A LATTICE IN MORE THAN TWO KAC–MOODY GROUPS IS ARITHMETIC

PIERRE-EMMANUEL CAPRACE* AND NICOLAS MONOD‡

ABSTRACT. Let $\Gamma < G_1 \times \cdots \times G_n$ be an irreducible lattice in a product of infinite irreducible complete Kac–Moody groups of simply laced type over finite fields. We show that if $n \geq 3$, then each G_i is a simple algebraic group over a local field and Γ is an S-arithmetic lattice. This relies on the following alternative which is satisfied by any irreducible lattice provided $n \geq 2$: either Γ is an S-arithmetic (hence linear) group, or Γ is not residually finite. In that case, it is even virtually simple when the ground field is large enough.

More general CAT(0) groups are also considered throughout.

1. INTRODUCTION

The theory of lattices in semi-simple Lie and algebraic groups has witnessed tremendous developments over the past fourty years. It has now reached a remarkably deep and rich status, notably thanks to the celebrated work of G. Margulis, whose main aspects may be consulted in [Mar91]. Amongst the followers and exegetes of Margulis' work, several authors extended the methods and results pertaining to this classical setting to broader classes of lattices in locally compact groups. It should be noted however that as of today there exists apparently no characterisation of the S-arithmetic lattices purely within the category of lattices in compactly generated locally compact groups.

It turns out that relatively few examples of compactly generated topologically simple groups are known to possess lattices; to the best of our knowledge, they are all *locally compact* CAT(0) *groups*. In fact, the only examples which are neither algebraic nor Gromov hyperbolic are all automorphism groups of non-Euclidean locally finite buildings. Amongst these, the most prominent family consists perhaps of the so-called **irreducible complete Kac–Moody groups** over finite fields constructed by J. Tits [Tit87] (see § 4.B below for more details).

We now proceed to describe our main result. To this end, fix a positive integer n.

For each $i \in \{1, \ldots, n\}$, let X_i be a proper CAT(0) space and $G_i < \mathrm{Is}(X_i)$ be a closed subgroup acting cocompactly.

Theorem 1.1. *Let $\Gamma < G_1 \times \cdots \times G_n$ be any lattice whose projection to each G_i is faithful. Assume that G_1 is an irreducible complete Kac–Moody group of simply laced type over a finite field.*

If $n \geq 3$, then each G_i contains a cocompact normal subgroup which is a compact extension of a semi-simple group over a local field, and Γ is an S-arithmetic lattice.

Key words and phrases. Lattice, locally compact group, arithmeticity, Kac–Moody group, building, non-positive curvature, CAT(0) space.

*F.N.R.S. Research Associate.

‡Supported in part by the Swiss National Science Foundation.

(The same conclusion holds if G_1 is instead assumed to be a non-Gromov-hyperbolic irreducible complete Kac–Moody group of 3-spherical type over a finite field of characteristic $\neq 2$.)

One can summarise the above result as follows: *As soon as $n \geq 3$ and one of the factors G_i is Kac–Moody as above, all G_i are topologically commensurable to semi-simple algebraic groups and the lattice is S-arithmetic.*

Remark 1.2. Our aim in Theorem 1.1 is to provide a statement without any restrictive assumptions on the lattice Γ, on the spaces X_i or on the groups G_i. Considerable complications are caused by the fact that Γ is not supposed finitely generated. It turns out *a posteriori* that only finitely generated lattices exist — as a consequence of arithmeticity.

Remark 1.3. The assumption on faithfulness of the projections of Γ to each factor G_i is a form of irreducibility. We refer to Section 2.B below for a discussion of the different possible definitions of irreducibility for lattices in products of locally compact groups.

The above theorem is a new manifestation of the phenomenon that "high rank" yields rigidity. Numerous other results support this vague statement, including the rank rigidity of Hadamard manifolds, the arithmeticity of lattices in higher rank semi-simple groups, or the fact that any irreducible spherical building of rank ≥ 3 (resp. affine building of dimension ≥ 3) is associated to a simple algebraic group (possibly over a skew field).

Theorem 1.1 will be established with the help of the following **arithmeticity vs. non-residual-finiteness alternative**.

Theorem 1.4. *Let $\Gamma < G_1 \times \cdots \times G_n$ be a lattice which is algebraically irreducible. Assume that G_1 is an infinite irreducible complete Kac–Moody group of simply-laced type over a finite field.*

If $n \geq 2$ then either Γ is an S-arithmetic group or Γ is not residually finite.

(The same conclusion holds if G_1 is instead assumed to be a non-Gromov-hyperbolic irreducible complete Kac–Moody group of 3-spherical type over a finite field of characteristic $\neq 2$.)

It is known that if $G = G_1 \times G_2$ is a product of two isomorphic complete Kac–Moody groups over a sufficiently large finite field, then G contains at least one irreducible non-uniform lattice (see [Rém99], [CG99]). In [CR09], this specific example is shown to be simple provided G_1 and G_2 are non-affine (and without any other restriction on the type). Theorem 1.4 shows in particular that, under appropriate assumptions, *all* irreducible lattices in G are virtually simple. More precisely, we have the following **arithmeticity vs. simplicity alternative** which, under more precise hypotheses, strengthens the alternative from Theorem 1.4.

Corollary 1.5. *Let $G = G_1 \times \cdots \times G_n$, where G_i is an infinite irreducible Kac–Moody group of simply-laced type over a finite field $\mathbf{F}_{q_i}$ (or a non-affine non-Gromov-hyperbolic irreducible complete Kac–Moody group of 3-spherical type over a finite field $\mathbf{F}_{q_i}$ of characteristic $\neq 2$). Let $\Gamma < G$ be a topologically irreducible lattice; if Γ is not uniform, assume in addition that $q_i \geq 1764^{d_i}/25$ for each i, where d_i denotes the maximal rank of a finite Coxeter subgroup of the Weyl group of G_i. If $n \geq 2$, then exactly one of the following assertions holds:*

(i) *Each G_i is of affine type and Γ is an arithmetic lattice.*
(ii) *$n = 2$ and Γ is virtually simple.*

It is important to remark that, in most cases, a group G satisfying the hypotheses of any of the above statements does *not* admit any uniform lattice (see Remark 4.4 below).

Corollary 1.6. *Let $G = G_1 \times \cdots \times G_n$, where G_i is an infinite irreducible non-affine Kac–Moody groups of simply-laced type over a finite field $\mathbf{F}_{q_i}$ (or a non-affine non-Gromov-hyperbolic irreducible complete Kac–Moody group of 3-spherical type over a finite field $\mathbf{F}_{q_i}$ of characteristic $\neq 2$). Assume that $q_i \geq 1764^{d_i}/25$ for each i, where d_i denotes the maximal rank of a finite Coxeter subgroup of the Weyl group of G_i.*

If $n \geq 2$, then any topologically irreducible lattice of G has a discrete commensurator, and is thus contained in a unique maximal lattice.

Our proof of Theorems 1.1 and 1.4 builds upon the general methods developed in [CM09a, CM09b] for studying lattices in isometry groups of non-positively curved spaces. Our treatment of the residual-finiteness/simplicity dichotomy is inspired by the work of Burger–Mozes for tree lattices [BM00].

Acknowledgement. We are grateful to the anonymous referee for his/her useful comments.

2. Lattices in non-positive curvature

2.A. **The set-up.** We now introduce the setting for this section and the subsequent ones. The situation will differ from the very simple assumptions made in the Introduction; indeed our first task in the proof of Theorems 1.1 and 1.4 will be to reduce them to the set-up below.

Fix an integer $n \geq 2$. For each $i \in \{1, \dots, n\}$, let X_i be an irreducible proper CAT(0) space not isometric to the real line; irreducibility of X_i means that it does not split (non-trivially) as a direct product. It follows that X_i has no Euclidean factor. We also assume that the boundary ∂X_i is finite-dimensional for the Tits topology (though this assumption will only be used in later sections).

Let further $G_i < \mathrm{Is}(X_i)$ be a compactly generated closed subgroup without global fixed point at infinity. We assume that G_i acts **minimally** in the sense that there is no invariant closed convex proper subspace of X_i. We point out that this assumption is automatically fulfilled upon passing to some subspace since there is no fixed point at infinity, compare Remark 36 in [Mon06].

We set $G = G_1 \times \cdots \times G_n$ and $X = X_1 \times \cdots \times X_n$. Finally, let $\Gamma < G$ be a lattice.

It was established in [CM09b] that the following "Borel density" holds.

Proposition 2.1. *The action of Γ and its finite index subgroups on X is minimal and without fixed points at infinity.*

Proof. This is a special case of Theorem 2.4 in [CM09b]. □

2.B. **Irreducible lattices and residual finiteness.** Let $G = G_1 \times \cdots \times G_n$ be a locally compact group. The following properties provide several possible definitions of irreducibility for a lattice Γ in the product $G = G_1 \times \cdots \times G_n$, which we shall subsequently discuss.

- **(Irr1)** The projection of Γ to any proper sub-product of G is dense (and all G_i are non-discrete). In this case Γ is called **topologically irreducible**.
- **(Irr2)** The projection of Γ to each factor G_i is injective.
- **(Irr3)** The projection of Γ to any proper sub-product of G is non-discrete.

(Irr4) Γ has no finite index subgroup which splits as a direct product of two infinite subgroups. In this case Γ is called **algebraically irreducible**.

It turns out, as is well known, that if each factor G_i is a semi-simple linear algebraic group, then all four properties (Irr1)–(Irr4) are equivalent. We shall show that, in the setting of § 2.A, the implications (Irr2)⇒(Irr3)⇒(Irr4) do hold. If one assumes furthermore that each G_i is topologically simple and compactly generated, then each G_i has trivial quasi-centre by [BEW08, Theorem 4.8] and, hence, one has (Irr1)⇒(Irr2) in that case. However, even in the setting of § 2.A, *none* of the implications (Irr2)⇒(Irr1), (Irr3)⇒(Irr2) or (Irr4)⇒(Irr3) holds true.

