Groups of piecewise projective homeomorphisms

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The group of piecewise projective homeomorphisms of the line provides straightforward torsion-free counter-examples to the so-called von Neumann conjecture. The examples are so simple that many additional properties can be established.

Introduction

In 1924, Banach and Tarski accomplished a rather paradoxical feat. They proved that a solid ball can be decomposed into five pieces which are then moved around and reassembled in such a way as to obtain *two* balls identical with the original one [6]. This wellnigh miraculous duplication was based on Hausdorff's 1914 work [18].

In his 1929 study of the Hausdorff–Banach–Tarski paradox, von Neumann introduced the concept of amenable groups [37]. Tarski readily proved that amenability is the *only* obstruction to paradoxical decompositions [34, 35]. However, the known paradoxes relied more prosaically on the existence of non-abelian free subgroups. Therefore, the main open problem in the subject remained for half a century to find non-amenable groups without free subgroups. Von Neumann's name was apparently attached to it by Day in the 1950s. The problem was finally solved around 1980: Ol′shanskiĭ proved the non-amenability of the Tarski monsters that he had constructed [28, 29, 30]; Adyan showed that his work on Burnside groups yields non-amenability [3, 4]. Finitely presented examples were constructed another twenty years later by Ol′shanskiĭ–Sapir [27]. There are several more recent counter-examples [15, 31, 32].

Given any subring $A < \mathbf{R}$, we shall define a group $G(A)$ and a subgroup $H(A) < G(A)$ of piecewise projective transformations. Those will provide concrete, uncomplicated new examples with many additional properties. Perhaps ironically, our short proof of non-amenability ultimately relies on basic free groups of matrices, as in Hausdorff's 1914 paradox, even though the Tits alternative [36] shows that the examples cannot be linear themselves.

Construction

I saw the pale student of unhallowed arts kneeling beside the thing he had put together.
Mary Shelley, *Frankenstein*
(introduction to the 1831 edition)

Consider the natural action of the group $\mathrm{PSL}_2(\mathbf{R})$ on the projective line $\mathbf{P}^1 = \mathbf{P}^1(\mathbf{R})$. We endow $\mathbf{P}^1$ with its $\mathbf{R}$-topology making it a topological circle. We denote by G the group of all homeomorphisms of $\mathbf{P}^1$ which are piecewise in $\mathrm{PSL}_2(\mathbf{R})$, each piece being an interval of $\mathbf{P}^1$, with finitely many pieces. We let $H < G$ be the subgroup fixing the point $\infty \in \mathbf{P}^1$ corresponding to the first basis vector of $\mathbf{R}^2$. Thus H is left-orderable since it acts faithfully on the topological line $\mathbf{P}^1 \setminus \{\infty\}$, preserving orientations. It follows in particular that H is torsion-free.

Given a subring $A < \mathbf{R}$, we denote by $P_A \subseteq \mathbf{P}^1$ the collection of all fixed points of all hyperbolic elements of $\mathrm{PSL}_2(A)$. This set is $\mathrm{PSL}_2(A)$-invariant and is countable if A is so. We define $G(A)$ to be the subgroup of G given by all elements that are piecewise in $\mathrm{PSL}_2(A)$ with all interval endpoints in P_A. We write $H(A) = G(A) \cap H$, which is the stabilizer of ∞ in $G(A)$.

The main result of this article is the following, which relies on a new method for proving non-amenability.

Theorem 1. *The group $H(A)$ is non-amenable if $A \neq \mathbf{Z}$.*

The next result is a sequacious generalization of the corresponding theorem of Brin–Squier about piecewise affine transformations [7] and we claim no originality.

Theorem 2. *The group H does not contain any non-abelian free subgroup. Thus, $H(A)$ inherits this property for any subring $A < \mathbf{R}$.*

Thus already $H = H(\mathbf{R})$ itself is a counter-example to the von Neumann conjecture. Writing $H(A)$ as the directed union of its finitely generated subgroups, we deduce:

Corollary 3. *For $A \neq \mathbf{Z}$, the groups $H(A)$ contain finitely generated subgroups that are simultaneously non-amenable and without non-abelian free subgroups.*

Further properties The groups $H(A)$ seem to enjoy a number of additional interesting properties, some of which are weaker forms of amenability. In the last section, we shall prove the following five propositions (and recall the terminology). Here $A < \mathbf{R}$ is an arbitrary subring.

