THE CUP PRODUCT OF BROOKS QUASIMORPHISMS

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ABSTRACT. We prove the vanishing of the cup product of the bounded cohomology classes associated to any two Brooks quasimorphisms on the free group. This is a consequence of the vanishing of the square of a universal class for tree automorphism groups.

1. INTRODUCTION

Although bounded cohomology found a great variety of applications, it remains so mysterious that even for a (non-abelian) free group F of finite rank, we do not know much about it.

More precisely, beyond the trivial case of $\mathrm{H}^1_\mathrm{b}(F,\mathbf{R}) = 0$, it is known that both $\mathrm{H}^2_\mathrm{b}(F,\mathbf{R})$ and $\mathrm{H}^3_\mathrm{b}(F,\mathbf{R})$ are infinite-dimensional. However, $\mathrm{H}^{n\geq 4}_\mathrm{b}(F,\mathbf{R})$ remains completely unknown; in particular, we do not know whether $\mathrm{H}^4_\mathrm{b}(F,\mathbf{R})$ vanishes or not.

The first infinite family of non-trivial classes in $\mathrm{H}^2_\mathrm{b}(F,\mathbf{R})$ are provided by **Brooks quasimorphisms** [1] (anticipated by Johnson [7, 2.8] and Rhemtulla [12]); we recall their definition. Pick any reduced word w in a choice of free generators for F and consider the counting function $f_w\colon F \to \mathbf{R}$ defined on $g \in F$ by

$$f_w(g) = \sharp\{\text{occurrences of } w \text{ in } g\} - \sharp\{\text{occurrences of } w \text{ in } g^{-1}\}.$$

If w is reduced to one letter (or trivial), then f_w is a homomorphism. In all other cases, f_w is a quasimorphism and defines a non-trivial class $\beta_w \in \mathrm{H}^2_\mathrm{b}(F,\mathbf{R})$ unless w is conjugated to a power of a letter. The space spanned by all these β_w is infinite-dimensional [1][8] and is dense in $\mathrm{H}^2_\mathrm{b}(F,\mathbf{R})$ for a suitable topology of pointwise convergence [4, 5.7]. (Following Brooks, we allow overlaps when counting occurrences, whilst other authors do not; see [5, p. 251] for the density in our setting.)

The aim of this note is to show that the cup product of any two elements in this dense sub-space vanishes in $\mathrm{H}^4_\mathrm{b}(F,\mathbf{R})$.

Theorem 1. *Let $\beta_w, \beta_{w'} \in \mathrm{H}^2_\mathrm{b}(F,\mathbf{R})$ be the bounded cohomology classes associated to two Brooks quasimorphisms on F.*

Then $\beta_w \smile \beta_{w'} = 0$ in $\mathrm{H}^4_\mathrm{b}(F,\mathbf{R})$.

We were informed by N. Heuer that he independently obtained a similar result [6] by methods completely different from ours.

We can give a rather transparent proof of Theorem 1 by realizing bounded cohomology with the *aligned chains* that we introduced in [2]. This simplifies the combinatorics and allows us to exhibit a natural explicit coboundary for the cup product.

Moreover, we can carry out this task at once for all w, w' simultaneously — by working instead with the **universal class** $[\omega]$ that we now proceed to define (similar constructions were considered in [10, §2], in [11, 7.11] and in [3, §9]).

Let $T = (V, E)$ be a locally finite tree with Serre's conventions, which means in particular that an element of E represents an *oriented* edge and that E is endowed with a fixed-point-free involution $e \mapsto \bar{e}$ reversing the orientation. We denote by P the set of **paths**, namely sequences $p = (e_1, \ldots, e_n)$ of successive edges $e_i \in E$ without backtracking. The **reverse** path is $\overline{p} = (\bar{e}_n, \ldots, \bar{e}_1)$ and n is the **length** of p. Given two vertices x, y we denote by $[x, y]$ the path connecting them. The **path module** $\mathbf{R}_{\mathrm{alt}}[P]$ is the $\mathrm{Aut}(T)$-module of all elements of the free vector space $\mathbf{R}[P]$ that change sign when replacing a path by its reverse. We define an $\mathrm{Aut}(T)$-equivariant map $\vartheta\colon V^2 \to \mathbf{R}_{\mathrm{alt}}[P]$ by setting

$$\vartheta(x_0, x_1)(p) = \pm 1$$

if p (respectively $\overline{p}$) is contained as a sub-path in $[x_0, x_1]$, and 0 in all other cases. We define

$$\omega = d\vartheta\colon V^3 \to \mathbf{R}_{\mathrm{alt}}[P]$$

as the coboundary of ϑ. We recall here that d will always be the usual alternating sum of the maps omitting the individual variables; we refer to the preliminaries below for explicit values of ω.

