# APPENDIX ON A QUESTION OF KAZHDAN AND YOM DIN

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Abstract. This is an appendix to Eli Glasner's *On a question of Kazhdan and Yom Din* [Gla21] in the Benjy Weiss birthday IJM volume. I am grateful to Eli for his comments and for welcoming this appendix.

To Benjy, with admiration.

David Kazhdan and Alexander Yom Din asked the following [Gla21].

**Question.** Fix $\delta > 0$. Let $V$ be a Banach space equipped with a linear isometric action of a group $G$.

Let $\alpha$ be an element of the dual $V^*$ such that $\|\alpha\| = 1$ and $\|g\alpha - \alpha\| \le \delta/10$ for all $g \in G$.

(i) Does there exist a $G$-invariant $\beta \in V^*$ with $\|\alpha - \beta\| \le \delta$?

(ii) Specifically, does this hold for $V^* = \mathscr{B}(H)$, the space of bounded operators of a Hilbert space $H$?

(The predual $V$ for (ii) is the space of trace-classs operators.)

In [Gla21] Eli Glasner proved (among many other things) that Question (i) has a negative answer in general, including for some weak-* closed subspaces of $\mathscr{B}(H)$, suggesting that perhap (ii) also has a negative answer.

**Theorem.** *Question* (ii) *does indeed have a negative answer.*

*Proof.* We shall use ideas introduced by Bożejko–Fendler [BF91] in the context of Herz–Schur multipliers and uniformly bounded representations. This leads to a concrete and self-contained construction as follows.

Let $G$ be the free group on an infinite set $S$ of generators. For any non-trivial $x \in G$, denote by $x_-$ the element obtained by erasing the right-most letter in the reduced word for $x$. The Hilbert space $H$ for question (ii) will be $H = \ell^2(G)$ with Hilbert basis $(\delta_x)_{x\in G}$. The $G$-action on $\mathscr{B}(H)$ is the adjoint representation associated to the left regular representation $\lambda$ on $\ell^2(G)$. That is, for $T \in \mathscr{B}(H)$ and $g \in G$, we define $g.T = \lambda(g) \circ T \circ \lambda(g)^{-1}$. This is indeed isometric and induced by an isometric representation on the predual $V$. Below, all norms are understood in $V^* = \mathscr{B}(H)$ unless specified otherwise with a subscript such as $\|\cdot\|_{\ell^2}$.

Given $n > 0$, we choose a subset $S_n \subseteq S$ of $n$ generators. We define $T_n \in \mathscr{B}(H)$ by requiring $T_n\delta_x = \delta_{x_-}$ if $x_-^{-1}x \in S_n^{\pm 1}$ (with $x \neq e$) and $T_n\delta_x = 0$ in all other cases; this defines a bounded operator.

Fix $g \in G$. Then $T_n$ *almost* commutes with $\lambda(g)$ because $(gx)_- = g(x_-)$ holds unless $x$ is completely cancelled by $g$ (or $x$ is already trivial). Let us compute the commutator when this complete cancellation occurs. Write $g$ in reduced form as $g = s_1^{\epsilon_1} \cdots s_k^{\epsilon_k}$ with $k \in \mathbf{N}$, $s_i \in S$ and $\epsilon_i = \pm 1$ (we can assume $k > 0$). Then $x = s_k^{-\epsilon_k} \cdots s_{h+1}^{-\epsilon_{h+1}}$ for some $h < k$. Thus $gx = s_1^{\epsilon_1} \cdots s_h^{\epsilon_h}$.

Now we have

$$\Big(T_n \circ \lambda(g) - \lambda(g) \circ T_n\Big)\delta_x = \delta_{s_1^{\epsilon_1}\cdots s_{h-1}^{\epsilon_{h-1}}} - \delta_{s_1^{\epsilon_1}\cdots s_{h+1}^{\epsilon_{h+1}}}$$

provided $s_h$ and $s_{h+1}$ are in $S_n$ (otherwise the corresponding term simply does not occur). This implies the estimate

$$\Big\|T_n \circ \lambda(g) - \lambda(g) \circ T_n\Big\| \leq 2 \qquad \forall\, g \in G\ \forall\, n$$

because, $g$ being fixed, only one $x \in G$ gives rise to the pair of elements $s_1^{\epsilon_1}\cdots s_{h-1}^{\epsilon_{h-1}}$ and $s_1^{\epsilon_1}\cdots s_{h+1}^{\epsilon_{h+1}}$. In other words, the commutator is a sum of two partial isometries, whence the estimate. (A more high-tech argument with Riesz–Thorin interpolation can be found in [BF91, §2].)

At this point we have elements $T_n \in V^* = \mathscr{B}(H)$ with $\|g.T_n - T_n\| \leq 2$ for all $g \in G$. We now examine the distance

$$d(T_n, V^{*G}) := \inf\Big\{\|T_n - \beta\| :\ \beta \text{ is } G\text{-fixed}\Big\}$$

from $T_n$ to any $G$-fixed point in $\mathscr{B}(H)$.

We have $T_n \mathbf{1}_{S_n^{\pm 1}} = 2n\delta_e$ and use that $\beta$ commutes with $\lambda$ to compute

$$\begin{aligned} 2n = \|T_n \mathbf{1}_{S_n^{\pm1}}\|_{\ell^2} &\leq \|\beta \mathbf{1}_{S_n^{\pm1}}\|_{\ell^2} + \|T_n - \beta\| \cdot \|\mathbf{1}_{S_n^{\pm1}}\|_{\ell^2} \\ &= \|\lambda(\mathbf{1}_{S_n^{\pm1}})\beta\delta_e\|_{\ell^2} + \sqrt{2n}\,\|T_n - \beta\| \\ &\leq \|\lambda(\mathbf{1}_{S_n^{\pm1}})\| \cdot \|\beta\delta_e\|_{\ell^2} + \sqrt{2n}\,\|T_n - \beta\| \\ &\leq \|\lambda(\mathbf{1}_{S_n^{\pm1}})\| \cdot \|T_n - \beta\| + \sqrt{2n}\,\|T_n - \beta\|, \end{aligned}$$

where at the end we used $T_n\delta_e = 0$. The norm $\|\lambda(\mathbf{1}_{S_n^{\pm1}})\|$ is well-known to be $2\sqrt{2n-1}$, since it is a spectral radius that can be computed by enumerating paths in a tree, see Thm. 3 in [Kes59]. To be very precise: we can consider $H = \ell^2(G)$ as a multiple of the regular representation of the subgroup $F_n$ generated by $S_n$ and Kesten's computation takes place in this $\ell^2(F_n)$; the resulting norm coincides with $\|\lambda(\mathbf{1}_{S_n^{\pm1}})\|$ in $\ell^2(G)$. Now the above inequalities imply

$$d(T_n, V^{*G}) > \frac{\sqrt{2n}}{3} \qquad \forall\, n.$$

We have therefore our answer to the original question by defining $\alpha \in V^* = \mathscr{B}(H)$ to be the unit vector corresponding to $T_n$ and letting $n \to \infty$. □

## References

[BF91] Marek Bożejko and Gero Fendler, *Herz-Schur multipliers and uniformly bounded representations of discrete groups.*, Arch. Math. **57** (1991), no. 3, 290–298.

[Gla21] Eli Glasner, *On a question of Kazhdan and Yom Din*, To appear in Israel J. Math., 2021.

[Kes59] Harry Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. **92** (1959), 336–354.

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