A crucial point in the proof of Theorem 1.4 is that, nevertheless, the implication (Irr4)⇒(Irr2) becomes true provided the lattice Γ is residually finite, see Proposition 2.4 below.

From now on, we retain the notation of § 2.A.

The following result implies that (Irr2)⇒(Irr3)⇒(Irr4).

Proposition 2.2.

(i) *If the projection of Γ to each factor G_i is faithful, then the projection of Γ to any proper sub-product of G is non-discrete.*

(ii) *If the projection of Γ to any proper sub-product of G is non-discrete, or if the projection to at least one factor G_i is faithful, then Γ is algebraically irreducible.*

Proof. (i) Assume that the projection of Γ to each factor G_i is faithful and consider any (non-trivial) regrouping of factors $G = G' \times G''$; we need to show that the projection of Γ to G' is not discrete. Assume thus that the latter is discrete. In that case, Lemma I.1.7 in [Rag72] ensures that $\Gamma \cap (\{1\} \times G'')$ is a lattice in $\{1\} \times G''$. In particular it is non-trivial. Therefore the projection of Γ to G' cannot be faithful (a fortiori to some G_i).

(ii) Suppose that some finite index subgroup $\Gamma_0 < \Gamma$ admits a splitting. Proposition 2.1 implies in particular that Γ_0 acts on X as well as on each factor X_j, minimally and without fixed points at infinity. These are exactly the assumptions necessary to apply the splitting theorem of [Mon06]. The latter provides a splitting of X as $X = Y \times Z$ which, by the canonicity of the geometric decomposition $X \cong X_1 \times \cdots \times X_n$, must correspond to some regrouping of irreducible factors of X. In other words we have a non-empty subset $J \subsetneq \{1, \dots, n\}$ such that $Y = \prod_{j \in J} X_j$ and $Z = \prod_{j \notin J} X_j$. The desired result now follows from the fact that the respective Γ_0-actions on Y and Z are discrete but not faithful. □

Examples showing that the implication (Irr3)⇒(Irr2) fails in the setting of § 2.A can be obtained as extensions of arithmetic lattices by products or free groups, using similar constructions as in [CM09b, §6.C] (suggested by Burger–Mozes). The following result shows that (Irr4)⇒(Irr3) provided that the lattice Γ is finitely generated.

Proposition 2.3. *Assume that Γ is finitely generated and algebraically irreducible. Then the projection of Γ to any proper sub-product of $G = G_1 \times \cdots \times G_n$ has non-discrete image.*

Proof. See Theorem 4.2(i) in [CM09b]. □

We shall now describe an example showing that the implication (Irr4)⇒(Irr3) fails to hold if one removes the hypothesis that Γ be finitely generated. This also illustrates some of the technical difficulties that are unavoidable in the proof of our main results, since we deal with general (*i.e.* possibly non-uniform infinitely generated) lattices.

Example. Let $A = \bigoplus_{n\geq 0} \mathbf{Z}/2$ and $M = A * A$. Then M can be realised as a non-uniform lattice in the group $\mathrm{Aut}(T_3)$ of the regular tree of degree 3. To see this, one can express M as the fundamental group of a graph of groups as follows. Let λ^+ and λ^- be two copies of the simplicial half-line, and let us index the respective vertices of λ^+ and λ^- by the strictly positive integers in the natural order. On both λ^+ and λ^-, we attach the group $(\mathbf{Z}/2)^n$ to the vertex n and to the edge joining n to $n+1$. The embedding of the edge group attached with $[n, n+1]$ to the vertex group attached with $n+1$ is the natural inclusion of $(\mathbf{Z}/2)^n$ in $(\mathbf{Z}/2)^{n+1}$ defined by $x \mapsto (x, 0)$. Finally, we join the vertex 1 of λ^+ to the vertex 1 of λ^- by an edge to which we attach the trivial group. In this way, we obtain a graph of groups, which is simplicially isomorphic to the line. Its fundamental group is isomorphic to M and acts on the universal cover T_3 as a non-uniform lattice.

Similarly, we set $B = \bigoplus_{n\geq 0} \mathbf{Z}/3$ and view the group $N = B * B$ acting as a non-uniform lattice on the regular tree T_4 of degree 4 using the same construction, but replacing $\mathbf{Z}/2$ by $\mathbf{Z}/3$.

Now we define an action of M by automorphisms on N. Clearly A acts on B by non-trivial automorphisms componentwise, so that the semi-direct product $A \ltimes B$ is isomorphic to $\bigoplus_{n\geq 0}(\mathbf{Z}/2 \ltimes \mathbf{Z}/3)$: in each coordinate, the group $\mathbf{Z}/2$ acts on $\mathbf{Z}/3$ as the (unique) non-trivial automorphism. This action extends naturally to a diagonal action of A on $B \times B$ which, post-composed with the embedding of sets $B \times B \hookrightarrow B * B$, defines an action of A on the generators of $N = B * B$ preserving all the defining relations. Thus A acts on $N = B * B$ by automorphisms. Precomposing this with the natural quotient map $M = A * A \to A$ which annihilates the second free factor, we obtain a homomorphism

$$\alpha : M \to \mathrm{Aut}(N).$$

Since the M-action on N preserves the graph of group decomposition of N, it extends to an M-action by automorphisms on T_4 which, by a slight abuse of notation, we also denote by α. As a subgroup of $\mathrm{Aut}(T_4)$, the group $\alpha(M)$ fixes pointwise a line; the closure of M in $\mathrm{Aut}(T_4)$ is a compact subgroup Q isomorphic to $\prod_{\mathbf{Z}} \mathbf{Z}/2$.

Set now

$$\Gamma = N \rtimes_\alpha M \qquad \text{and} \qquad G = \mathrm{Aut}(T_4) \times \mathrm{Aut}(T_3).$$

We have already defined an embedding $f_4 : \Gamma \to \mathrm{Aut}(T_4)$ and a lattice embedding of M in $\mathrm{Aut}(T_3)$. Precomposing the latter with the quotient map $\Gamma \to M$, we obtain a homomorphism $f_3 : \Gamma \to \mathrm{Aut}(T_3)$ whose image is the lattice $M < \mathrm{Aut}(T_3)$. Finally, we define an injective homomorphism

$$f : \Gamma \to G : \gamma \mapsto (f_4(\gamma), f_3(\gamma)).$$

The image $f(\Gamma)$ is discrete. Moreover, since the image of $f(\Gamma)$ is a lattice in $\mathrm{Aut}(T_3)$ and the kernel of the projection of $f(\Gamma)$ to $\mathrm{Aut}(T_3)$ is a lattice in $\mathrm{Aut}(T_4)$, it follows that $f(\Gamma)$ is a lattice in G.

Remark that Γ is algebraically irreducible since no finite index subgroup of M is normal in Γ. The projection of Γ to $\mathrm{Aut}(T_3)$ is discrete while its projection to $\mathrm{Aut}(T_4)$ is not, since its closure is isomorphic to $N \rtimes Q$. This shows that Proposition 2.3 does not hold for lattices which are not finitely generated.

We finish this subsection with a crucial ingredient in the proof of Theorem 1.4 which shows that the implication (Irr4)⇒(Irr2) does however hold under the extra assumption that the lattice Γ is residually finite — even if it is not finitely generated.

Proposition 2.4. *Assume that Γ is residually finite and algebraically irreducible. Then the projection of Γ to each G_i is faithful.*

Proof. In the special case when Γ is finitely generated, we obtained this result in Theorem 4.10 of [CM09b]. In the present level of generality, we can write Γ as the union of an ascending sequence of finitely generated subgroups $(\Gamma_j)_{j\geq 0}$ because Γ is countable since $\mathrm{Is}(X)$ is second countable.

We let H_i denote the closure of the projection of Γ to G_i. Upon reordering the factors, we may assume that there is some $s \in \{0, \dots, n\}$ such that H_i is discrete if and only if $i > s$. We remark that if $s = 0$, then H_i is discrete for all i. Therefore, Lemma I.1.6 from [Rag72] ensures that $H_1 \times \cdots \times H_n$ is a lattice in $G = G_1 \times \cdots \times G_n$ and that the index of Γ in $H_1 \times \cdots \times H_n$ is finite. Therefore the product group $(\Gamma \cap H_1) \times \cdots \times (\Gamma \cap H_n)$ has finite index in Γ, contradicting the fact that Γ is algebraically irreducible. Thus $s > 0$ as asserted.

By [CM09a, Corollary 1.11], each G_i is either totally disconnected or an adjoint simple non-compact Lie group. By the definition of s, the group H_i is non-discrete for all $i \leq s$ and, hence, dense in every connected factor of $H_1 \times \cdots \times H_s$ by Borel density [Bor60, 4.2] (see also [Mar91, II.6.2]). Thus, for all $i \leq s$, the group H_i is a non-discrete closed subgroup which coincides with G_i if the latter is not totally disconnected.

Let $I \subseteq \{1, \dots, s\}$ be any non-empty subset. We claim that if the projection of Γ to $\prod_{i\notin I} H_i$ is not faithful, then the projection of Γ to $\prod_{i\in I} H_i$ is discrete.

In order to establish the claim, we let C denote the closure of the projection of Γ to $\prod_{i\in I} H_i$ and let

$$N = \Gamma \cap (\prod_{i\in I} H_i \times \prod_{i\notin I}\{1\}) < H_1 \times \cdots \times H_n.$$

Then C is totally disconnected; this is shown in the proof of Theorem 4.10 in [CM09b] by arguments that do not depend on the finite generation of Γ, but use the fact that H_i is either totally disconnected or a connected simple Lie group for all $i \in I$.