Proposition 4. *All L^2-Betti numbers of $H(A)$ and of $G(A)$ vanish.*

Proposition 5. *The group $H(A)$ is inner amenable.*

Proposition 6. *The group H is bi-orderable and hence so are all its subgroups. It follows that there is no non-trivial homomorphism from any Kazhdan group to H.*

Proposition 7. *Let $E \subseteq \mathbf{P}^1$ be any subset. Then the subgroup of $H(A)$ which fixes E pointwise is co-amenable in $H(A)$ unless E is dense (in which case the subgroup is trivial).*

Proposition 8. *If $H(A)$ acts by isometries on any proper CAT(0) space, then either it fixes a point at infinity or it preserves a Euclidean subspace.*

One can also check that $H(A)$ satisfies no group law and has vanishing properties in bounded cohomology (see below).

Non-amenability

An obvious difference between the actions of $\mathrm{PSL}_2(A)$ and of $H(A)$ on $\mathbf{P}^1$ is that the latter group fixes ∞ whilst the former does not. The next proposition shows that this is the only difference as far as the orbit structure is concerned.

Reserved for Publication Footnotes

Proposition 9. *Let $A < \mathbf{R}$ be any subring and let $p \in \mathbf{P}^1 \setminus \{\infty\}$. Then*

$$\mathrm{PSL}_2(A) \cdot p \ \subseteq\ \{\infty\} \cup H(A) \cdot p.$$

Thus, the equivalence relations induced by the actions of $\mathrm{PSL}_2(A)$ and of $H(A)$ on $\mathbf{P}^1$ coincide when restricted to $\mathbf{P}^1 \setminus \{\infty\}$.

Proof. We need to show that given $g \in \mathrm{PSL}_2(A)$ with $gp \neq \infty$, there is an element $h \in H(A)$ such that $hp = gp$. We assume $g\infty \neq \infty$ since otherwise $h = g$ will do. Equivalently, we need an element $q \in G(A)$ fixing gp and such that $q\infty = g\infty$, writing $h = q^{-1}g$. It suffices to find a hyperbolic element $q_0 \in \mathrm{PSL}_2(A)$ with $q_0\infty = g\infty$ and whose fixed points $\xi_\pm \in \mathbf{P}^1$ separate gp from both ∞ and $g\infty$, see Figure 1. Indeed, we can then define q to be the identity on the component of $\mathbf{P}^1 \setminus \{\xi_\pm\}$ containing gp, and define q to coincide with q_0 on the other component.

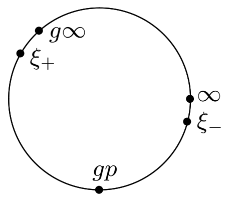

Fig. 1. The desired configuration of $\xi_\pm$

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix representative of g; thus, $a, b, c, d \in A$ and $ad - bc = 1$. The assumption $g\infty \neq \infty$ implies $c \neq 0$ and thus we can assume $c > 0$. Let q_0 be given by $\begin{pmatrix} a & b + ra \\ c & d + rc \end{pmatrix}$ with $r \in A$ to be determined later; thus $q_0\infty = g\infty$. This matrix is hyperbolic as soon as $|r|$ is large enough to ensure that the trace $\tau = a + d + rc$ is larger than 2 in absolute value. We only need to show that a suitable choice of r will ensure the above condition on $\xi_\pm$. Notice that ∞ and $g\infty$ lie in the same component of $\mathbf{P}^1 \setminus \{\xi_\pm\}$ since q_0 preserves these components and sends ∞ to $g\infty$. In conclusion, it suffices to prove the following two claims: (1) as $|r| \to \infty$, the set $\{\xi_\pm\}$ converges to $\{\infty, g\infty\}$; (2) changing the sign of r (when $|r|$ is large) will change the component of $\mathbf{P}^1 \setminus \{\infty, g\infty\}$ in which $\xi_\pm$ lie (we need it to be the component of gp). The claims can be proved by elementary dynamical considerations; we shall instead verify them explicitly.

The fixed points $\xi_\pm$ are represented by the eigenvectors $\begin{pmatrix} x_\pm \\ c \end{pmatrix}$, where $x_\pm = \lambda_\pm - d - rc$ and where $\lambda_\pm = (\tau \pm \sqrt{\tau^2 - 4})/2$ are the eigenvalues. Now $\lim_{r \to +\infty} \lambda_+ = +\infty$ implies $\lim_{r \to +\infty} \lambda_- = 0$ since $\lambda_+ \lambda_- = 1$ and therefore $\lim_{r \to +\infty} x_- = -\infty$. Similarly, $\lim_{r \to -\infty} x_+ = +\infty$ (Figure 1 depicts the case $r > 0$). This already proves claim (2) and half of claim (1). Since $g\infty = [a\!:\!c]$, it only remains to verify that both $\lim_{r \to +\infty} x_+$ and $\lim_{r \to -\infty} x_-$ converge to a, which is a direct computation. □

