VARIATIONS ON A THEME BY HIGMAN

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ABSTRACT. We propose elementary and explicit presentations of groups that have no amenable quotients and yet are SQ-universal. Examples include groups with a finite $K(\pi, 1)$, no Kazhdan subgroups and no Haagerup quotients.

1. INTRODUCTION

In 1951, G. Higman defined the group

$$\mathrm{Hig}_n = \big\langle a_i \ (i \in \mathbf{Z}/n\mathbf{Z}) : [a_{i-1}, a_i] = a_i \big\rangle \tag{1}$$

and proved that for $n \geq 4$ it is infinite without non-trivial finite quotient [9]. Since the presentation (1) is explicit and simple, A. Thom suggested that Hig_n is a good candidate to contradict approximation properties for groups and proved such a result in [21]. Perhaps the most elusive approximation property is still *soficity* [7, 22]; but a non-sofic group would in particular not be residually *amenable*, a statement we do not know for the Higman groups (cf. also [8]). The purpose of this note is to propound variations of Higman's construction with no non-trivial amenable quotients at all.

There are several known sources of groups without amenable quotients since it suffices to take a (non-amenable) *simple* group to avoid all possible quotients. However, as Thelonius Sphere Monk observed, *simple ain't easy*. To wit, one had to wait until the break-through of Burger–Mozes [2, 3] for simple groups of *type F*, i.e. admitting a finite $K(\pi, 1)$. Before this, no torsion-free finitely presented simple groups were known.

The examples below are of a completely opposite nature because they admit a wealth of quotients: indeed, like Hig_n, they are *SQ-universal*, i.e. contain any countable group in a suitable quotient. It follows that they have uncountably many quotients [14, §III], despite having no amenable quotients.

We shall start with the easiest examples, whose cyclic structure is directly inspired by (1). Below that, we propose a cleaner construction, starting from copies of $\mathbf{Z}$ only, which might be a better candidate to contradict approximation properties; the price to pay is to replace the cycle by a more complicated graph.

Disclaimer. No claim is made to produce the first examples of groups with a hodgepodge of sundry properties (for instance, if G is a Burger–Mozes group, then $G * G$ satisfies many properties of G_n in Theorem 2 below, though with "amenable" instead of "Haagerup").
Our goal is to suggest transparent presentations for which the stated properties are explicit and their proofs effective.

1.A. Starting from large groups. Given a group K, an element $x \in K$ and a positive integer n, we define the group

$$K^{(n,x)} = \big\langle K_i \ (i \in \mathbf{Z}/n\mathbf{Z}) : [x_{i-1}, x_i] = x_i \big\rangle,$$

where K_i, x_i denote n independent copies of K, x. Thus, $K^{(n,e)} = K^{*n}$ and $\mathrm{Hig}_n = \mathbf{Z}^{(n,1)}$.

We recall that a group is *normally generated* by a subset if no proper normal subgroup contains that subset. Following the ideas of Higman and Schupp, we obtain:

Proposition 1. *Let K be a group normally generated by an element x of infinite order and let $n \geq 4$.*

(i) If K has no infinite amenable quotient (e.g. if K is Kazhdan), then $K^{(n,x)}$ has no non-trivial amenable quotient.
(ii) If K is finitely presented, torsion-free, type F_∞, or type F, then $K^{(n,x)}$ has the corresponding property.
(iii) Every countable group embeds into some quotient of $K^{(n,x)}$.

Remark. Suppose that $\mathscr{C}$ is any class of groups closed under taking subgroups. The proof of (i) shows: if every quotient of K in $\mathscr{C}$ is finite, then $K^{(n,x)}$ has no non-trivial quotient in $\mathscr{C}$. For instance, if K is Kazhdan, then $K^{(n,x)}$ has no non-trivial quotient with the Haagerup property [4].