We now assume that N is non-trivial and need to deduce that C is discrete. Since Γ has trivial amenable radical [CM09b, Corollary 2.7] the normal subgroup $N \trianglelefteq \Gamma$ is not locally finite and, hence, we can assume upon discarding the first few indices in the filtration $(\Gamma_j)_{j\geq 0}$ that $\Gamma_j \cap N$ is infinite for each $j \geq 0$. Furthermore, since Γ has no fixed point ∂X by Proposition 2.1 and since ∂X is compact when endowed with the cone topology, we can moreover assume that Γ_j has no fixed point in ∂X for all j. Finally, let $Q < C$ be a compact open subgroup and denote by $C_j < C$ the subgroup generated by Q and the image of Γ_j.

By construction the group C_j is compactly generated, it acts without fixed point at infinity and the intersection of C_j with the image of N in C is an infinite discrete normal subgroup of C_j. Since $\Gamma \cap \big((\prod_{i\notin I} H_i) \cdot C_j\big)$ projects densely to C_j, we deduce from [CM09b, Proposition 4.8] that $[C_j \cap N, C_j^{(\infty)}] = 1$, where $C_j^{(\infty)}$ denotes the intersection of all open normal subgroups of C_j. We recall that a non-compact group of isometries of a proper CAT(0) space X acting without fixed point at infinity has a compact centraliser in $\mathrm{Is}(X)$ (though this is an overkill, it follows *e.g.* from the splitting theorem in [Mon06]); hence $C_j^{(\infty)}$ is compact.

On the other hand, the group C_j possesses a maximal compact normal subgroup because it acts without fixed point at infinity; this follows *e.g.* from Corollary 5.8 in [CM09a], the compact subgroup being the kernel of the C_j-action on a minimal subspace. Therefore, it follows from [CM09a, Proposition 6.12] that $C_j^{(\infty)}$ is non-compact whenever C_j is non-discrete. This shows that C_j is discrete. By construction C_j is open in C, thus C is discrete as well. This proves the claim.

We shall now establish that $s = n$. To this end, we notice that the projection of Γ to $G_{s+1} \times \cdots \times G_n$ cannot be faithful since it has discrete image (see [Rag72, Lemma I.1.7]). Applying the claim with the set $I = \{1, \dots, s\}$, we infer that the closure C of the projection of Γ to $H_1 \times \cdots \times H_s$ is discrete. By the definition of s, we also observe that the closure D of the projection of Γ to $H_{s+1} \times \cdots \times H_n$ is discrete. Thus Γ is contained in the discrete subgroup $C \times D < (H_1 \times \cdots \times H_s) \times (H_{s+1} \times \cdots \times H_n)$. Since Γ is a lattice, it must therefore have finite index in $C \times D$. Therefore Γ has a finite index subgroup which splits as a direct product, which contradicts the hypothesis that Γ is algebraically irreducible. This contradiction confirms that $s = n$.

Finally, assume for a contradiction that the projection of Γ to G_k is not faithful for some $k \in \{1, \dots, n\}$. We then invoke the claim above to the set $I = \{1, \dots, n\} \setminus \{k\}$. From the claim, we infer that the projection C' of Γ to $\prod_{i \neq k} H_i$ is discrete. Therefore, this projection cannot be faithful (see [Rag72, Lemma I.1.7]). We can then invoke the claim one more time, now with the set $I = \{k\}$. This implies that the projection D' of Γ to H_k is discrete, contradicting $s = n$. □

2.C. **Open subgroups.** Recall that, given a lattice Λ in a locally compact group H and any open subgroup $P < H$, the intersection $\Lambda \cap P$ is a lattice in P; indeed a Haar measure for P may be obtained by restricting the Haar measure of H. Furthermore, if Λ is uniform in H, so is $\Lambda \cap P$ in P. We shall frequently take advantage of this basic observation and study the intersection $\Gamma \cap P$ for various open subgroups $P < G$.

Lemma 2.5. *Let H, J be locally compact groups, $\Lambda < H \times J$ a lattice, $P < H$ an open subgroup and $\Lambda_P = \Lambda \cap (P \times J)$. Then any intermediate group $\Lambda_P < \Lambda' < \Lambda$ is a lattice in $H' \times J$ and in $H' \times J'$, where H' and J' are the closure of the projection of Λ' to H and J respectively.*

Proof. Let H^* be the the closure of the projection of Λ to H and $P^* = P \cap H^*$. Then Λ is a lattice in $H^* \times J$ by [Rag72, I.1.6]. Moreover, Λ_P is a lattice in $P^* \times J$ projecting densely to P^* since P is open; in particular, $P^* \subseteq H'$. Let $F \subseteq P^* \times J$ be a (left) fundamental domain for Λ_P in $P^* \times J$.

We claim that the Λ'-translates of F cover $H' \times J$. Pick thus any (h_0, j_0) in $H' \times J$. Since P^* is open in H', there is (h_1, j_1) in Λ' such that $h_1 h_0 \in P^*$. Since $(h_1 h_0, j_1 j_0) \in P^* \times J$, there is (h_2, j_2) in Λ_P such that $(h_2 h_1 h_0, j_2 j_1 j_0) \in F$. Since $(h_2 h_1, j_2 j_1) \in \Lambda'$, this proves the claim.

Since Λ' is discrete and since the Haar measures of $P^* \times J$ extend to Haar measures of $H' \times J$, we conclude that Λ' is indeed a lattice in $H' \times J$. Applying again [Rag72, I.1.6], we deduce that it is also a lattice in $H' \times J'$. □

We now return to our geometric setting.

Proposition 2.6. *Let $P < G_1$ be an open subgroup and set*

$$\Gamma_P = \Gamma \cap (P \times G_2 \times \cdots \times G_n).$$

Assume that the projection of Γ_P to some G_i with $i \geq 2$ is faithful. Then Γ_P is algebraically irreducible.

Proof. In order to argue as in Proposition 2.2, we need to show that the Γ_P-action on X_i is minimal and without fixed point at infinity.

We claim that without loss of generality we may assume that G_1 is totally disconnected. Indeed, otherwise by Corollary 1.11 in [CM09a] the group G_1 is an almost connected simple Lie group. In that case the open subgroup $P < G_1$ has finite index in G_1 and hence Γ_P has finite index in Γ. The claimed statement is thus a case of Proposition 2.2.

In view of the claim, we assume that G_1 is totally disconnected; hence so is P. Therefore there exists a compact open subgroup $U < P$ (see [Bou71, III.4 No 6]). Then the group

$$\Gamma_U = \Gamma \cap (U \times G_2 \times \cdots \times G_n)$$

is a lattice in $U \times G_2 \times \cdots \times G_n$ and thus its projection to $H := G_2 \times \cdots \times G_n$ is a lattice as well. Since the H-action on $Y := X_2 \times \cdots \times X_n$ is minimal and without fixed point at infinity, so is the Γ_U-action by Proposition 2.1. Now we deduce *a fortiori* that the Γ_P-action on Y and hence on X_i is minimal and without fixed point at infinity. □

Corollary 2.7. *Let $P < G_1$ be any open subgroup and set*

$$\Gamma_P = \Gamma \cap (P \times G_2 \times \cdots \times G_n).$$

If Γ is algebraically irreducible and residually finite, then so is Γ_P.

Proof. By Proposition 2.4 the projection of Γ to each G_i is faithful. Thus we may apply Proposition 2.6. □

2.D. **Cofinite embeddings of semi-simple groups.** We do not know if a semi-simple algebraic group can appear as a subgroup of finite covolume in a locally compact group without being cocompact[1]. We shall prove that this does not happen in the CAT(0) setting.

Proposition 2.8. *Let H be a locally compact group and $L < H$ a closed subgroup of finite covolume which is a compact extension of a semi-simple algebraic group. Suppose that H admits a cocompact proper continuous isometric action on some* CAT(0) *space.*

Then H/L is compact.

Morover, if the semi-simple group has no rank one factors, then upon factoring out a (unique maximal) compact normal subgroup, H is a group of automorphismsm of the semi-simple algebraic group.

The following fact is well-known.

Lemma 2.9. *A group of isometries preserving a non-zero finite measure on a complete* CAT(0) *space fixes a point.*

Proof. Let G be the group, X the space and μ the measure. There is a bounded set $B \subseteq X$ such that $\mu(B) > \mu(X)/2$. Therefore $gB \cap B \neq \varnothing$ for all $g \in G$. It follows that G has a bounded orbit and hence a fixed point by Cartan's circumcentre principle [BH99, II.2.8]. □

Lemma 2.10. *Let H be a locally compact group containing a closed subgroup of finite covolume which is a compact extension of a semi-simple algebraic group. Then any continuous isometric H-action on a proper* CAT(0) *space preserves a non-empty closed convex subset with trivial Euclidean factor.*

[1]Note added in proof: we have been informed that Bader–Furman–Sauer address this question in forthcoming work.

Proof. Invoking repeteadly the canonical Euclidean decomposition [BH99, II.6.15], it suffices to prove that any continuous isometric H-action on any $\mathbf{R}^d$ has a fixed point. Let $L < H$ be the given subgroup with $K_0 \lhd L$ compact normal such that L/K_0 is semi-simple. The non-empty subspace of K_0-fixed points is an affine subspace preserved by L; we therefore have an isometric action of the semi-simple group L/K_0 on some $\mathbf{R}^{d'}$.

It is well-known that all such L/K_0-actions have a fixed point. Therefore L fixes a point in $\mathbf{R}^d$, which implies by Lemma 2.9 that H also fixes a point. □

Proof of Proposition 2.8. Let X be a CAT(0) space as in the statement; it is necessarily proper. Since H acts cocompactly, it has a minimal convex invariant subspace and thus we can assume X minimal upon factoring out the compact kernel of the H-action on that subspace. We note in passing that this kernel is a (necessarily unique) maximal compact normal subgroup of H.

Lemma 2.10 implies that X has trivial Euclidean factor. Moreover, we claim that H has no fixed point at infinity. Indeed, otherwise by minimality the corresponding Busemann character $H \to \mathbf{R}$ would be non-trivial. This however would produce a non-trivial character of L which would thus descend non-trivially to the semi-simple group, which is absurd.