In order to view ω as a cocycle in bounded cohomology, we need to specify a norm on $\mathbf{R}_{\mathrm{alt}}[P]$; of course, ϑ should be unbounded for this norm since otherwise the class of ω would be trivial. The specific norm is however not too relevant; one property we want is that, when restricted to the free vector space on the set of paths of length n, it is equivalent to the ℓ^1-norm $\|\cdot\|_{n,1}$. One explicit choice is the norm $\|\cdot\|_{\mathrm{path}} = \sum_{n \geq 1} \frac{1}{n!} \|\cdot\|_{n,1}$ whose normalisation factor $1/n!$ is an arbitrary way to ensure uniform boundedness statements in the proofs.

Furthermore, we write $\mathscr{P}$ for the completion of $\mathbf{R}_{\mathrm{alt}}[P]$. Indeed, even though our arguments will be explicit and finitary, the general tools of continuous bounded cohomology work best with *Banach* spaces.

Suppose given a choice of free generators for the free group F. The corresponding Cayley graph for F is then a tree T; moreover, there is a natural embedding of F into the automorphism group of the tree T. We view ω as a cocycle for the continuous bounded cohomology $\mathrm{H}^*_{\mathrm{cb}}$ of the locally compact group $\mathrm{Aut}(T)$.

Moreover, every path in T is labelled by a reduced word in F since T is a Cayley graph. Thus, given a reduced word w, we can define an F-invariant bounded linear form λ_w on $\mathbf{R}_{\mathrm{alt}}[P]$, hence also on $\mathscr{P}$, by specifying its values on individual paths as follows:

$$\lambda_w(p) = \begin{cases} 1 & \text{if } w \text{ labels } p, \\ -1 & \text{if } w \text{ labels } \overline{p}, \\ 0 & \text{otherwise.} \end{cases}$$

This definition ensures that if $g \in F$ labels $[x_0, x_1]$, then

$$\lambda_w \circ \vartheta(x_0, x_1) = f_w(g).$$

Therefore, we deduce immediately the following relation between the universal class $[\omega]$ and individual quasimorphisms.

Proposition 2. *Let $\beta_w \in \mathrm{H}^2_{\mathrm{b}}(F, \mathbf{R})$ be the bounded cohomology class associated to a Brooks quasimorphism on F for the chosen generators. Then β_w is the image of the class of ω under the map*

$$\mathrm{H}^2_{\mathrm{cb}}\big(\mathrm{Aut}(T), \mathscr{P}\big) \xrightarrow{\mathrm{rest}} \mathrm{H}^2_{\mathrm{b}}(F, \mathscr{P}) \xrightarrow{(\lambda_w)_*} \mathrm{H}^2_{\mathrm{b}}(F, \mathbf{R}),$$

where the first arrow is the restriction map and the second is induced by λ_w. $\square$

The cup product of two elements of $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{Aut}(T), \mathscr{P})$ is a class in $\mathrm{H}^4_{\mathrm{cb}}$ with values in the tensor product module $\mathscr{P} \otimes \mathscr{P}$, which we can also (projectively) complete to $\mathscr{P} \mathbin{\widehat{\otimes}} \mathscr{P}$ (see the preliminaries for the norm). The naturality of the cup product now implies:

Corollary 3. *Given two reduced words w and w', we keep all the above notation.*
Then $[\omega] \smile [\omega]$, viewed as a class with coefficients in $\mathscr{P} \mathbin{\widehat{\otimes}} \mathscr{P}$, is mapped to $\beta_w \smile \beta_{w'}$

$$\mathrm{H}^4_{\mathrm{cb}}\big(\mathrm{Aut}(T), \mathscr{P} \mathbin{\widehat{\otimes}} \mathscr{P}\big) \longrightarrow \mathrm{H}^4_{\mathrm{b}}(F, \mathbf{R})$$

under the restriction followed by $(\lambda_w \otimes \lambda_{w'})_*$. □

In view of Corollary 3, Theorem 1 is now an immediate consequence of the following vanishing result for the square of the universal class $[\omega]$.