We recall that a measurable equivalence relation with countable classes is *amenable* if there is an a.e. defined measurable assignment of a mean on the orbit of each point in such a way that the means of two equivalent points coincide. We refer e.g. to [12] and [20] for background on amenable equivalence relations. It follows from this definition that any relation produced by a measurable action of a (countable) amenable group is amenable, by push-forward of the mean [33, 1.6(1)]. An a.e. free action of a countable group is amenable in Zimmer's sense [40, 4.3] if and only if the associated relation is amenable; see [2, Thm. A].

Proof of Theorem 1. Let $A \neq \mathbf{Z}$ be a subring of $\mathbf{R}$. Then A contains a countable subring $A' < A$ which is dense in $\mathbf{R}$. Since $H(A')$ is a subgroup of $H(A)$, we can assume that A itself is countable dense. Now $H(A)$ is a countable group and $\Gamma := \mathrm{PSL}_2(A)$ is a countable dense subgroup of $\mathrm{PSL}_2(\mathbf{R})$.

It is proved in Théorème 3 of [10] that the equivalence relation on $\mathrm{PSL}_2(\mathbf{R})$ induced by the multiplication action of Γ is non-amenable; see also Remarks 10 and 11 below. Equivalently, the Γ-action on $\mathrm{PSL}_2(\mathbf{R})$ is non-amenable. Viewing $\mathbf{P}^1$ as a homogeneous space of $\mathrm{PSL}_2(\mathbf{R})$, it follows that the Γ-action on $\mathbf{P}^1$ is non-amenable. Indeed, amenability is preserved under extensions, see [39, 2.4] or [2, Cor. C]. This action is a.e. free since any non-trivial element has at most two fixed points. Thus the relation induced by Γ on $\mathbf{P}^1$ is non-amenable. Restricting to $\mathbf{P}^1 \setminus \{\infty\}$, we deduce from Proposition 9 that the relation induced by the $H(A)$-action is also non-amenable. (Amenability is preserved under restriction [20, 9.3], but here $\{\infty\}$ is a null-set anyway.) Thus $H(A)$ is a non-amenable group. □

Remark 10. *We recall from [10] that the non-amenability of the Γ-relation on $\mathrm{PSL}_2(\mathbf{R})$ is a general consequence of the existence of a non-discrete non-abelian free subgroup of Γ. Thus the main point of our appeal to [10] is the existence of this non-discrete free subgroup, but this is much easier to prove directly in the present case of $\Gamma = \mathrm{PSL}_2(A)$ than for general non-discrete non-soluble Γ.*

Remark 11. *Here is a direct argument avoiding all the above references in the examples of $A = \mathbf{Z}[\sqrt{2}]$ or $A = \mathbf{Z}[1/\ell]$, where ℓ is prime. We show directly that the Γ-action on $\mathbf{P}^1$ is not amenable. We consider Γ as a lattice in $L := \mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$ in the first case and in $L := \mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{Q}_\ell)$ in the second case, both times in such a way that the Γ-action on $\mathbf{P}^1$ extends to the L-action factoring through the first factor. If the Γ-action on $\mathbf{P}^1$ were amenable, so would be the L-action (by co-amenability of the lattice). But of course L does not act amenably since the stabilizer of any point contains the (non-amenable) second factor of L.*

The non-discreteness of A was essential in our proof, thus excluding $A = \mathbf{Z}$.

Problem 12. *Is $H(\mathbf{Z})$ amenable?*

The group $H(\mathbf{Z})$ is related to Thompson's group F, for which the question of (non-)amenability is a notorious open problem. Indeed F seems to be historically the first candidate for a counter-example to the so-called von Neumann conjecture. The relation is as follows: if we modify the definition of $H(\mathbf{Z})$ by requiring that the breakpoints be rational, then all its elements are automatically C^1 and the resulting group is conjugated to F. The corresponding relation holds between $G(\mathbf{Z})$ and Thompson's group T. These facts are attributed to a remark of Thurston around 1975 and a very detailed exposition can be found in [22].

H is a free group free group

We shall largely follow [7, § 3], the main difference being that we replace commutators by a non-trivial word in the *second* derived subgroup of a free group on two generators.