By Theorem 2.4 in [CM09b], the L-action on X is minimal and without fixed point at infinity. In particular, L has no non-trivial compact normal subgroup and we can decompose it into its simple factors $L = L_1 \times \cdots \times L_n$. Each L_i is non-compact and we can assume $n \geq 1$ since otherwise H is compact (actually trivial at this point).

In view of Addendum 1.8 in [CM09a] we can write $X = X_1 \times \cdots \times X_n$, where each L_i acts minimally on X_i; the finite-dimensionality of ∂X holds by Theorem C in [Kle99] since H acts cocompactly. Moreover, each ∂X_i is finite-dimensional and each L_i has full limit set in ∂X_i because the two corresponding statements for ∂X and the L-action on X hold: the latter by Proposition 2.9 in [CM09b], using again cocompactness of H.

We can now apply Theorem 7.4 in [CM09a] and deduce that each L_i acts cocompactly on X_i; indeed the proof of *loc. cit.* even provides a quasi-isometry bewteen X_i and the model space (symmetric space or Bruhat–Tits building) of L_i. Thus L acts cocompactly on X, which implies that L is cocompact in H.

Theorem 7.4 in [CM09a] also provides a Tits-isometric identification of ∂X_i with the boundary of the model space of L_i. Assuming now that each L_i has rank at least two, we can apply Tits' rigidity theorem (Theorem 5.18.4 in [Tit74]) and deduce that $\mathrm{Is}(X_i)$ is the group of isometries of the model space, which coincides with the group of automorphisms of the associated semi-simple group. □

3. Presence of an algebraic factor

3.A. **Algebraic factors in general.** Following Margulis [Mar91, IX.1.8], we shall say that a simple algebraic group $\mathbf{G}$ defined over a field k is **admissible** if *none* of the following holds:

- $\mathrm{char}(k) = 2$ and $\mathbf{G}$ is of type A_1, B_n, C_n or F_4;
- $\mathrm{char}(k) = 3$ and $\mathbf{G}$ is of type G_2.

A semi-simple group will be called admissible if all its simple factors are.

Theorem 3.1. *Let k be a local field and $\mathbf{G}$ an adjoint admissible connected semi-simple k-group without k-anisotropic factors. Let X be a proper* CAT(0) *space without Euclidean factor and $H < \mathrm{Is}(X)$ a closed totally disconnected subgroup acting minimally and without fixed point at infinity. Let $\Gamma < \mathbf{G}(k) \times H$ be any lattice; in case $\mathrm{rank}_k \mathbf{G} = 1$ and $\mathrm{char}(k) > 0$, we assume in addition Γ cocompact.*

If Γ projects faithfully to the simple factors of $\mathbf{G}(k)$, then H is a semi-simple algebraic group upon passing to a finite covolume subgroup containing the image of Γ. Furthermore, Γ is finitely generated.

Remark 3.2. There is a similar statement without the CAT(0) space X in Theorem 5.20 of [CM09b], but at the cost of assuming H compactly generated and assuming that Γ projects densely to H. In general, we do not know how to prove *a priori* that the closure of the projection of Γ to H is compactly generated, even if we assume H compactly generated. In the above theorem, compact generation of H follows *a posteriori* from the statement. In fact, the bulk of the proof given below is concerned with addressing this very issue.

We begin with a geometric finiteness result that will allow us to *rule out phenomena of adélic type* in the proof of Theorem 3.1; for its own sake, we provide more generality.

Proposition 3.3. *Let X be a proper* CAT(0) *space and $H < \mathrm{Is}(X)$ a closed subgroup acting minimally and without fixed point at infinity. Let $\{H_n\}$ be a non-decreasing family of closed subgroups of H such that the closure of the union of all H_n is co-amenable in H.*

Then there is $N \in \mathbf{N}$ such that no H_n can be a compact extension of a direct product of more than N non-compact factors.

Proof. Let $X = Y \times \mathbf{R}^d$ be the maximal Euclidean decomposition, so that $\mathrm{Is}(X) = \mathrm{Is}(Y) \times \big(\mathbf{O}(d) \ltimes \mathbf{R}^d\big)$, see Theorem II.6.15 in [BH99]. Arguing by contradiction, we can assume that each H_n has a compact normal subgroup K_n such that H_n/K_n can be decomposed as a direct product of n non-compact factors. We claim that we can pass to a further subsequence and regroup factors so that all n factors have non-compact image in $\mathrm{Is}(Y^{K_n})$. Indeed, each H_n/K_n acts on a Euclidean subspace of $\mathbf{R}^d$, namely its K_n-fixed points. This implies that at most d of the non-compact factors of H_n/K_n have a non-compact image in $(\mathbf{R}^d)^{K_n}$; thus at least $n-d$ factors have non-compact image in $\mathrm{Is}(Y^{K_n})$, which implies the claim.

Since the closure $H_\infty < H$ of the union of all H_n is co-amenable, it has no fixed point in ∂Y by Proposition 2.1 in [CM09b]. Therefore, by compactness of ∂Y, we can further assume that none of the H_n has a fixed point in ∂Y. It follows that each H_n admits some minimal non-empty closed convex invariant subspace $Y_n \subseteq Y$ and that moreover the union $Z_n \subseteq Y$ of all such subspaces splits isometrically and equivariantly as $Z_n \cong Y_n \times T_n$, where the "space of components" T_n is a *bounded* CAT(0) space endowed with the trivial H_n-action; for all this, see Remark 39 in [Mon06].

We claim that the sequence $\{T_n\}$ is of non-increasing diameter. Indeed, let $t, t' \in T_{n+1}$; then both $Y_{n+1} \times \{t\}$ and $Y_{n+1} \times \{t'\}$ contain some, *a priori* several, minimal H_n-subspaces. We denote by $s, s' \in T_n$ the elements corresponding to some arbitrary such choices $Y_n \times \{s\} \subseteq Y_{n+1} \times \{t\}$ and $Y_n \times \{s'\} \subseteq Y_{n+1} \times \{t'\}$. Now we have $d(t,t') \leq d(s,s')$ and the claim follows.

In view of the claim, we may choose a sequence of points $y_n \in Y_n$ that remains bounded. Notice that K_n acts trivially on Y_n. Our assumption on H_n/K_n together with the splitting theorem from [Mon06] shows that Y_n admits a splitting as a product of n non-compact factors. In particular, we can choose n geodesic rays issuing from y_n and spanning a Euclidean n-dimensional quadrant. Having Euclidean quadrants of unbounded dimension but based at the points y_n which remain in a bounded set contradicts the local compactness of Y. □

Proof of Theorem 3.1. In view of the nature of the statement, we may and shall replace H by the closure of the projection of Γ, which has finite covolume in H. By Theorem 2.4 in [CM09b], H still acts minimally and without fixed point at infinity. In particular, we can assume it non-compact since otherwise it is trivial, in which case the statement is empty except for the finite generation of Γ; the latter would still follow as explained below for Γ_U, which coincides with Γ when H is trivial.

As we shall see, given the results we proved in [CM09b], the main step here is to prove the following.

Main claim: the lattice Γ is finitely generated.

To this end, let $U < H$ be a compact open subgroup, which exists by [Bou71, III.4 No 6]. Since Γ projects injectively to $\mathbf{G}(k)$, we can consider $\Gamma_U = \Gamma \cap (\mathbf{G}(k) \times U)$ as a lattice in $\mathbf{G}(k)$. Moreover, Γ_U is irreducible in $\mathbf{G}(k)$ since it projects injectively to the simple factors (recalling that the various notions of irreducibility coincide in the case of lattices in semi-simple groups). Our assumptions imply that Γ_U is finitely generated; indeed, we recall the argument given in [CM09b, 5.20]: either we have simultaneously $\mathrm{char}(k) > 0$ and $\mathbf{G}$ is simple of k-rank one, in which case we assumed Γ cocompact, so that Γ_U is cocompact in the compactly generated group $\mathbf{G}(k)$ and hence finitely generated [Mar91, I.0.40]; otherwise, Γ_U is known to be finitely generated by applying, as the case may be, either Kazhdan's property, or the theory of fundamental domains, or the cocompactness of p-adic lattices — we refer to Margulis, Sections (3.1) and (3.2) of Chapter IX in [Mar91].

We choose a non-decreasing sequence $\{\Gamma_n\}$ of finitely generated subgroups with $\Gamma_U < \Gamma_n < \Gamma$ and which exhaust all of Γ. We denote by $G_n < \mathbf{G}(k)$ the closure of the projection of Γ_n to $\mathbf{G}(k)$ and by $H_n < H$ the closure of the projection of Γ_n to H. Notice that the closure of the union of all H_n coincides with the closure of the projection of Γ to H and thus is all of H in view of our preliminary reduction.

We claim that Γ_n is a topologically irreducible lattice in $G_n \times H_n$ upon discarding the first few n. Lemma 2.5 shows that Γ_n is indeed a lattice in $G_n \times H_n$ and hence the point to verify is that G_n, H_n are both non-discrete.

If all G_n are discrete, they are lattices in $\mathbf{G}(k)$ and thus Γ_U has finite index in Γ_n since the projection of Γ to $\mathbf{G}(k)$ is faithful; in particular, H_n is compact and hence fixes a point in X. Considering the nested sequence of H_n-fixed points in the compactification $\overline{X}$, we deduce by compactness that H fixes a point in $\overline{X}$. This is impossible since H fixes no point at infinity and is non-compact.

If H_n is discrete, then $\Gamma_n \cap (G_n \times 1)$ is a lattice (see Theorem 1.13 in [Rag72]). Viewing it in G_n, it is a normal (hence cocompact) lattice since it is normalised by the projection of Γ_n. However, G_n does not admit a normal lattice when it is non-discrete. Indeed, being Zariski-dense in $\mathbf{G}$ (by Borel density applied to Γ_U), it contains the group $\mathbf{G}_\alpha(k)^+$ for some simple factor $\mathbf{G}_\alpha$ by [Pra77] and the latter is simple [Tit64] (and non-discrete). The claim that Γ_n is irreducible in $G_n \times H_n$ is proved.