Theorem 4. *Let T be a locally finite tree.*
Then the class of $\omega \smile \omega$ vanishes in $\mathrm{H}^4_{\mathrm{cb}}\big(\mathrm{Aut}(T), \mathscr{P} \mathbin{\widehat{\otimes}} \mathscr{P}\big)$.

The remainder of this note is devoted to the proof of Theorem 4.

2. Preliminaries

Recall that the **cup product** of two cochains $\alpha\colon X^{p+1} \to A$ and $\beta\colon X^{q+1} \to B$ (on an arbitrary set X) ranging in coefficient modules A and B is the cochain

$$\alpha \smile \beta\colon X^{p+q+1} \longrightarrow A \otimes B, \quad (\alpha \smile \beta)(x_0, \ldots, x_{p+q}) = \alpha(x_0, \ldots, x_p) \otimes \beta(x_p, \ldots, x_{p+q})$$

ranging in $A \otimes B$. Thus, if A and B are normed vector spaces and if both α and β are bounded, then $\alpha \smile \beta$ is bounded for any cross-norm on $A \otimes B$. We refer to [13] and recall that the **projective** cross-norm is defined for $c \in A \otimes B$ by $\|c\|_\pi = \inf \sum_{i=1}^n \|a_i\| \cdot \|b_i\|$, where the infimum is over all decompositions $c = \sum_{i=1}^n a_i \otimes b_i$. The corresponding completion is denoted by $A \mathbin{\widehat{\otimes}} B$. Since $\| \cdot \|_\pi$ is the largest cross-norm, the vanishing result of Theorem 4 with respect to $\| \cdot \|_\pi$ implies the corresponding vanishing for any other cross-norm.

We say that a path p is **carried** by a path q, and write $p \sqsubset q$, if either p or $\overline{p}$ is contained in q as a sub-path. We attach a sign ± 1 to these two cases, referred to as the **orientation** of p relative to q. We define the **interior** $\mathrm{Int}(p) \subseteq V$ of a path p to consist of all the vertices of the path except its two extremities.

Recall that any three vertices $x_0, x_1, x_2 \in V$ determine a **center** $c \in V$ characterized as the unique common vertex of all $[x_i, x_j]$. Given a path p, the definition of ω now shows that $\omega(x)(p) = \pm 1$ when p is carried by some $[x_i, x_j]$ and $c \in \mathrm{Int}(p)$, and that $\omega(x)(p)$ vanishes otherwise.

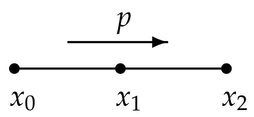
A path can contain at most $n - 1$ sub-paths of length n containing a given vertex in their interior. Therefore, considering all three configurations and two orientations, we can bound the norm of ω by

$$\|\omega(x)\|_{\mathrm{path}} \leq 3 \cdot 2 \cdot \sum_{n \geq 1} \frac{1}{n!}(n-1) = 6,$$

witnessing that ω is indeed uniformly bounded.

Recall that a $(q+1)$-tuple $(x_0, \ldots, x_q) \in V^{q+1}$ is **aligned** if the vertices $x_0, \ldots, x_q$ are contained in some geodesic segment of T. This tuple is furthermore said to be **coherent** if these $q + 1$ vertices are distinct and in increasing order for one of the two linear orders induced on $\{x_0, \ldots, x_q\}$ by any such segment. We denote by $V^{q+1}_{\mathrm{coh}} \subseteq V^{q+1}$ the set of coherent aligned tuples.