The *support* $\mathrm{supp}(g)$ of an element $g \in H$ denotes the set $\{p : gp \neq p\}$, which is a finite union of open intervals. Any subgroup of H fixing some point $p \in \mathbf{P}^1$ has two canonical homomorphisms to the metabelian stabilizer of p in $\mathrm{PSL}_2(\mathbf{R})$

given by left and right germs. Therefore, we deduce the following elementary fact, wherein $\langle f, g\rangle$ denotes the subgroup of H generated by f and g.

Lemma 13. *If $f, g \in H$ have a common fixed point $p \in \mathbf{P}^1$, then any element of the second derived subgroup $\langle f, g\rangle''$ acts trivially on a neighbourhood of p.* □

Theorem 2 is an immediate consequence of the following more precise statement.

Theorem 14. *Let $f, g \in H$. Either $\langle f, g\rangle$ is metabelian or it contains a free abelian group of rank two.*

Proof. We suppose that $\langle f, g\rangle$ is not metabelian, so that there is a word w in the second derived subgroup of a free group on two generators such that $w(f, g) \in H$ is non-trivial. We now follow faithfully the proof of Theorem 3.2 in [7], replacing $[f, g]$ by $w(f, g)$. For the reader's convenience, we sketch the argument; the details are on page 495 of [7] (or [8, p. 232]). Applying Lemma 13 to all endpoints p of the connected components of $\mathrm{supp}(f) \cup \mathrm{supp}(g)$, we deduce that the closure of $\mathrm{supp}(w(f, g))$ is contained in $\mathrm{supp}(f) \cup \mathrm{supp}(g)$. This implies that some element of $\langle f, g\rangle$ will send any connected component of $\mathrm{supp}(w(f, g))$ to a disjoint interval. The needed element might depend on the connected component. However, upon replacing $w(f, g)$ by another non-trivial element $w_1 \in \langle f, g\rangle''$ with minimal number of intersecting components with $\mathrm{supp}(f) \cup \mathrm{supp}(g)$, some element h of $\langle f, g\rangle$ sends the whole of $\mathrm{supp}(w_1)$ to a set disjoint from it. The corresponding conjugate $w_2 := h w_1 h^{-1}$ will commute with w_1 and indeed these two elements generate freely a free abelian group. □

As pointed out to us by Cornulier, the above argument can be pushed so that w_1 and h generate a wreath product $\mathbf{Z} \wr \mathbf{Z}$, compare [17, Thm. 21] for the piecewise linear case.

Lagniappe

Proof of Proposition 4. We refer to [11] for the L^2-Betti numbers $\beta^n_{(2)}$, $n \in \mathbf{N}$. Fix a large integer n and let $\Gamma = G(H)$ or $H(A)$. Choose a set $F \subseteq P_A$ of $n+1$ distinct points and let $\Lambda < \Gamma$ be the pointwise stabilizer of F. Any intersection Λ^* of any (finite) number of conjugates of Λ is still the pointwise stabilizer of a finite set F^* containing $m \geq n+1$ points. The definition of $G(A)$ shows that Λ^* is the product of m infinite groups. The Künneth formula [11, § 2] implies $\beta^i_{(2)}(\Lambda^*) = 0$ for all $i = 0, \ldots, m-1$. In this situation, Theorem 1.3 of [5] asserts $\beta^i_{(2)}(\Gamma) = 0$ for all $i \leq m-1$. □

A subgroup K of a group J is called *co-amenable* if there is an J-invariant mean on J/K. Equivalent characterizations, generalizations and unexpected examples can be found in [16] and [25].

Recall that a group J is *inner amenable* if there is a conjugacy-invariant mean on $J \setminus \{e\}$. It is equivalent to exhibit such a mean that is invariant under the second derived subgroup J'' since the latter is co-amenable in J. Thus, Proposition 5 is a consequence of the stronger fact that $H(A)$ is "{asymptotically commutative}-by-metabelian" in a sense inspired by [38] as follows.

Proposition 15. *Let $A < \mathbf{R}$ be any subring. For any finite set $S \subseteq H(A)''$ there is a non-trivial element $h_S \in H(A)$ commuting with each element of S.*

Indeed, any accumulation point of this net of point-masses at h_S is $H(A)''$-invariant.

Proof of Proposition 15. By the argument of Lemma 13, there is a neighbourhood of ∞ on which all elements of S are trivial. Thus is suffices to exhibit a non-trivial element h_S of $H(A)$ which is supported in this neighbourhood. Notice that $\mathrm{PSL}_2(\mathbf{Z})$ contains hyperbolic elements with both fixed points $\xi_\pm$ arbitrarily close to ∞, and on the same side. For instance, conjugate $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ by $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for sufficiently large $n \in \mathbf{N}$. We choose such an element h_0 with $\xi_\pm$ in the given neighbourhood and define h_S to be trivial on the component of $\mathbf{P}^1 \setminus \{\xi_\pm\}$ containing ∞ and to coincide with h_0 on the other component. □

A group is called *bi-orderable* if it carries a bi-invariant total order. The construction below is completely standard, compare e.g. [8, p. 233] for a first-order version of our second-order argument.