Example. The group $K = \mathbf{SL}_d(\mathbf{Z})$ is an infinite, finitely presented (even type F_∞) Kazhdan group for all $d \geq 3$ and the Steinberg relations show that it is normally generated by any elementary matrix (with coefficient 1). Alternatively, the Steinberg group itself $K = \mathbf{St}_d(\mathbf{Z})$ has the same properties (it is Kazhdan because it is a finite extension of $\mathbf{SL}_d(\mathbf{Z})$, see e.g. [13, 10.1]). This gives us the following presentations of SQ-universal type F_∞ groups without Haagerup quotients:

$$S_{d,n} = \left\langle E_i^{p,q} \ (i \in \mathbf{Z}/n\mathbf{Z}, 1 \leq p \neq q \leq d) : \begin{array}{l} [E_i^{p,q}, E_i^{q,r}] = E_i^{p,r} \ (p \neq r \neq q) \\ [E_i^{p,q}, E_i^{r,s}] = e \ (q \neq r, p \neq s \neq r) \\ [E_{i-1}^{1,2}, E_i^{1,2}] = E_i^{1,2} \end{array} \right\rangle.$$

The choice of the pair $(1,2)$ is arbitrary and any other elementary matrix for x gives an isomorphic group. If we use the Magnus–Nielsen presentation [10, 20] of $\mathbf{SL}_d(\mathbf{Z})$ instead of the Steinberg group, we have to add the relations $(E_i^{1,2}(E_i^{2,1})^{-1}E_i^{1,2})^4 = e$.

These groups are not, however, torsion-free. Although congruence subgroups of $\mathbf{SL}_d(\mathbf{Z})$ are torsion-free (and even type F by [17]), the latter are never normally generated by a single element because they have large abelianizations.

This construction can be transposed to other Chevalley groups.

Notice that if in addition K is *just infinite*, like for instance $K = \mathbf{SL}_d(\mathbf{Z})$ for d odd [12], then this construction shows that K embeds into all non-trivial quotients of $K^{(n,x)}$, such as for instance the simple quotients obtained from maximal normal subgroups.

1.B. An example built from Z. Consider the semi-direct product

$$L = (\mathbf{Z}[1/2])^2 \rtimes (\mathbf{Z} \times F_2)$$

where the generator h of $\mathbf{Z}$ acts on $(\mathbf{Z}[1/2])^2$ by multiplication by 2, and the generators u, v of the free group F_2 act by multiplication by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

respectively. In particular the group L is torsion-free, linear and finitely presented. It is generated by $\{x, y, h, u, v\}$ where (x, y) is the standard basis of $\mathbf{Z}^2$.

We define a group G_n by fusing together n copies L_i of L in a circular fashion along the corresponding generators as follows:

$$G_n = \big\langle L_i : (h_i, u_i, v_i) = (y_{i-2}, x_{i-2}, y_{i-1}), i \in \mathbf{Z}/n\mathbf{Z} \big\rangle. \tag{2}$$

It is easy to write down an explicit presentation of G_n. Observe first that L, with our choice of generators, has a presentation with the following set R of relations

$$\begin{aligned} R(x, y, h, u, v): \quad & e = [x, y] = [x, u] = [y, v] = [h, u] = [h, v], \\ & [h, x] = [u, y] = x, \ [h, y] = [v, x] = y. \end{aligned}$$

Now (2) is equivalent to the finite presentation

$$G_n = \big\langle x_i, y_i : R(x_i, y_i, y_{i-2}, x_{i-2}, y_{i-1}), i \in \mathbf{Z}/n\mathbf{Z} \big\rangle. \tag{3}$$

We find these groups more elementary than $K^{(n,x)}$ (with Kazhdan K) and hope that they will be easier to use in applications. In return, we have to work more than before to deduce some of the following properties.