Below, we shall be particularly interested in the above description of $\omega(x)$ specialized to coherent triples $x \in V^3_{\mathrm{coh}}$. In that case, $\omega(x)(p) = \pm 1$ if $x_1 \in \mathrm{Int}(p)$ and p is carried by $[x_0, x_2]$, with the sign given by the orientation of p relative to $[x_0, x_2]$, and vanishes in all other cases.



$$\Longrightarrow \quad \omega = 1.$$

3. A COHERENT RESOLUTION

Let E be any isometric Banach $\mathrm{Aut}(T)$-module and recall that $\mathrm{H}^q_{\mathrm{cb}}(\mathrm{Aut}(T), E)$ can be computed with the (non-augmented) complex $\ell^\infty(V^{q+1}, E)^{\mathrm{Aut}(T)}$ of $\mathrm{Aut}(T)$-equivariant elements of the resolution

$$0 \longrightarrow E \longrightarrow \ell^\infty(V, E) \longrightarrow \ell^\infty(V^2, E) \longrightarrow \ell^\infty(V^3, E) \longrightarrow \cdots \tag{i}$$

(see e.g. [9, 4.5.2]). There is a natural restriction map to the complex $\ell^\infty(V^{q+1}_{\mathrm{coh}}, E)$ on coherent tuples, but we warn the reader that *the latter is not a resolution* of E.

Example 5. *Take $E = \mathbf{R}$ and fix a path p of length one. Then the map $(x_0, x_1) \mapsto |\vartheta(x_0, x_1)(p)|$ belongs to $\ell^\infty(V^2_{\mathrm{coh}})$ and cannot be a coboundary since it is not antisymmetric. On the other hand, one checks readily that it is a cocycle in the complex $\ell^\infty(V^3_{\mathrm{coh}})$.*

Recall that an element of $\ell^\infty(V^{q+1}, E)$ is called **alternating** if any permutation σ of the variables corresponds to the multiplication by the signature $\mathrm{sign}(\sigma)$. We denote by τ_q the permutation of $\{0, \ldots, q\}$ that reverses the order and observe that its signature is $(-1)^{\lfloor \frac{q+1}{2} \rfloor}$, where $\lfloor \cdot \rfloor$ denotes the integer part. Consider the $\mathrm{Aut}(T)$-equivariant involution $\hat{\tau}_q$ of $\ell^\infty(V^{q+1}_{\mathrm{coh}}, E)$ defined by $\hat{\tau}_q(\alpha)(x) = \mathrm{sign}(\tau_q)\alpha(x^{\tau_q})$. Being an involution, it induces an eigenspace decomposition

$$\ell^\infty(V^{q+1}_{\mathrm{coh}}, E) = \ell^\infty_+(V^{q+1}_{\mathrm{coh}}, E) \oplus \ell^\infty_-(V^{q+1}_{\mathrm{coh}}, E)$$

which is preserved by $\mathrm{Aut}(T)$. Although $\ell^\infty(V^{q+1}_{\mathrm{coh}}, E)$ is not a resolution, we have:

Proposition 6. *The sub-complex*

$$0 \longrightarrow E \longrightarrow \ell^\infty_+(V^1_{\mathrm{coh}}, E) \longrightarrow \ell^\infty_+(V^2_{\mathrm{coh}}, E) \longrightarrow \ell^\infty_+(V^3_{\mathrm{coh}}, E) \longrightarrow \cdots \tag{ii}$$

is a resolution. Moreover, the map

$$A_q \circ \mathrm{rest}\colon \ell^\infty(V^{q+1}, E) \longrightarrow \ell^\infty_+(V^{q+1}_{\mathrm{coh}}, E)$$

from (i) *to* (ii) *obtained by restriction followed by the projection $A_q = (\hat{\tau}_q + \mathrm{Id})/2$ yields an isomorphism between $\mathrm{H}^q_{\mathrm{cb}}(\mathrm{Aut}(T), E)$ and the cohomology of the complex*

$$0 \longrightarrow \ell^\infty_+(V^1_{\mathrm{coh}}, E)^{\mathrm{Aut}(T)} \longrightarrow \ell^\infty_+(V^2_{\mathrm{coh}}, E)^{\mathrm{Aut}(T)} \longrightarrow \ell^\infty_+(V^3_{\mathrm{coh}}, E)^{\mathrm{Aut}(T)} \longrightarrow \cdots \tag{iii}$$

