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Group Theory

Negative curvature from a cohomological viewpoint and cocycle superrigidity

Nicolas Monod [a], Yehuda Shalom [b]

[a] *Department of Mathematics, University of Chicago, 5734 S. University Avenue, Chicago, IL 60637, USA*
[b] *School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel*

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Abstract

In the framework of general negatively curved spaces, we present new superrigidity results and introduce new techniques based on bounded cohomology. This applies to irreducible lattices, and more generally to cocycles, of products of arbitrary locally compact groups. Together with a new vanishing result for higher rank groups, this also generalizes and unifies all previously known results in that direction. The non-vanishing results provide a large class of examples for our results on orbit equivalence rigidity (Monod and Shalom, Ann. of Math., in press). We prove the 'toy-case' of actions on trees. ***To cite this article: N. Monod, Y. Shalom, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

La courbure strictement négative d'un point de vue cohomologique et superrigidité des cocycles. Nous proposons de nouvelles méthodes cohomologiques pour établir des énoncés de superrigidité dans le cadre général des espaces métriques à courbure strictement négative. Nos résultats s'appliquent aux réseaux irréductibles, ou plus généralement aux cocycles, pour des produits de groupes localement compacts généraux. Avec le concours d'un nouveau théorème d'annulation, on subsume et généralise de la sorte tous les résultats qui allaient dans ce sens ; en outre, les énoncés de non annulation fournissent une vaste classe d'exemples pour nos résultats en équivalence orbitale (Monod et Shalom, Ann. of Math., in press). Nous donnons une preuve dans le cas particulièrement simple des arbres. ***Pour citer cet article : N. Monod, Y. Shalom, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

1. Introduction

The following result associates a cohomological invariant to groups Γ acting on any metric space X that is *negatively curved* in a very general sense. Recall that $\Gamma < \mathrm{Isom}(X)$ is called *elementary* if it has bounded orbits in X or if it preserves a point or a pair in the boundary at infinity ∂X. When X has bounded geometry, this is equivalent to the amenability of the closure of Γ in H.

E-mail addresses: monod@math.uchicago.edu (N. Monod), yeshalom@post.tau.ac.il (Y. Shalom).

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Theorem 1.1. *Let $H = \mathrm{Isom}(X)$, where X is one of the following*: (i) *Any proper* $\mathrm{CAT}(-1)$ *space*; (ii) *Any Gromov-hyperbolic graph of bounded valency*; (iii) *Any Gromov-hyperbolic proper geodesic metric space on which H acts cocompactly.*

Then for any non-elementary countable subgroup (*not necessarily closed*) *$\Gamma < H$, the space $\mathrm{H}^2_{\mathrm{b}}(\Gamma, L^2(H))$ is non-zero.*

(The complete proof of (i) will be found in [5]; the main additional ingredient needed for (ii) and (iii) is a joint work with Mineyev [3].) We will show how this non-vanishing of $\mathrm{H}^2_{\mathrm{b}}$ (defined below) can be used for rigidity results. On a different level, we propose the following class of groups $\mathcal{C}_{\mathrm{reg}}$ as a cohomological notion of negative curvature:

$$\mathcal{C}_{\mathrm{reg}} := \{\Gamma\colon\ \mathrm{H}^2_{\mathrm{b}}(\Gamma, \ell^2(\Gamma)) \neq 0\}.$$

Our work [6] on orbit/measure equivalence rigidity applies to this class. Theorem 1.1 implies:

Corollary 1.2. *Let H be as above and $\Gamma < H$ be any discrete non-elementary subgroup. Then $\Gamma \in \mathcal{C}_{\mathrm{reg}}$.*

Locally finite trees yield a particular case of the above discussion. However, in view of Bass–Serre theory, it is important to deal with general trees:

Theorem 1.3. *Let Γ be a countable group with a non-elementary action on a simplicial tree. Denote by E the set of edges, endowed with the corresponding Γ-action. Then $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \ell^2(E \times E))$ is non-zero.*

In particular, if Γ is any free product $\Gamma = A * B$ (with $A \neq 1$ and $|B| > 2$) then $\Gamma \in \mathcal{C}_{\mathrm{reg}}$.