Theorem 2. *Let $n \geq 8$.*

- *(i) The group G_n has no non-trivial Haagerup quotient.*
- *(ii) Any quotient with a $\frac{1}{36}$-Følner set for the generators x_i, y_i is trivial.*
- *(iii) The only Kazhdan subgroup of G_n is the trivial group.*
- *(iv) The group G_n admits a finite $K(\pi, 1)$.*
- *(v) The group G_n can be constructed starting from copies of $\mathbf{Z}$, using amalgamated free products, semi-direct products and HNN-extensions.*
- *(vi) Every countable group embeds into some quotient of G_n if $n \geq 9$.*
- *(vii) The groups G_m are trivial for $m \leq 4$ and $m = 6$.*

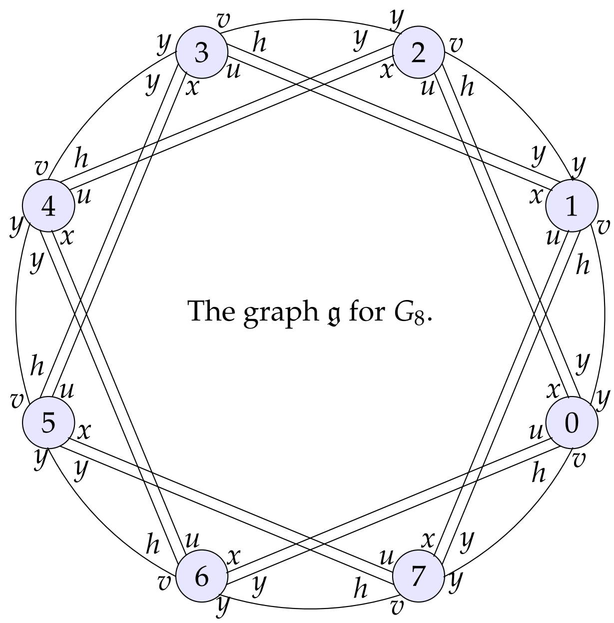
The restriction $n \geq 9$ is probably not needed in (vi) but makes it very easy to check Schupp's criterion for SQ-universality. We have not elucidated G_5 and G_7, but Laurent Bartholdi kindly informed us that G_5 is trivial according to a brief conversation with GAP.

Scholium. We should like to point out a general type of presentations subsuming the examples above. Consider a group L and two finite sets $A, P \subseteq L$. We think of elements in A as "active", whilst those in P are "passive". Consider furthermore a transitive labelled oriented graph $\mathfrak{g}$ whose edges are labelled by $P \times A$. To every vertex i of $\mathfrak{g}$ we associate an independent copy L_i of L. We then form the group

$$G = \big\langle L_i, i \in \mathfrak{g} : p_j = a_k \text{ if } \exists\, (p, a)\text{-labelled edge from } j \text{ to } k \big\rangle.$$

In order to get a manageable group from this presentation, we would like to ensure at the very least that each L_i embeds. A favourable case is when A is a basis for a free subgroup in L and the edges spread the passive elements of P_j incoming to a vertex k over copies L_j for suitably distinct js. (In our case, we allowed a commutation in A_k because it was going to hold also among the corresponding P_j.)

The trade-off is that this spreading should remain limited compared to the girth of the cycles in $\mathfrak{g}$ along which we can cut the amalgamation scheme. Higman's groups and the groups $K^{(n,x)}$ use a simple n-cycle for $\mathfrak{g}$; as for G_n, we depict its graph in the figure below for $n = 8$; the orientation is implicit from the labelling.



Notation. Our convention for commutators is $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$; Higman used a different convention for (1) but this does not affect the group Hig_n. Given a subset E of a group H, we denote the subgroup it generates by $\langle E \rangle$ or by $\langle E \rangle_H$ when H needs to be clarified.

2. Proof of Proposition 1

This proposition really is just a variation on the work of Higman and Schupp. For (i), we start by recalling the following.