We can now apply Theorem 5.1 from [CM09b] and deduce that H_n is a compact extension of a semi-simple algebraic group without compact factors. In fact, this reference allows a priori for a possibly infinite discrete direct factor in H_n which is also virtually a direct factor of Γ_n, but in the case at hand this contradicts the fact that it is Zariski dense in a simple algebraic group, namely any simple factor $\mathbf{G}_\alpha$ (since it contains Γ_U which is Zariski-dense in $\mathbf{G}$ by Borel density).

We claim that the obtained semi-simple quotient of H_n is a direct factor of the quotient associated to H_{n+1}.

Indeed, Margulis' commensurator arthmeticity [Mar91, 1.(1)] shows that Γ_U is S-arithmetic and hence the projection of Γ is contained in $\mathbf{G}(K)$ for some global field K over which $\mathbf{G}$ is defined by Theorem 3.b in [Bor66] (see also [Wor07, Lemma 7.3]). An examination of the proof of Theorem 5.1 in [CM09b] shows that the semi-simple quotient of H_n is the product of the non-compact semi-simple factors of all $\mathbf{G}(K_v)$, where v ranges over the set of places of K for which Γ_n is unbounded. This proves the claim.

Proposition 3.3 applies and we deduce from the previous claim that the sequence of the semi-simple quotients associated to $\{H_n\}$ stabilises. In view of the above discussion, it follows that $\Gamma < \mathbf{G}(K)$ is in fact itself S-arithmetic. In view of the assumptions on $\mathbf{G}$ and of the results in Section 3.2 of Chapter IX in [Mar91], this S-arithmetic group is finitely generated if it is irreducible. Since Γ projects injectively into the simple factors of $\mathbf{G}(k)$, irreducibility follows. (Alternatively, argue as in the proof of Proposition 2.2.) This concludes the proof of the main claim.

We now have $\Gamma = \Gamma_n$ for n large enough; in particular, $H_n = H$ and the proof is complete. □

3.B. **Reduction to the totally disconnected case.** Retain the notation of § 2.A. The following result will later allow us to focus on the case where each G_i is totally disconnected.

Proposition 3.4. *Assume that the projection of Γ to each G_i is faithful.*

If G is not totally disconnected, then each G_i contains a closed subgroup H_i of finite covolume which is a simple algebraic group over a local field and Γ is S-arithmetic. If in addition G acts cocompactly on X, then G_i/H_i is compact.

Proof. Define H_i as the closure of the projection of Γ to G_i; we shall focus on the statements about G_i and H_i, since the arithmeticity of Γ will then follow by Margulis' results (see Theorem (A) in Chapter IX of [Mar91]).

If the identity component G° is non-trivial, then the same holds for some G_i. Upon renumbering the G_i's, we may and shall assume that G_i is totally disconnected if and only if $i > k$ for some $k \in \{1, \dots, n\}$. By [CM09a, Corollary 1.11], it follows that G_i is a non-compact simple Lie group with trivial centre for each $i \in \{1, \dots, k\}$.

By Proposition 2.2(i), the group H_i is non-discrete for each i. In particular we have $H_i = G_i$ for each $i \leq k$ by Borel density [Bor60], since a Zariski-dense subgroup of a simple Lie group is either discrete or dense. Furthermore, since H_i has finite covolume in G_i, it follows from [CM09b, Theorem 2.4] that H_i acts minimally without fixed point at infinity on X_i. In particular it has no non-trivial compact normal subgroup. Now Theorem 3.1 implies that

$$H := G_1 \times \cdots \times G_k \times H_{k+1} \times \cdots \times H_n$$

is a semi-simple algebraic group. In view of Proposition 2.8, if G_i acts cocompactly on X_i, then so does H_i. □

It will be convenient to have the following *ad hoc* simpler variant of Proposition 3.4; it is essentially just a shortcut available in positive characteristic.

Proposition 3.5. *Let k be a local field of positive characteristic and $\mathbf{G}$ an adjoint connected absolutely almost simple k-group of positive k-rank. Let X be a proper* CAT(0) *space without Euclidean factor and let $H < \mathrm{Is}(X)$ be a closed subgroup acting cocompactly, minimally and without fixed point at infinity.*

If there is any lattice $\Gamma < \mathbf{G}(k) \times H$ that projects faithfully to $\mathbf{G}(k)$, then H is totally disconnected.

Proof. Theorem 1.6 in [CM09a] implies that H is of the form $H = S \times D$, where S is a connected semi-simple Lie group and D is totally disconnected. Let $U < D$ be a compact open subgroup and observe that the lattice $\Gamma_U < \mathbf{G}(k) \times S \times U$ (as considered in 2.C) still projects injectively to $\mathbf{G}(k)$. Suppose for a contradiction that S is non-compact. Then we have obtained a lattice in $\mathbf{G}(k) \times S$ which is irreducible and S-arithmetic in view of Margulis' arithmeticity [Mar91]. This is absurd since the characteristics of the fields of definition do not agree. □

3.C. **Arithmeticity of residually finite lattices.** We remain in the setting of § 2.A.

Theorem 3.6. *Suppose that the lattice $\Gamma < G = G_1 \times \cdots \times G_n$ is algebraically irreducible. Assume that G_1 possesses an open subgroup P which is a compact extension of a non-compact admissible simple algebraic group $\mathbf{H}$ over a local field k. In case k has positive characteristic and $\mathbf{H}$ has k-rank one, we assume in addition Γ cocompact.*

If Γ is residually finite, then each G_i contains a closed subgroup of finite covolume which is a compact extension of a simple algebraic group over a local field, and Γ is S-arithmetic.

Proof. By Proposition 2.4, the projection of Γ to each G_i is faithful. Therefore, in view of Proposition 3.4, we can assume that G is totally disconnected. Set

$$\Gamma_P = \Gamma \cap (P \times G_2 \times \cdots \times G_n).$$

By assumption P has a compact normal subgroup K such that $P/K = \mathbf{H}(k)$. Now Γ_P maps onto a lattice in the product

$$\mathbf{H}(k) \times G_2 \times \cdots \times G_n$$

and this map has finite kernel. Since Γ_P is residually finite, we can assume that the kernel is trivial upon replacing Γ_P with a finite index subgroup; we henceforth consider Γ_P as a lattice in the above product.

The projection of Γ_P to $\mathbf{H}(k)$ is faithful since we have already recorded that Γ projects injectively to G_1. Therefore, we can apply a first time Theorem 3.1 to Γ_P and deduce in particular for each $i \geq 2$ that G_i is an admissible semi-simple algebraic group upon replacing it by a closed subgroup of finite covolume containing the image of Γ_P. In fact, these groups are simple in view of the irreducibility of X_i (*e.g.* by the splitting theorem). We write $G_i = \mathbf{G}_i(k_i)$ for $i \geq 2$ and also note that Γ_P is irreducible (*e.g.* by Proposition 2.2).

We now return to the lattice $\Gamma < G$ with the intention to apply a second time Theorem 3.1, but reversing the rôles of G_1 and $G_2 \times \cdots \times G_n$. We point out that the simple groups $\mathbf{G}_i$ are all admissible since both the absolute type and characteristic are constant over all factors in view of the fact that the S-arithmetic group Γ_P is irreducible. However, we have no guarantee that the technical assumption made on $\mathbf{H}$ holds for $\mathbf{G}_i$. It can indeed fail and likewise the finite generation used in the proof of Theorem 3.1 for Γ_U is also known to fail. We shall now circumvent this difficulty.

The group Γ_P is finitely generated by the above application of Theorem 3.1. We consider a non-decreasing sequence of finitely generated groups Γ_j starting with $\Gamma_0 = \Gamma_P$ and exhausting Γ. We denote by L_j and R_j the closure of the projection of Γ_j to

$$G_1 \times \cdots \times G_{n-1} \quad \text{and} \quad G_n$$

respectively. Lemma 2.5 shows that Γ_j is a lattice in $L_j \times R_j$. It is topologically irreducible in the former product since already the projections of Γ_0 are non-discrete (*e.g.* by Proposition 2.3). Theorem 5.1 in [CM09b] implies that L_j is a compact extension of a semi-simple group S_j. We write $Q = P \times G_2 \times \cdots \times G_{n-1}$, wherein $Q = P$ is understood if $n = 2$. The group $Q \cap L_j$ is open in L_j and non-compact since it contains L_0 which is of finite covolume in the non-compact group Q. Therefore the image of $Q \cap L_j$ in S_j contains S_j^+ by [Pra82, thm. (T)] (or by an application of Howe–Moore). Since S_j^+ is cocompact in S_j (see [BT73, 6.14]) we conclude that $Q \cap L_j$ is cocompact in L_j and hence has finite index. This shows that L_0 has finite index in L_j. Therefore, S_0 has finite index in S_j for all j; it follows, since S_0^+ is simple [Tit64], that S_j normalises S_0^+. We denote by L_0^+ the preimage of S_0^+ in L_0 and set $\Gamma_0^+ = \Gamma \cap (L_0^+ \times G_n)$. Then Γ_0^+ has finite index in Γ_0 since the latter is finitely generated and since S_0/S_0^+ is a virtually Abelian torsion group [BT73, 6.14]. Moreover, Γ_j normalises Γ_0^+ for all j in view of corresponding statement for $S_0^+ < S_j$ above.