Proof. Following [2], we denote by $\ell^\infty_{\mathscr{A}}(V^{q+1}, E)$ the sub-space of alternating maps defined on aligned tuples. The restriction to coherent tuples thus induces an isomorphism

$$\ell^\infty_{\mathscr{A}}(V^{q+1}, E) \cong \ell^\infty_{+}(V^{q+1}_{\mathrm{coh}}, E).$$

Therefore, the first statement is simply a reformulation of Corollary 8 of [2]. Moreover, as observed there, the modules $\ell^\infty_{\mathscr{A}}(V^{q+1}, E)$ are relatively injective in the sense of bounded cohomology because the $\mathrm{Aut}(T)$-action on the set of aligned tuples is proper, see [9, 4.5.2]. More precisely, $\ell^\infty_{\mathscr{A}}(V^{q+1}, E)$ is a direct factor of the larger space without the alternation condition, to which [9, 4.5.2] applies, and one concludes as in [9, 7.4.5] by an alternation map.

A direct computation using the relation $\mathrm{sign}(\tau_q) \cdot \mathrm{sign}(\tau_{q+1}) = (-1)^{q+1}$ shows that $\hat{\tau}_q$ is a chain map. In particular, $\hat{\tau}_q$ automatically preserves the decomposition $\ell^\infty_{\pm}(V^{q+1}_{\mathrm{coh}}, E)$ and A_q is a chain map as well. Now the second statement follows by general cohomological principles (see e.g. §7.2 in [9]). $\square$

4. A primitive for the square of ω on coherent tuples

We define an $\mathrm{Aut}(T)$-equivariant map

$$B\colon V^4_{\mathrm{coh}} \longrightarrow \mathbf{R}_{\mathrm{alt}}[P] \otimes \mathbf{R}_{\mathrm{alt}}[P]$$

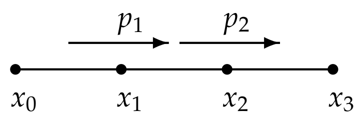
by setting, for any coherent 4-tuple x and any paths $p_1, p_2 \in P$,

$$B(x)(p_1, p_2) = \pm 1$$

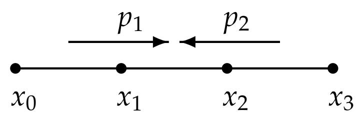
whenever all the following hold:

- both p_1 and p_2 are carried by $[x_0, x_3]$,
- the interior of p_1 and of p_2 are disjoint,
- $x_i \in \mathrm{Int}(p_i)$ for each $i = 1, 2$.

In that case, the sign ± 1 is the product of the orientations of p_1 and of p_2 relative to $[x_0, x_3]$. All this is perhaps much more intuitive in a picture, drawn for two of the four orientation possibilities:



$\Longrightarrow \quad B = 1.$



$\Longrightarrow \quad B = -1.$

In all other cases, we set $B(x)(p_1, p_2) = 0$.

Proposition 7. *We have $dB(x) = \omega \smile \omega(x)$ for every coherent 5-tuple x.*

Proof. Let $p_1, p_2 \in P$. By definition,

$$\omega \smile \omega(x)(p_1, p_2) = \omega(x_0, x_1, x_2)(p_1) \cdot \omega(x_2, x_3, x_4)(p_2).$$

Thus, $\omega \smile \omega(x)(p_1, p_2) \neq 0$ if and only if all the following hold:

$$\text{(iv)} \qquad \begin{cases} x_1 \in \mathrm{Int}(p_1) \ \text{ and } \ p_1 \sqsubset [x_0, x_2], \\ x_3 \in \mathrm{Int}(p_2) \ \text{ and } \ p_2 \sqsubset [x_2, x_4]. \end{cases}$$

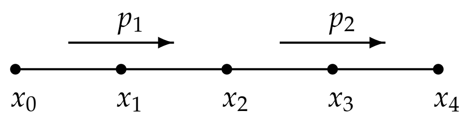
As for dB, we observe that $dB(x)(p_1, p_2) = 0$ unless possibly

$$\text{(v)} \qquad \begin{cases} p_1, p_2 \text{ have disjoint interior and are carried by } [x_0, x_4], \\ x_1 \text{ or } x_2 \in \mathrm{Int}(p_1), \\ x_2 \text{ or } x_3 \in \mathrm{Int}(p_2). \end{cases}$$

In the case when Conditions (v) are not satisfied, Conditions (iv) are not either; therefore in that case dB and $\omega \smile \omega$ agree since they both vanish.