Lemma 3 (Higman's circular argument)**.** *Let f be a homomorphism from* Hig_n *to another group. If $f(a_i)$ has finite order for some i, then f is trivial.*

Proof (see also [15, p. 547]). The relations in (1) imply inductively that $f(a_i)$ has finite order $r_i \geq 1$ for *all* i. Suppose for a contradiction that $r_i > 1$ for some i, hence for all i by the relations (1). Let p be the smallest prime dividing any r_j. The relation $a_{j-1}^{r_{j-1}} a_j a_{j-1}^{-r_{j-1}} = a_j^{2^{r_{j-1}}}$ implies that $2^{r_{j-1}} - 1$ is a multiple of r_j and hence of p. In particular, $p \neq 2$ and the order $s > 1$ of 2 in $(\mathbf{Z}/p\mathbf{Z})^\times$ divides r_{j-1}. This contradicts the choice of p because $s \leq p - 1$. □

Suppose now that f is a homomorphism from $K^{(n,x)}$ to an amenable group. The image of K_i in $K^{(n,x)}$ is mapped by f to a finite group, so that in particular $f(x_i)$ has finite order for all i. Since we have a homomorphism $\mathrm{Hig}_n \to K^{(n,x)}$ sending a_i to x_i, we deduce from Lemma 3 that $f(x_i)$ is in fact trivial. Since K is normally generated by x, it follows that $f(K_i)$ is trivial. We conclude that f is trivial because the various K_i generate $K^{(n,x)}$.

The two other points follow once we re-construct $K^{(n,x)}$ as a suitable amalgam. Recall that x has infinite order; thus

$$L = \langle K, h : [h, x] = x \rangle \tag{4}$$

is an HNN-extension; we define L_i, h_i similarly. Now

$$H = \big\langle L_0, L_1 : x_0 = h_1 \big\rangle$$

is a free product with amalgamation (because x_0 has infinite order) and therefore, using also the HNN-structure of (4), it follows that $\langle h_0, x_1 \rangle$ is a free group on h_0, x_1. Likewise, since $n \geq 4$, we deduce that

$$H' = \big\langle L_2, \ldots L_{n-1} : x_2 = h_3, \ldots, x_{n-2} = h_{n-1} \big\rangle$$

is a (successive) free product with amalgamation and that h_2, x_{n-1} are a basis of a free group in H'. Therefore, we obtain $K^{(n,x)}$ by amalgamating H and H' over the groups $\langle h_0, x_1 \rangle$ and $\langle x_{n-1}, h_2 \rangle$ by identifying the free generators in the order given here.

Now the finiteness properties of (ii) all follow since $K^{(n,x)}$ was obtained from copies of K by finitely many HNN-extensions and amalgamated free products (see e.g. [6, §7]). As for SQ-universality, the method devised by P. Schupp for Higman's group can be applied here. More precisely, we shall use the following criterion:

Lemma 4 (P. Schupp)**.** *Consider a free product with amalgamation $A *_C B$ with C non-trivial. If A contains an element t of order at least three such that t and C generate a free product $\langle t \rangle * C$ in A, then $A *_C B$ is SQ-universal.*

Proof. Theorem II in [19] applies to this situation, as explained in the paragraph immediately following that theorem. (In the terminology of that reference, any two distinct non-trivial powers of t are a *blocking pair* for C in A; such powers exist since t has order ≥ 3.) ☐

We apply this criterion to $A = H$ and $C = \langle h_0, x_1 \rangle$. The element $t = x_0^{-1} x_1 h_0 x_1^{-1} x_0$ satisfies the above condition because t, h_0, x_1 form a basis of a free group; this is proved in Lemma 4.3 of [19], noting that we have a canonical inclusion $H_2 \to H$ for the group H_2 considered in [19].

