially, then there is a non-trivial Γ-equivariant splitting $X = X_1 \times X_2$ such that the projection of Γ to at least one of the two factors is discrete.

  By [CM09a, Th. 1.10] and [CM09b, Cor. 2.7], on each irreducible non-Euclidean factor of X, either the projection of Γ' is trivial or the projection of Γ'' is trivial. Thus there is a canonical decomposition $X = \mathbf{R}^n \times X'_1 \times \cdots \times X'_p \times X''_1 \times \cdots \times X''_q$ such that X'_i and X''_j are non-Euclidean irreducible for all i, j, and moreover the Γ'-action on X''_j (resp. the Γ''-action on X'_i) is trivial.

  Assume first that $X = \mathbf{R}^n \times X'_1 \times \cdots \times X'_p$. Then Γ'' is a discrete subgroup of $\mathrm{Is}(\mathbf{R}^n)$. Since Γ acts cocompactly on $\mathbf{R}^n$, its action is minimal, so the direct factor Γ'' must be infinite. Hence Γ'' is virtually isomorphic to $\mathbf{Z}^k$ for some $k \geq 1$, and the desired splitting of X is afforded by Theorem 2(ii). Assume now that $X''_1 \times \cdots \times X''_q$ is not reduced to a single point. Let G_1 be the closure of the projection of Γ to $\mathrm{Is}(\mathbf{R}^n \times X'_1 \times \cdots \times X'_p)$. Then Γ' is a finitely generated discrete normal subgroup of G_1, so the centraliser $\mathscr{Z}_{G_1}(\Gamma')$ is open in G_1, and acts trivially on X'_i for all i by [CM09b, Cor. 2.7]. Notice that $\Gamma' \cap \mathscr{Z}_{G_1}(\Gamma') = \mathscr{Z}(\Gamma')$ is finitely generated, since it acts properly on $\mathbf{R}^n$. If $\mathscr{Z}(\Gamma')$ is infinite, then Γ virtually normalizes a subgroup isomorphic to $\mathbf{Z}^k$ for some $k \geq 1$, and we conclude again by invoking Theorem 2(ii). If $\mathscr{Z}(\Gamma')$ is finite, then $\mathscr{Z}_{G_1}(\Gamma')$ is compact (because it contains $\mathscr{Z}(\Gamma')$ as a lattice), hence trivial since it is normal in the group G_1, which acts minimally on $\mathbf{R}^n \times X'_1 \times \cdots \times X'_p$. Therefore

G_1 is discrete, and we obtain the desired splitting by setting $X_1 = \mathbf{R}^n \times X_1' \times \cdots \times X_p'$ and $X_2 = X_1'' \times \cdots \times X_q''$. This concludes the proof.

We record at this occasion that Theorem 4.2(i) *does not* generalise to infinitely generated lattices. Indeed, consider the lattice Γ in $\mathrm{Is}(\mathbf{R}) \times \mathrm{Aut}(T)$ constructed in Example 7 above; by construction, Γ projects discretely to $\mathrm{Aut}(T)$. Therefore Γ does not satisfy the conclusion of Theorem 4.2(i) even though it is irreducible as an abstract group; the latter fact follows readily from the classification of commuting elements in free products alluded to in Example 7.

- Lemma 4.7 holds under the additional assumption that X' has no Euclidean factor.
- Theorem 4.10 holds since it concerns finitely generated residually finite lattices, in which case the Euclidean factor theorem holds by Theorem 2(iv) above. The references to Lemma 4.7 are harmless since we are in the setting where the additional assumption necessary for Lemma 4.7 holds.
- In Theorem 4.11, the hypothesis that Γ and Λ do not split virtually a $\mathbf{Z}^n$ factor should be replaced by the condition that they do not commensurate a $\mathbf{Z}^n$ subgroup. By Theorem 2(i) this implies that the ambient CAT(0) spaces X and Y do not have non-trivial Euclidean factors, and the proof goes through without changes.
- Corollary 4.13 holds without changes. Indeed, although its proof currently relies on Theorem 3.8, this dependence can be avoided by using Theorem 2(i), together with a similar argument as in the proof of Theorem 4.11.
- Theorem 6.1 holds; the statement of part (ii) contains typos and should be replaced by the following: *There is a normal subgroup $\Gamma_D \subseteq \Gamma$ which is either finite or infinitely generated, and such that the quotient Γ/Γ_D is an arithmetic lattice in a product of semi-simple Lie and algebraic groups.* In the proof, the definition of Γ_D has to be slightly modified, since it can happen that X' has a non-trivial Euclidean factor $\mathbf{R}^n$. We write $\mathrm{Is}(X') = S \times A \times D$ as in the proof of Theorem 2 above. Since Γ is irreducible by hypothesis, it follows from [CM09b, Th. 4.2(i)], together with the Borel density theorem, that the projection of Γ to S is dense. We replace A and D by the closures of the respective projections of Γ.