Suppose now that Conditions (v) are satisfied. By symmetry, we can assume that the orientation of p_1 and p_2 are compatible with the orientation of $[x_0, x_4]$ (and hence of $[x_0, x_3]$ and of $[x_1, x_4]$). Since p_1 and p_2 have disjoint interior, x_2 is contained in at most one of $\mathrm{Int}(p_1)$ or $\mathrm{Int}(p_2)$; we can suppose that it is not contained in $\mathrm{Int}(p_1)$, the other case being completely analogous. We have now three cases:

First case: $x_1 \in \mathrm{Int}(p_1)$, $x_2 \notin \mathrm{Int}(p_1) \cup \mathrm{Int}(p_2)$ and $x_3 \in \mathrm{Int}(p_2)$.

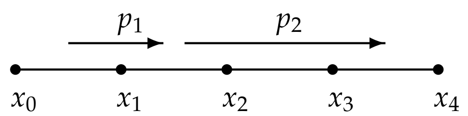


The value of $\omega \smile \omega(x)(p_1, p_2)$ is $+1$, while the only non-zero summand in

$$dB(x)(p_1, p_2) = \Sigma_{i=0}^{4}(-1)^i B(\ldots, \widehat{x_i}, \ldots)$$

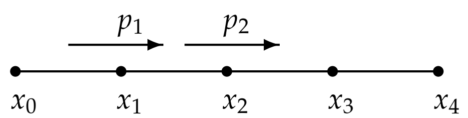
is the one for $i = 2$, which is indeed also $+1$.

Second case: $x_1 \in \mathrm{Int}(p_1)$ and $x_2, x_3 \in \mathrm{Int}(p_2)$.



Condition (iv) is not satisfied and hence $\omega \smile \omega$ vanishes. As for dB, only the summands for $i = 2$ and $i = 3$ are non-zero and cancel out to give $dB(x)(p_1, p_2) = 0$.

Third case: $x_1 \in \mathrm{Int}(p_1)$, $x_2 \in \mathrm{Int}(p_2)$ and $x_3 \notin \mathrm{Int}(p_2)$.



Again, condition (v) is not satisfied and $\omega \smile \omega$ vanishes. As for dB, only the summands for $i = 3$ and $i = 4$ are non-zero and cancel out to give $dB(x)(p_1, p_2) = 0$. $\square$

5. PROOF OF THEOREM 4

We first verify that the primitive B is bounded.

Lemma 8. *The map B is uniformly bounded on V^4_{coh} with respect to the projective norm $\|\cdot\|_\pi$.*

Proof. Given a path p, denote by $\underline{p}$ the element $p - \overline{p}$ of $\mathbf{R}_{\mathrm{alt}}[P]$. Then $\|\underline{p}\|_{\mathrm{path}} = 2/n!$ if p has length n. Fix now $x \in V^4_{\mathrm{coh}}$. By definition of the projective cross-norm, we can bound $\|B(x)\|_\pi$ by $\sum(\|\underline{p_1}\|_{\mathrm{path}} \cdot \|\underline{p_2}\|_{\mathrm{path}})$, where the sum runs over all pairs of paths (p_1, p_2) on which $B(x)$ does not vanish, but taking only one representative of the possible orientations. Arguing as in our estimate for the norm of ω, we have at most $(n_1 - 1)(n_2 - 1)$ such pairs whenever we fix the length n_i of each p_i. We conclude that $B(x)$ has norm at most

$$\sum_{n_1, n_2} \frac{4(n_1-1)(n_2-1)}{n_1! n_2!} = 4\left(\sum_n \frac{n-1}{n!}\right)^2 = 4.$$

□

At this point, we conclude that $A_3(B)$ belongs to $\ell^\infty_+(V^4_{\mathrm{coh}}, \mathscr{P})$. Since A_* is a chain map (as pointed out in the proof of Proposition 6), we deduce from Proposition 7 that we have $A_4(\omega \smile \omega) = dA_3(B)$. Now Proposition 6 implies that the class of $\omega \smile \omega$ vanishes, completing the proof of Theorem 4. □

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