3. Proof of Theorem 2

We now turn to the groups L and G_n defined in part 1.B of the Introduction and fix some more notation. Denote by $\mathrm{Heis}(\alpha, \beta, \zeta)$ the (discrete) Heisenberg group with generators α, β and central generator ζ. More precisely, it is defined by the relations $[\alpha, \beta] = \zeta$ and $[\zeta, \alpha] = [\zeta, \beta] = e$. For instance, $\{v, x, y\}$ (or just $\{v, x\}$) generate a copy of $\mathrm{Heis}(v, x, y)$ in L.

We shall use repeatedly, but tacitly, the following fundamental property of a free product with amalgamation $A *_C B$. If $A' < A$ and $B' < B$ are subgroups whose intersections with C yield the same subgroup $C' < C$, then the canonical map $A' *_{C'} B' \to A *_C B$ is an embedding [16, 8.11].

We embed L into a larger group J generated by L together with an additional generator z by defining the following free product with amalgamation:

$$J = L *_{\langle h,v \rangle} \mathrm{Heis}(h, z, v). \tag{5}$$

Although h and v already occur in our definition of L, there is no ambiguity since they form a basis of a copy of $\mathbf{Z}^2$ both in L and in $\mathrm{Heis}(h, z, v)$. In particular, L is indeed canonically embedded in J.

When we want to consider normal forms for this amalgamation (cf. [18, §1] or Thm. 4.4 in [11]), it is convenient that there are very nice coset representatives of $\langle h, v\rangle$ in each factor. Indeed, in $\mathrm{Heis}(h,z,v)$, we can simply take the group $\langle z\rangle$. In L written as

$$L = (\mathbf{Z}[1/2])^2 \rtimes (\langle h\rangle \times \langle u, v\rangle),$$

we can take as set of representatives the group $(\mathbf{Z}[1/2])^2 \rtimes K$, where $K \lhd \langle u, v\rangle$ is the kernel of the morphism killing v.

As before, we shall denote by J_i a family of independent copies of J. We further denote by z_i the corresponding additional generator. Then we have an equivalent presentation of G_n given by

$$\left\langle J_i : \begin{array}{l} v_{i-1} = h_i \\ x_{i-1} = z_i \\ y_{i-1} = v_i \\ z_{i-1} = u_i \end{array} \quad \forall\, i \in \mathbf{Z}/n\mathbf{Z} \right\rangle. \tag{6}$$

The advantage is that each relation involves only *successive* indices $i-1$ and i.

We define inductively the groups D_r for $r \in \mathbf{N}$, starting with $D_0 = J_0$, by the presentation

$$D_r = \left\langle D_{r-1}, J_r : \begin{array}{l} v_{r-1} = h_r \\ x_{r-1} = z_r \\ y_{r-1} = v_r \\ z_{r-1} = u_r \end{array} \right\rangle.$$

We claim that this is in fact a free product with amalgamation of D_{r-1} and J_r. More precisely, we claim that the subgroups of J given respectively by

$$Q = \langle v, x, y, z\rangle_J \ \text{ and } \ T = \langle h, z, v, u\rangle_J \tag{7}$$

are isomorphic under matching their generators in the order listed in (7). This claim, transported to the various J_i, implies in particular by induction that D_r is indeed a free product with amalgamation $D_r \cong D_{r-1} *_{Q_{r-1}=T_r} J_r$, where Q_i, T_i denote the corresponding subgroups of J_i.

To prove the claim, we note first that the structure of Q is revealed by observing which subgroups are generated by $\{v, x, y\}$ and by $\{v, z\}$ in the amalgamation (5) defining J. Both intersect $\langle h, v\rangle$ exactly in $\langle v\rangle$ and thus Q is itself a free product with amalgamation $Q = \mathrm{Heis}(v,x,y) *_{\langle v\rangle} \langle v, z\rangle_J$ with $\langle v, z\rangle_J \cong \mathbf{Z}^2$.