  Let $F < \Gamma$ be a commensurated free abelian subgroup of rank n provided by Theorem 2(i). Upon replacing F by a finite index subgroup we may assume that F acts properly by translations on the Euclidean factor. Let M be the normal closure of F in Γ, and N be the kernel of the projection of Γ to $S \times A$. Since the projection of F to S is trivial, the projection of M to S is trivial as well. Define $\Gamma_D = M \cdot N$.

  Observe that N is a discrete normal subgroup of D. Thus Γ/N embeds as a lattice in $S \times A \times (D/N)$. The image MN/N of M now embeds in A (because M has trivial image in S), and is thus an abelian normal subgroup of Γ/N. Let M_A (resp. M_D) denote the closure of the projection of M to A (resp. to D/N). Let $A' = A/M_A$ and $D' = (D/N)/M_D$. Then A' is compact and D' is totally disconnected. We next show that the natural image of M, which is isomorphic to MN/N, is a lattice in $M_A \times M_D$. To this end, remark first that MN/N is discrete. Moreover it contains the group F whose projection to M_A is a lattice and whose projection to M_D is compact. Since the projection of M to M_D is dense by construction, it follows that MN/N is cocompact in $M_A \times M_D$. This confirms that MN/N is a lattice in $M_A \times M_D$. We deduce that $\Gamma/\Gamma_D \cong (\Gamma/N)/(MN/N)$ is a cocompact lattice in $S \times A' \times D'$, hence also in $S \times D'$ since A' is compact. Notice moreover that the image of the projection of Γ/Γ_D to S is dense (because it coincides with the image of the projection of Γ). Similarly, the image of the projection of Γ/Γ_D to D' is dense (by the definition of D'). The conclusion of Theorem 6.1 therefore holds as a consequence of [CM09b, Th. 5.18].

It remains to show that Γ_D cannot be infinite and finitely generated. When A is trivial (i.e. X' has no Euclidean factor) the original argument goes through. Otherwise the group M defined above is non-trivial. If Γ_D is finitely generated, then so is MN/N, since it is a quotient of Γ_D. Thus the projection of Γ to A normalises a finitely generated cocompact group of translations. It follows from Lemma 5 that the projection of Γ to A is virtually abelian. By Theorem 2(iii) this implies that Γ virtually decomposes as a direct product, contradicting the hypotheses.

- Theorems 6.2 and 6.3 hold without changes.
- Theorem 6.6 holds without changes, since the hypotheses imply that Γ is finitely generated and residually finite, in which case the Euclidean factor theorem holds by Theorem 2(iv) above.

Acknowledgements. We are grateful to Ian Leary and Ashot Minasyan for sending us their preprint [LM] and for alerting us to the inconsistency between their beautiful results and our mistaken claim. We also thank them for comments on an earlier version of this note.

References

[Abe74] Herbert Abels, *Specker-Kompaktifizierungen von lokal kompakten topologischen Gruppen*, Math. Z. **135** (1973/74), 325–361.

[BL05] Andreas Balser and Alexander Lytchak, *Centers of convex subsets of buildings*, Ann. Global Anal. Geom. **28** (2005), no. 2, 201–209.

[Bro94] Kenneth S. Brown, *Cohomology of groups*, Springer-Verlag, New York, 1994.

[CCMT15] Pierre-Emmanuel Caprace, Yves Cornulier, Nicolas Monod, and Romain Tessera, *Amenable hyperbolic groups*, J. Europ. Math. Soc. **17** (2015), no. 11, 2903–2947.

[CKRW] Pierre-Emmanuel Caprace, Peter H. Kropholler, Colin D. Reid and Phillip Wesolek, *On the residual and profinite closures of commensurated subgroups*, preprint, to appear in Math. Proc. Cambridge Philos. Soc.

[CM09a] Pierre-Emmanuel Caprace and Nicolas Monod, *Isometry groups of non-positively curved spaces: structure theory*, J Topology **2** (2009), no. 4, 661–700.

[CM09b] ———, *Isometry groups of non-positively curved spaces: discrete subgroups*, J Topology **2** (2009), no. 4, 701–746.

[CM11] ———, *Decomposing locally compact groups into simple pieces*, Math. Proc. Cambridge Philos. Soc. **150** (2011), no. 1, 97–128.

[CM13] ———, *Fixed points and amenability in non-positive curvature*, Math. Ann. **356** (2013), no. 4, 1303–1337.

[HR63] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis. Vol. I*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[LM] Ian J. Leary and Ashot Minasyan, *Commensuration HNN-extensions: non-positive curvature and biautomaticity*, preprint, `arXiv:1907.03515` (2019).

[MKS66] Wilhelm Magnus, Abraham Karrass and Donald Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Interscience Publishers, New York-London-Sydney, 1966.

[Rag72] Madabusi Santanam Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68.

[Wei40] André Weil, *L'intégration dans les groupes topologiques et ses applications*, Hermann et Cie., Paris 1940.

UCL – Math, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium
Email address: `pe.caprace@uclouvain.be`

EPFL, 1015 Lausanne, Switzerland
Email address: `nicolas.monod@epfl.ch`