At this point we have a natural map $\Gamma \to \mathrm{Aut}(S_0^+)$ whose image normalises Γ_0^+. In combination with the injective map $\Gamma \to G_n$, we have a realisation of Γ in the normaliser of Γ_0^+ in the algebraic group $\mathrm{Aut}(S_0^+) \times \mathbf{G}_n(k_n)$. Since Γ_0^+ is a lattice in the latter, it follows from Borel's density theorem that Γ_0^+ has finite index in its normaliser, see [Mar91, II.6.3]. In conclusion, Γ_0^+ has finite index in Γ and thus both sequences Γ_j and L_j are eventually constant, completing the proof. □

4. Kac–Moody groups

4.A. **A lemma on Coxeter–Dynkin diagrams.** A theorem of G. Moussong characterises the Gromov hyperbolic Coxeter groups in terms of their Coxeter diagram. In fact, Moussong's result says that a finitely generated Coxeter group is Gromov hyperbolic if and only if it does not contain any subgroup isomorphic to $\mathbf{Z} \times \mathbf{Z}$. The latter property can easily be detected on the Coxeter diagram of G, since any subgroup isomorphic to $\mathbf{Z} \times \mathbf{Z}$ is conjugate into a special parabolic subgroup of W which is either of affine type or which decomposes as the direct product of two infinite special subgroups. It turns out that for some specific families of Coxeter groups, the presence of a $\mathbf{Z} \times \mathbf{Z}$-subgroup always guarantees the presence of an affine parabolic.

Lemma 4.1. *Let (W, S) be a crystallographic Coxeter system of simply laced or 3-spherical type, with S finite. Then W is Gromov hyperbolic if and only if W contains no parabolic subgroup of affine type.*

Recall that a Coxeter group W is called **crystallographic** if its natural geometric representation in $\mathbf{R}^n$ preserves a lattice (see [Bou68]). This property is known to be equivalent to each of the following conditions:

- The Coxeter numbers which appear in a Coxeter presentation of W belong to $\{2, 3, 4, 6, \infty\}$.
- W is the Weyl group of some Kac–Moody Lie algebra.

In particular, if W is the Weyl group of a Kac–Moody group over any field, then W is crystallographic.

Proof of Lemma 4.1. We may assume that W is irreducible. If W possesses a parabolic subgroup of affine type, then it contains a $\mathbf{Z} \times \mathbf{Z}$-subgroup and cannot be hyperbolic. Assume now that W is not hyperbolic. In view of Moussong's theorem, all we need to show is that if W contains two infinite special subgroups W_I, W_J which mutually commute, then it also contains a parabolic subgroup of affine type. Without loss of generality we may assume that W_I and W_J are *minimal* infinite special subgroups, namely that every proper special subgroup of W_I or W_J is finite. The list of minimal infinite Coxeter groups is known and may be found in Exercises 13–17 for § 4 in Chapter V from [Bou68]. It turns out that every such a Coxeter group is either affine or is defined by a diagram belonging to a short list, the members of which have size ≤ 5. A short glimpse at this list shows that none of them is simply laced. Furthermore, only three of them are 3-spherical crystallographic, namely those depicted in Figure 1.

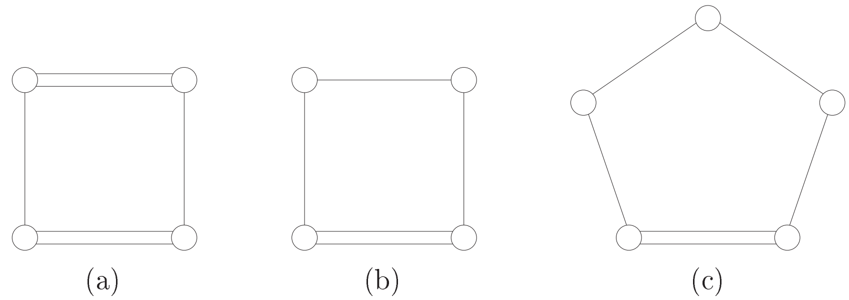

FIGURE 1. Minimal non-spherical 3-spherical Dynkin diagrams

We now consider a path of minimal possible length joining I to J in the Coxeter diagram of W and consider the Coxeter diagram induced on the union of the vertex set P of this path together with $I \cup J$.

If any vertex in $P - (I \cup J)$ is linked by an edge to a vertex of I or J belonging to an edge labelled 4, then the Coxeter diagram contains a subdiagram of type $\tilde{B}_2$ or $\tilde{C}_2$ and we are done. Similarly, if some vertex of $P - (I \cup J)$ is linked to a vertex of I or J by an edge labelled 4, then we are done as well. Thus we may assume that all labels of edges linking a vertex in $P - (I \cup J)$ to a vertex in $I \cup J$ are 3, and that all such edges are not adjacent to an edge labelled 4 in I or J.

Now it follows that if more than one vertex of I or J is linked by an edge to vertex in $P-(I \cup J)$, then the diagram contains a subdiagram of type $\tilde{A}_n$ and we are done. It remains to consider the case where each vertex of $P - (I \cup J)$ is linked to at most one vertex in I and J. In that case, it is readily seen that the diagram contains a subdiagram of type $\tilde{C}_n$. This finishes the proof. □

Another useful and well-known fact is the following.

Lemma 4.2. *Let (W, S) be an irreducible non-spherical non-affine Coxeter system such that for each proper subset $J \subset S$, the special subgroup W_J is either spherical or affine. Then $|S| \leq 10$.*

Proof. See Exercises 13–17 for § 4 in Chapter V from [Bou68]. □

4.B. **Complete Kac–Moody groups and their buildings.** Let G be a complete adjoint Kac–Moody group over a finite field $\mathbf{F}_q$. Such a group may be obtained as follows. Start with a Kac–Moody–Tits functor $\mathcal{G}$ associated to a Kac–Moody root datum of adjoint type, as defined in [Tit87] (see also [Rém02] for non-split versions). Thus $\mathcal{G}$ is a group functor on the category of commutative rings. Its value on any field k is a group $\mathcal{G}(k)$ which admits two natural uniform structures. Completing $\mathcal{G}(k)$ with respect to any of these yields a totally disconnected topological group $\mathbf{G}(k)$ which contains $\mathcal{G}(k)$ as a dense subgroup, see [CR09, § 1.2]. When $k = \mathbf{F}_q$ is a finite field of order q, then $\mathbf{G}(k)$ is locally compact.

We remark that the functor $\mathbf{G}$ may be obtained by a different construction, due to Olivier Mathieu [Mat89], which yields not only a group functor but an ind-group scheme.

We assume henceforth that $k = \mathbf{F}_q$ and set $G = \mathbf{G}(\mathbf{F}_q)$. The group G possesses a BN-pair with B compact open in G. This BN-pair yields a locally finite building X of thickness $q + 1$ on which G acts faithfully, continuously and properly by automorphisms. Furthermore X possesses a natural realisation as a CAT(0) space whose isometry group contains Aut(X) as a closed subgroup [Dav98]. By a slight abuse of notation, we shall not distinguish between X and its CAT(0) realisation. Thus we view G as a closed subgroup of $\mathrm{Is}(X)$. The G-action on X is transitive on the chambers and the chambers are compact. In particular G stabilises a minimal closed convex invariant nonempty subspace which we may view as a CAT(0) realisation of X on which G acts minimally. There is thus no loss of generality in assuming that G acts minimally on X.

The fact G has no fixed point at infinity may be established in several different ways. The conceptually easiest one is the following. It is shown in [CR09, Lemma 9] that the derived group $[G, G]$ is dense in G. It follows that if G fixed a point ξ in the boundary at infinity ∂X, then G would stabilise each horoball centered at that point, contradicting the minimality of the action. Another way to obtain this statement is by using the fact that a Coxeter group has no fixed point at infinity in its natural action on the associated CAT(0) cell complex as one sees by considering the numerous reflections (or else by applying [CM09b, Theorem 3.14]). Since for any apartment $A \subset X$ the $\mathrm{Stab}_G(A)$-action on A is isomorphic to the natural action of the Weyl group on its cell complex, and since apartments are convex, it follows again that G has no fixed point at infinity.

Finally, since X has a cocompact isometry group it has finite-dimensional Tits boundary by [Kle99, Theorem C]. This discussion shows in particular that the group G_1 appearing in the statement of Theorem 1.1 satisfies the set-up described in § 2.A by considering its natural action on the associated building.

Furthermore, it turns out that G is topologically simple [CR09, Proposition 11]. In addition, if the ground field $\mathbf{F}_q$ has order $q \geq 1764^d/25$, where d denotes the dimension of the building X, and if W is 2-spherical, then G has Kazhdan's property (T) [DJ02, Corollary G]. Notice that the dimension of X is bounded above by the maximal rank of a finite Coxeter subgroup of W, see [Dav98].

Lemma 4.3. *Let G be an irreducible complete Kac–Moody group of adjoint type over a finite field $\mathbf{F}_q$. Assume that the Weyl group W of G is infinite and simply laced or 3-spherical but not Gromov hyperbolic. Then G contains an open subgroup P which is a compact extension of a simple algebraic group over a local field of characteristic $p = \mathrm{char}\,\mathbf{F}_q$ and rank ≥ 2. Furthermore, if W is simply laced or if $\mathrm{char}\,\mathbf{F}_q \neq 2$, then the latter simple group is admissible.*

Proof. By Lemma 4.1, the group Weyl group W possesses a special parabolic subgroup of (irreducible) affine type W_J. Let $P_J < G$ be a parabolic subgroup of type W_J. Thus $P_J = B \cdot W_J \cdot B$, where B denotes the Borel subgroup of G, namely the B-subgroup of the BN-pair. In particular P_J contains the compact open subgroup B and is thus open. The subgroup

$$K_J = \bigcap_{g \in P_J} gBg^{-1}$$

is a compact normal subgroup of P_J. The quotient P_J/K_J is a complete Kac–Moody group of type W_J over $\mathbf{F}_q$ (see [CR09, Proposition 11] and [CER08, § 5]). It follows from [Tit85, Appendix] (or else from the uniqueness theorem in [Tit87]) that P_J/K_J is a simple algebraic group whose Weyl group is the spherical Weyl group of W_J. This yields the desired conclusions. □

Remark 4.4. As pointed out by G. Margulis [Mar91, IX.1.6(viii)], it follows from the arithmeticity theorem, combined with [Har75, Korollar 1 p. 133], that if W_J is not of type $\tilde{A}_n$, then P_J does *not* admit any uniform lattice. (For type $\tilde{A}_n$, such lattices indeed exist, see [BH78] and [CS98].) If follows in particular that if W_J is not of type $\tilde{A}_n$, then *no* product of the form $G \times H$, where H is a totally disconnected locally compact group, possesses *any* uniform lattice.