As for T, given its relations, we have an epimorphism $Q \to T$ given by the above matching of generators; we need to show that it is in fact injective. To this end, consider that T is generated by its subgroups $\mathrm{Heis}(h,z,v)$ and $\langle h, v, u\rangle_J$. Since L is a factor of J, the latter is $\langle h, v, u\rangle_L \cong \mathbf{Z} \times F_2$. Thus T is an amalgamated free product $\mathrm{Heis}(h,z,v) *_{\langle h,v\rangle} \langle h, v, u\rangle_L$. The injectivity now follows. In conclusion, D_r is the following iterated free product with amalgamations:

$$D_r \cong J_0 *_{Q_0=T_1} J_1 *_{Q_1=T_2} \cdots *_{Q_{r-1}=T_r} J_r.$$

We also need to understand the intersection $Q \cap T$, which contains at least the group $\langle z, v\rangle_J \cong \mathbf{Z}^2$. In fact, this intersection is exactly $\langle z, v\rangle_J$. This follows by examining the normal form for the particularly simple choice of coset representatives made above.

As a consequence, we deduce that when $r \geq 3$, the subgroups T_0 and Q_r of

$$D_r \cong \big(J_0 *_{Q_0=T_1} J_1\big) *_{Q_1=T_2} \cdots *_{Q_{r-2}=T_{r-1}} \big(J_{r-1} *_{Q_{r-1}=T_r} J_r\big) \tag{8}$$

intersect trivially and hence generate a free product $T_0 * Q_r$.

Finally, to close the circle, we will use the assumption $n \geq 8$ and glue D_{n-5} with a copy D_3' of D_3 as follows. We shift indices in the D_3 factor to obtain the isomorphic group

$$D_3' = J_{n-4} *_{Q_{n-4}=T_{n-3}} \cdots *_{Q_{n-2}=T_{n-1}} J_{n-1}.$$

In D_3', the subgroups T_{n-4} and Q_{n-1} generate $T_{n-4} * Q_{n-1}$. Since we have constructed isomorphisms $T_0 \cong Q_{n-1}$ and $Q_{n-5} \cong T_{n-4}$, we have a corresponding isomorphism

$$\varphi\colon T_0 * Q_{n-5} \longrightarrow Q_{n-1} * T_{n-4}$$

and therefore we have a free product with amalgamation

$$D_{n-5} *_\varphi D_3'. \tag{9}$$

Since this is a rewriting of the presentation (6), we have indeed constructed G_n as an amalgam whenever $n \geq 8$. In particular, L_i is embedded in G_n.

At this point, we have established point (v) of Theorem 2, observing that $(\mathbf{Z}[1/2])^2 \rtimes \langle h \rangle$ is an HNN-extension of $\mathbf{Z} \times \mathbf{Z}$, that we can write

$$L \cong \Big((\mathbf{Z}[1/2])^2 \rtimes \langle h \rangle \Big) \rtimes (\mathbf{Z} * \mathbf{Z})$$

and that Heisenberg groups have the form $(\mathbf{Z} \times \mathbf{Z}) \rtimes \mathbf{Z}$.

On the other hand, point (iv) follows from (v), see e.g. [6, §7]. As for (iii), we only need to recall that Kazhdan groups have Serre's property FA [5, §6.a]. This implies that any Kazhdan subgroup of G_n can be recursively constrained into the factors of any amalgam. By (v), we finally reach $\mathbf{Z}$, which has no non-trivial Kazhdan subgroup.

For (vi), we indulge in the expedience of $n \geq 9$. This allows us to see from the decomposition (8) applied to $r = n - 5 \geq 4$ that we have a free product

$$\langle T_0, u_2 x_2, Q_r \rangle_{D_r} = T_0 * \langle u_2 x_2 \rangle * Q_r.$$

Indeed, reasoning within J, we see that $\langle ux \rangle$ intersects both Q and T trivially (and is infinite). Therefore, we can apply Schupp's criterion stated in Lemma 4 to $A = D_r$, $C = T_0 * Q_r$ and $t = u_2 x_2$. We conclude that G_n is SQ-universal.