Proof of Proposition 6. Choose an orientation of $\mathbf{P}^1 \setminus \{\infty\}$ and define a (right) germ at a point p to be positive if either its first derivative is > 1 or if it is $= 1$ but the second derivative is > 0. Then define the set H_+ of positive elements of H to consist of all transformations whose first non-trivial germ (starting from ∞ along the orientation) is positive. Now H_+ is a conjugacy invariant sub-semigroup and $H \setminus \{e\}$ is $H_+ \sqcup H_+^{-1}$; this means that H_+ defines a bi-invariant total order.

Suppose now that we are given a homomorphism from a Kazhdan group to H. Its image is then a Kazhdan subgroup $K < H$. Kazhdan's property implies that K is finitely generated. It has been known for a long time that any non-trivial finitely generated bi-orderable group has a non-trivial homomorphism to $\mathbf{R}$: this follows ultimately from Hölder's 1901 work [19] by looking at maximal convex subgroups and is explained in [21, § 2]. But this is impossible for a Kazhdan group. □

Lemma 16. *For any $p \in \mathbf{P}^1 \setminus \{\infty\}$ there is a sequence $\{g_n\}$ in $H(\mathbf{Z})$ such that $g_n q$ converges to ∞ uniformly for q in compact subsets of $\mathbf{P}^1 \setminus \{p\}$.*

Proof. It suffices to show that for any open neighbourhoods U and V of p and ∞ respectively in $\mathbf{P}^1$, there is $g \in H(\mathbf{Z})$ which maps $\mathbf{P}^1 \setminus U$ into V. Since the collection of pairs of fixed points of hyperbolic elements of $\mathrm{PSL}_2(\mathbf{Z})$ is dense in $\mathbf{P}^1 \times \mathbf{P}^1$, we can find hyperbolic matrices $h_1, h_2 \in \mathrm{PSL}_2(\mathbf{Z})$ with repelling fixed points r_i in $U \setminus \{p\}$ and attracting fixed points a_i in $V \setminus \{\infty\}$ and such that the cyclic order is $\infty, a_1, r_1, p, r_2, a_2$. Now we define g to be a sufficiently high power of h_1 on the interval $[a_1, r_1]$ (for the above cyclic order), of h_2 on the interval $[r_2, a_2]$ and the identity elsewhere. □

Proof of Proposition 7. Let K be the pointwise stabilizer of a non-dense subset $E \subseteq \mathbf{P}^1$; it suffices to find a mean invariant under $H(A)''$. Let $\{g_n\}$ be the sequence provided by Lemma 16 for p an interior point of the complement of E. Any accumulation point of the sequence of point-masses at $g_n K$ in $H(A)/K$ will do. Indeed, since any $g \in H(A)''$ is trivial in a neighbourhood of ∞, we have $g_n^{-1} g g_n \in K$ for n large enough. □

The existence of two (or more) *commuting* co-amenable subgroups is also a weak form of amenability. It is the key in the argument cited below.

Proof of Proposition 8. Consider two disjoint non-empty open sets in $\mathbf{P}^1$. The pointwise stabilizers of their complement commute with each other and are co-amenable by Proposition 7. In this situation, Corollary 2.2 of [9] yields the desired conclusion. □

The properties used in this section show immediately that $H(A)$ fulfills the criterion of [1, Thm. 1.1] and thus satisfies no group law.

Combining Theorems 1 and 2 with the main result of [24], we conclude that the wreath product $\mathbf{Z} \wr H$ is a torsion-free non-unitarisable group without free subgroups. We can replace it by a finitely generated subgroup upon choosing a non-amenable finitely generated subgroup of H. This provides some new examples towards Dixmier's problem, unsolved since 1950 [13, 14, 26].

Finally, we mention that our argument from Proposition 6.4 in [23] applies to show that the bounded cohomology $\mathrm{H}^n_{\mathrm{b}}(H(A), V)$ vanishes for all $n \in \mathbf{N}$ and all mixing unitary representations V. More generally, it applies to any semi-separable coefficient module V unless all finitely generated subgroups of $H(A)''$ have invariant vectors in V (see [23] for details and definitions). This should be contrasted with the fact that amenability is characterized by the vanishing of bounded cohomology with all dual coefficients.

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