Notice furthermore that the condition that every special subgroup of W of affine type be of type $\tilde{A}_n$ is a very strong one. For example, if W is 3-spherical, then only finitely many Coxeter diagrams are possible for W. This shows that in general one should not expect G (or $G \times H$) to possess any uniform lattice.

5. Completion of the proofs

5.A. **Reduction of the hypotheses.** For the purposes of this last section, let us define a complete Kac–Moody group $G = \mathbf{G}(\mathbf{F}_q)$ over a finite field $\mathbf{F}_q$ with Weyl group W to be **admissible** if any of the following two conditions holds:

— W is simply laced.
— $\mathrm{char}(\mathbf{F}_q) \neq 2$ and W is 3-spherical but not Gromov hyperbolic.

Notice that the Weyl group W of $\mathbf{G}$ is Gromov hyperbolic if and only if $\mathbf{G}(F)$ is Gromov hyperbolic for each finite field F. Indeed $\mathbf{G}(F)$ acts properly and cocompactly on a building of type W, and it is known that a building is Gromov hyperbolic if and only if its Weyl group is so (see *e.g.* [Dav98]).

Although we have already used the term admissible in a different context in § 3.A, the above definition will cause no confusion. Indeed, given a Kac–Moody group $\mathbf{G}$ of affine type over $\mathbf{F}_q$ (equivalently $\mathbf{G}(\mathbf{F}_q)$ is isomorphic to a semi-simple algebraic group $\mathbf{H}$ over a field k of formal power series with coefficients in $\mathbf{F}_q$), if $\mathbf{G}(\mathbf{F}_q)$ is admissible in the above sense then $\mathbf{H}(k)$ is admissible in the sense of § 3.A.

We now proceed to relate the broad assumptions of the Introduction to the setting considered in § 2.A.

Let $n \geq 2$; for each $i \in \{1, \dots, n\}$, let X_i be a proper CAT(0) space and $G_i < \mathrm{Is}(X_i)$ be a closed subgroup acting cocompactly. We recall that cocompactness implies that G_i is compactly generated (see *e.g.* Lemma 22 in [MMS04]). Assume that G_1 is an admissible irreducible Kac–Moody group as discussed above. Set $G = G_1 \times \cdots \times G_n$ and $X = X_1 \times \cdots \times X_n$. Finally, let $\Gamma < G$ be a lattice whose projection to each G_i is faithful. We assume G_1 infinite; this hypothesis was not made in Theorem 1.1 but the latter is otherwise trivial since Γ would be finite.

Proposition 5.1. *The space X has trivial Euclidean factor and G has no fixed point at infinity.*

Moreover, for each $i \in \{1, \dots, n\}$, there is a closed normal subgroup of finite index $G_i^* \trianglelefteq G_i$, a proper CAT(0) *space $Y_i = Y_{i,1} \times \cdots \times Y_{i,k_i}$, where each $Y_{i,j}$ is irreducible $\neq \mathbf{R}$ with finite-dimensional boundary and a continuous proper map $G_i^* \to \mathrm{Is}(Y_{i,1}) \times \cdots \times \mathrm{Is}(Y_{i,k_i})$ which yields a cocompact minimal G_i^*-action on Y_i without fixed point at infinity. Finally, for all i, j the image of G_i^* in $\mathrm{Is}(Y_{i,j})$ is either totally disconnected or a connected simple Lie group.*

Proof. Since the G_i-action on X_i is cocompact, there is a non-empty closed convex G_i-invariant subset $Y_i \subseteq X_i$ on which the induced G_i-action is minimal. This action is proper and remains cocompact, which implies that the boundary ∂Y_i is finite-dimensional (Theorem C in [Kle99]). Corollary 5.3(ii) in [CM09a] now states that Y_i possesses a decomposition $Y_i = \mathbf{R}^{d_i} \times Y_{i,1} \times \cdots \times Y_{i,k_i}$, where $Y_{i,j}$ is an irreducible proper CAT(0) space, such that

$$\mathrm{Is}(Y_i) = \mathrm{Is}(\mathbf{R}^{d_i}) \times \Big(\big(\prod_{j=1}^{k_i} \mathrm{Is}(Y_{i,j}) \big) \rtimes F \Big),$$

where F is a finite permutation group of possibly isometric factors. Thus G_i possesses a closed normal subgroup of finite index $G_i^* = \prod_{j=1}^{k_i} G_{i,j}$ whose induced action on Y_i is componentwise.

We now proceed to prove that $Y := Y_1 \times \cdots \times Y_n$ has no Euclidean factor, *i.e.* $d_i = 0$ for all i. Our assumption on G_1 implies $d_1 = 0$ (see [CH09]). Considering the canonical Euclidean decomposition of Y (see [BH99, II.6.15]), we write $Y \cong Y' \times \mathbf{R}^d$, where Y' has no Euclidean factor and $d = d_1 + \cdots + d_n$. We claim that all G^*-fixed points at infinity lie in $\partial \mathbf{R}^d$, where $G^* = \prod G_i^*$. To this end, we observe that Γ provides us with a lattice in G^* upon passing to a finite index subgroup; we still denote it by Γ. If Γ is finitely generated, the claim follows from Proposition 3.15 in [CM09b]; in general, it is a consequence of the unimodularity of G^*, a fact we establish in [CM].

We can now apply Theorem 1.6 from [CM09a] and deduce that each $G_{i,j}$ is either totally disconnected or a connected simple Lie group (modulo the compact kernel of its action on $Y_{i,j}$). Proposition 3.6 in [CM09b] states that when Γ is finitely generated, it virtually splits off an Abelian direct factor of $\mathbf{Q}$-rank d. The finite generation, however, is only used to provide a complementary factor to this Abelian subgroup; the existence of a normal Abelian subgroup $A \lhd \Gamma$ of $\mathbf{Q}$-rank d is established in general in *loc. cit.* We finally deduce that $d = 0$ from the fact that Γ projects injectively to the Kac–Moody group G_1 using [CH09].

At this point we have established that Y has trivial euclidean factor and that G^* has no fixed points at infinity. In particular G has no fixed points in $\partial X = \partial Y$ and X has no Euclidean factor either since Y has finite codiameter in X. □

5.B. **End of the proofs.**

Proof of Theorem 1.4. Retain the notation of the theorem. Then Lemma 4.3 ensures that G_1 possesses an open subgroup P which is a compact extension of an admissible simple algebraic group of rank ≥ 2 over a local field. We can assume Γ residually finite. The statement of Theorem 1.4 is not affected by the reductions of Proposition 5.1; therefore, Theorem 3.6 yields the desired conclusion. □

Proof of Theorem 1.1. We adopt the notation of the theorem. Countrary to Theorem 1.4, the irreducibility assumption is in the present case sensitive to replacing G_i with the subfactors $G_{i,j}$ of Proposition 5.1. However, since G_1 is irreducible, Proposition 2.2 implies that Γ is at least algebraically irreducible.

As above, Lemma 4.3 provides an open subgroup $P < G_1$ which is a compact extension of an admissible simple algebraic group $\mathbf{H}(k)$ of rank ≥ 2 over a local field k; we emphasise that k has positive characteristic.

The canonical image of $\Gamma_P = \Gamma \cap (P \times G_2 \times \cdots \times G_n)$ in $\mathbf{H}(k) \times G_2 \times \cdots \times G_n$ is a lattice and it projects injectively to $\mathbf{H}(k)$ in view of the corresponding assumption on Γ. Thus Proposition 3.5 implies that all G_i are totally disconnected. In particular G_n possesses a compact open subgroup U (see [Bou71, III.4 No 6]). Set

$$\Gamma_U = \Gamma \cap (G_1 \times \cdots \times G_{n-1} \times U).$$

By assumption, the projection of Γ_U to U is faithful. In particular Γ_U is residually finite since U is so, being a profinite group. Applying the faithfulness assumption to any other factor G_i, we further deduce that Γ_U intersects the compact group $1 \times \cdots \times 1 \times U$ trivially; therefore, we can view Γ_U as a lattice in the product

$$G_1 \times \prod_{j=1}^{k_2} G_{2,j} \times \cdots \times \prod_{j=1}^{k_{n-1}} G_{n-1,j}.$$

By Proposition 2.6, the group Γ_U is algebraically irreducible. Thus we can apply Theorem 3.6 and deduce that G_1 and each factor $G_{i,j}$ contains a closed subgroup of finite covolume which is a simple algebraic group over a local field.

We now return to the initial lattice Γ in G, which is algebraically irreducible by Proposition 2.2, and conclude the proof as in the end of the proof of Theorem 3.6. □

Proof of Corollary 1.5. Since Γ is irreducible and each G_i is topologically simple (as recalled in Section 4.B), it follows that the projection of Γ to each G_i is faithful. In view of Theorem 1.1, we may assume that $n = 2$ and that Γ is not residually finite. All we need to show is that Γ is virtually simple.

Since G_1 and G_2 are topologically simple and Γ is irreducible, it follows from [BS06, Theorem 1.1] that if Γ is uniform, then every non-trivial normal subgroup of Γ has finite index. If Γ is not uniform, then it has property (T) in view of our assumptions and, hence, the same conclusion on normal subgroups holds in view of [BS06, Theorem 1.3].