Turning to (i), we first observe that every generator in the presentation (3) functions as a self-destruct button for the group G_n, i.e. normally generates G_n.

Lemma 5. *Let f be a homomorphism from G_n to another group. If f sends some x_i or some y_i to the identity, then f is trivial.*

Proof. The element $u_i v_i^{-1} u_i$ conjugates x_i to y_i^{-1} and therefore we can assume that $f(y_i)$ is trivial. Since $y_i = v_{i+1}$, the relation $[v_{i+1}, x_{i+1}] = y_{i+1}$ implies inductively that $f(y_j)$ vanishes for all j. Conjugating by $u_j v_j^{-1} u_j$, we find that all generators in (3) are trivialized by f. □

Let now f be a homomorphism from G_n to some Haagerup group. The subgroup $\langle x, y \rangle$ of $\langle x, y \rangle \rtimes \langle u, v \rangle$ has the relative property (T). Indeed, the proof of the corresponding statement for $\mathbf{Z}^2 \rtimes \mathbf{SL}_2(\mathbf{Z})$ only depends on the image of $\mathbf{SL}_2(\mathbf{Z})$ in the automorphism group of $\mathbf{Z}^2$, see e.g. [1]. Therefore, $f(\langle x_i, y_i \rangle)$ is finite for all i.

On the other hand, the presentation (2) shows that we have a morphism $\mathrm{Hig}_n \to G_n$ defined by $a_i \mapsto y_{2i}$. By Higman's argument (Lemma 3), it follows that $f(y_{2i})$ is trivial for all i. We conclude from Lemma 5 that f is trivial.

For (ii), we use the explicit *relative Kazhdan pair* (S_0, ϵ_0) provided by M. Burger, Example 2 p. 40 in [1]. Here S_0 is a certain generating set of $\mathbf{Z}^2 \rtimes \mathbf{SL}_2(\mathbf{Z})$ and $\epsilon_0 = \sqrt{2 - \sqrt{3}}$. Being a relative Kazhdan pair means that any unitary representation of $\mathbf{Z}^2 \rtimes \mathbf{SL}_2(\mathbf{Z})$ with (S_0, ϵ_0)-invariant vectors admits $\mathbf{Z}^2$-invariant vectors, see [5]. We denote by $S = \{x, y, u, v\}$ our usual generators of $\mathbf{Z}^2 \rtimes F_2$ and write $\overline{S} = S \cup S^{-1} \cup \{e\}$; then (S, ϵ)-invariance is equivalent to $(\overline{S}, \epsilon)$-invariance. The set S_0 from [1, Ex. 2] is contained in $\overline{S}^3$ under the map $F_2 \to \mathbf{SL}_2(\mathbf{Z})$ and therefore every $(S, \epsilon_0/3)$-invariant vector is (S_0, ϵ_0)-invariant. Now (ii) follows because $\epsilon_0/3 > 1/6$ and because any (S, ϵ)-Følner set gives a $(S, \sqrt{\epsilon})$-invariant vector.

Remark. The corresponding argument provides also a lower bound on Følner constants for quotients of $K^{(n,x)}$ when K is Kazhdan.

It only remains to prove (vii). Consider again the homomorphism $\mathrm{Hig}_n \to G_n$ above. When n is even, this factors through a morphism $\mathrm{Hig}_{n/2} \to G_n$. Since Hig_r is trivial for $r \leq 3$ (see [9]), it follows that y_0 is trivial when $n = 4, 6$; now Lemma 5 shows that G_n is trivial. The same argument applied to the original map $\mathrm{Hig}_n \to G_n$ takes care of $n \leq 3$.

Acknowledgements. We are grateful to the anonymous referee for suggesting a more explicit reformulation of Schupp's criterion and to Laurent Bartholdi for informing us that G_5 is trivial.

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