Therefore Proposition 1 from [Wil71] ensures that Γ is virtually isomorphic to a direct product of finitely many isomorphic simple groups. Since Γ is irreducible as an abstract group by Proposition 2.2, we deduce that the latter direct product has a single simple factor. Thus Γ is virtually simple. □

Remark 5.2. As pointed out by the anonymous referee, the above arguments show also the following. Let Γ be a finitely generated group without non-trivial infinite index normal subgroup. Suppose that Γ acts by isometries faithfully, minimally and without fixed point at infinity on an irreducible proper CAT(0) space X. Then Γ is either residually finite or virtually simple.

Indeed, in view of the above quoted result of Wilson, it suffices to show that Γ does not have a finite index subgroup $\Gamma^* \cong \Gamma_1 \times \Gamma_2$ splitting as a product of two infinite groups Γ_i. Since X is irreducible, we can assume that it has no Euclidean factor for otherwise $X = \mathbf{R}$ in which case the statement is obvious. Therefore our "Borel density" in the generality of Proposition 2.1 (presently invoked with $n = 1$) implies that Γ^* still acts minimally and without fixed point at infinity. By the splitting theorem of [Mon06], this forces at least one of the Γ_i to act trivially, a contradiction.

New examples of groups to which the above applies are provided in unpublished work of Shalom–Steger.

Proof of Corollary 1.6. In view of Corollary 1.5, the assumption that the G_i's are of non-affine type implies that $n = 2$ in the above, and that any irreducible lattice Γ of G is virtually simple. The finite residual $\Gamma^{(\infty)}$ of Γ is thus a normal subgroup of finite index, and any subgroup of G commensurating Γ normalizes $\Gamma^{(\infty)}$. Thus $\mathrm{Comm}_G(\Gamma) = \mathscr{N}_G(\Gamma^{(\infty)})$.

Under the present hypotheses, the group G, and hence also Γ has Kazhdan's property (T). Thus Γ is finitely generated and, hence so is the lattice $\Gamma^{(\infty)}$. In view of [CM09b, Corollary 2.7], it follows that $\mathscr{N}_G(\Gamma^{(\infty)})$ is itself a lattice in G, which is thus the desired maximal lattice. □

5.C. **A lattice in a product of a simple algebraic group and a Kac–Moody group.** Let G be an irreducible complete Kac–Moody group of simply laced type over a finite field $\mathbf{F}_q$. It is shown in [Rém99] (see also [CG99] and [CR09]) that the group $G \times G$ contains an irreducible non-uniform lattice Γ, provided that q is larger than the rank r of the Weyl group of G. Assume now that G is not of affine type. By Lemma 4.3 G contains an open subgroup $P < G$ which possesses a compact normal subgroup K such that P/K is a simple algebraic group over a local field. As in the proof of Theorem 3.6, we may consider the group

$$\Gamma_P = \Gamma \cap (P \times G)$$

and view it as a lattice in $P/K \times G$. Furthermore Γ_P is irreducible (see Proposition 2.6) and one shows, as in the proof of Theorem 3.6, that Γ_P is finitely generated provided the ground field is large enough.

Since P/K acts cocompactly on the associated Bruhat–Tits building, we see in particular that Γ_P is an example of a CAT(0) lattice (in the sense of [CM08, CM09b]) in a product of an affine and a non-affine building.

Proposition 2.4 together with Theorem 3.6 imply that Γ_P is *not* residually finite. In particular its projection to the linear group P/K is *not* faithful since a finitely generated linear group is residually finite by [Mal40]. This example shows that the assumption on the faithfulness of the projections in Theorem 1.1 may not be removed.

References

[BEW08] Yiftach Barnea, Mikhail Ershov, and Thomas Weigel, *Abstract commensurators of profinite groups*, Preprint, 2008; to appear in Trans. Amer. Math. Soc.

[BH78] Armand Borel and Günter Harder, *Existence of discrete cocompact subgroups of reductive groups over local fields*, J. Reine Angew. Math. **298** (1978), 53–64.

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften 319, Springer, Berlin, 1999.

[BM00] Marc Burger and Shahar Mozes, *Lattices in product of trees*, Inst. Hautes Études Sci. Publ. Math. (2000), no. 92, 151–194 (2001).

[Bor60] Armand Borel, *Density properties for certain subgroups of semi-simple groups without compact components*, Ann. of Math. (2) **72** (1960), 179–188.

[Bor66] ———, *Density and maximality of arithmetic subgroups*, J. Reine Angew. Math. **224** (1966), 78–89.

[Bou68] Nicolas Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.

[Bou71] ———, *Éléments de mathématique. Topologie générale. Chapitres 1 à 4*, Hermann, Paris, 1971.

[BS06] Uri Bader and Yehuda Shalom, *Factor and normal subgroup theorems for lattices in products of groups*, Invent. Math. **163** (2006), no. 2, 415–454.

[BT73] Armand Borel and Jacques Tits, *Homomorphismes "abstraits" de groupes algébriques simples*, Ann. of Math. (2) **97** (1973), 499–571.

[CER08] Lisa Carbone, Mikhail Ershov, and Gordon Ritter, *Abstract simplicity of complete Kac–Moody groups over finite fields*, J. Pure Appl. Algebra **212** (2008), no. 10, 2147–2162.

[CG99] Lisa Carbone and Howard Garland, *Lattices in Kac–Moody groups*, Math. Res. Letters **6** (1999), 439–448.

[CH09] Pierre-Emmanuel Caprace and Frédéric Haglund, *On geometric flats in the CAT(0) realization of Coxeter groups and Tits buildings*, Canad. J. Math. **61** (2009), 740–761.

[CM08] Pierre-Emmanuel Caprace and Nicolas Monod, *Some properties of non-positively curved lattices*, C. R. Math. Acad. Sci. Paris **346** (2008), no. 15-16, 857–862.

[CM09a] Pierre-Emmanuel Caprace and Nicolas Monod, *Isometry groups of non-positively curved spaces: structure theory*, Journal of Topology **2** (2009), no. 4, 661–700.

[CM09b] Pierre-Emmanuel Caprace and Nicolas Monod, *Isometry groups of non-positively curved spaces: discrete subgroups*, Journal of Topology **2** (2009), no. 4, 701–746.

[CM] ———, *Fixed points and amenability in non-positive curvature*, In preparation.

[CR09] Pierre-Emmanuel Caprace and Bertrand Rémy, *Simplicity and superrigidity of twin building lattices*, Invent. Math. **176** (2009), 169–221.

[CS98] Donald I. Cartwright and Tim Steger, *A family of $\tilde{A}_n$-groups*, Israel J. Math. **103** (1998), 125–140.

[Dav98] Michael Davis, *Buildings are* CAT(0), Geometry and cohomology in group theory, London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, 1998, pp. 108–123.

[DJ02] Jan Dymara and Tadeusz Januszkiewicz, *Cohomology of buildings and of their automorphism groups.*, Invent. Math. **150** (2002), no. 3, 579–627.

[Har75] Günter Harder, *Über die Galoiskohomologie halbeinfacher algebraischer Gruppen. III*, J. Reine Angew. Math. **274/275** (1975), 125–138, Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, III.

[Kle99] Bruce Kleiner, *The local structure of length spaces with curvature bounded above*, Math. Z. **231** (1999), no. 3, 409–456.

[Mal40] Anatoly I. Mal'cev, *On isomorphic matrix representations of infinite groups*, Rec. Math. [Mat. Sbornik] N.S. **8 (50)** (1940), 405–422.

[Mar91] Gregory A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 17, Springer-Verlag, Berlin, 1991.

[Mat89] Olivier Mathieu, *Construction d'un groupe de Kac–Moody et applications. (Construction of a Kac-Moody group and applications).*, Compos. Math. **69** (1989), no. 1, 37–60 (French).

[MMS04] Igor Mineyev, Nicolas Monod, and Yehuda Shalom, *Ideal bicombings for hyperbolic groups and applications*, Topology **43** (2004), no. 6, 1319–1344.

[Mon06] Nicolas Monod, *Superrigidity for irreducible lattices and geometric splitting*, J. Amer. Math. Soc. **19** (2006), no. 4, 781–814.

[Pra77] Gopal Prasad, *Strong approximation for semi-simple groups over function fields*, Ann. of Math. (2) **105** (1977), no. 3, 553–572.

[Pra82] ———, *Elementary proof of a theorem of Bruhat-Tits-Rousseau and of a theorem of Tits*, Bull. Soc. Math. France **110** (1982), no. 2, 197–202.

[Rag72] Madabusi Santanam Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68.

[Rém99] Bertrand Rémy, *Construction de réseaux en théorie de Kac–Moody*, C. R. Acad. Sc. Paris **329** (1999), 475–478.

[Rém02] ———, *Groupes de Kac–Moody déployés et presque déployés (Split and almost split Kac–Moody groups)*, Astérisque. 277. Paris: Astérisque. vii, 348 p. EUR 53.00; $ 76.00 , 2002 (French).

[Tit64] Jacques Tits, *Algebraic and abstract simple groups*, Ann. of Math. (2) **80** (1964), 313–329.

[Tit74] ———, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin, 1974.

[Tit85] ———, *Groups and group functors attached to Kac-Moody data.*, Arbeitstag. Bonn 1984, Proc. Meet. Max-Planck-Inst. Math., Bonn 1984, Lect. Notes Math. 1111, 193-223 (1985)., 1985.

[Tit87] ———, *Uniqueness and presentation of Kac-Moody groups over fields*, J. Algebra **105** (1987), no. 2, 542–573.

[Wil71] John S. Wilson, *Groups with every proper quotient finite.*, Proc. Camb. Philos. Soc. **69** (1971), 373–391.

[Wor07] Kevin Wortman, *Quasi-isometric rigidity of higher rank S-arithmetic lattices.*, Geom. Topol. **11** (2007), 995–1048.

UCLouvain, 1348 Louvain-la-Neuve, Belgium
E-mail address: `pierre-emmanuel.caprace@uclouvain.be`

EPFL, 1015 Lausanne, Switzerland
E-mail address: `nicolas.monod@epfl.ch`