Corollary 1.4. *There are $2^{\aleph_0}$ non-isomorphic countable groups in $\mathcal{C}_{\mathrm{reg}}$ and any countable group embeds into a group in $\mathcal{C}_{\mathrm{reg}}$.*

The cohomological invariant that we construct combines well with the product formula for bounded cohomology of [1,4]; using the general functorial machinery established in there (and its connection to Poisson boundary theory), we prove the cocycle superrigidity theorem below. In order to formulate it, we observe that a natural generalization of *elementarity* from actions to cocycles $\alpha : G \times \Omega \to H$ is the existence of a measurable α-invariant map from Ω to bounded subsets in X or to points or pairs in ∂X. Further, when G is a product $\prod_j G_j$, we call its action on Ω *irreducible* if each subproduct $G'_i := \prod_{j \neq i} G_j$ acts ergodically. For instance, if $\Gamma < G$ is a lattice, then the G-action on $\Omega = G/\Gamma$ is irreducible if and only if the lattice is irreducible in the sense that it projects densely in each G_i.

Theorem 1.5. *Let $G = G_1 \times \cdots \times G_n$ be any locally compact σ-compact group with an irreducible measure preserving action on a standard probability space Ω. Let X be a space as in Theorem* 1.1 *and $\alpha : G \times \Omega \to H$ be a non-elementary measurable cocycle, where $H < \mathrm{Isom}(X)$ is any closed subgroup.*

Then there is a closed subgroup $H' < H$ and a normal compact subgroup $K \lhd H'$ such that α is cohomologous to a cocycle $\alpha' : G \times \Omega \to H'$ whose composition with the natural map $H' \to H'/K$ yields a continuous homomorphism $G \to H'/K$ which factors through some G_j.

Observe that if X is, say, $\mathrm{CAT}(-1)$, then the existence of H' and K simply amounts to saying that α is equivalent to a continuous homomorphism of G upon possibly restricting to an invariant convex subspace of X.

As usual, cocycle superrigidity implies superrigidity for homomorphisms. There again, the case of arbitrary trees can also be addressed. In this Note, we shall indicate the proof of the following result, which can serve as a “toy case” illustrating the general line of [5] without raising any of the technical (and geometric) issues occurring with general spaces [3,5].

Theorem 1.6. *Let $G_1, \ldots, G_n$ be locally compact σ-compact groups and let $\Gamma < G = G_1 \times \cdots \times G_n$ be an irreducible lattice acting non-elementarily on a simplicial tree. Then there is a Γ-invariant subtree on which the Γ-action extends continuously to a G-action, factoring through some G_j.*

In case Γ is a cocompact lattice in a product of compactly generated groups (or when all G_j are linear and simple) this result was proved by Shalom in [7].

We turn now to a vanishing theorem generalizing [1, Theorem 21]:

Theorem 1.7. *Let $\Gamma < G = \mathbf{G}(k)$ be a lattice, where k is a local field and $\mathbf{G}$ is a connected almost k-simple algebraic group defined over k with $\mathrm{rank}_k \mathbf{G} \geqslant 2$. Let V be a separable dual isometric Banach Γ-module. Then $\mathrm{H}^2_{\mathrm{b}}(\Gamma, V) \cong E^{\Gamma}$ if $k = \mathbf{R}$ and $\pi_1(G)$ is infinite, and $\mathrm{H}^2_{\mathrm{b}}(\Gamma, V) = 0$ in all other cases.*

In particular, $\mathrm{H}^2_{\mathrm{b}}(\Gamma, -)$ vanishes for every unitary Γ-representation without non-zero invariant vectors.

The latter pins down an important property of higher rank lattices in view of the following consequence of the non-vanishing results:

Corollary 1.8. *Let Γ be any countable group such that $\mathrm{H}^2_{\mathrm{b}}(\Gamma, V) = 0$ for every unitary Γ-representation V with $V^{\Gamma} = 0$. Let H be as in Theorem* 1.1. *Then every homomorphism $\Gamma \to H$ is elementary.*

Finally, we indicate some of the needed cohomological tools (for all the following, see [1,4]). Let G be a locally compact group and V a separable dual isometric continuous Banach G-module. The (continuous) bounded cohomology $\mathrm{H}^{\bullet}_{\mathrm{cb}}(G, V)$ is defined by the complex $C_{\mathrm{b}}(G^{\bullet+1}, V)^G$ of equivariant continuous bounded functions with the usual homogeneous coboundary: $d = \sum (-1)^j d_j$ where d_j omits the jth variable. Let B be a standard measure space with a G-action preserving the measure class; if the action in amenable in the sense of Zimmer [8], then $\mathrm{H}^{\bullet}_{\mathrm{cb}}(G, V)$ is also realized by the complex $L^{\infty}_{\mathrm{alt}}(B^{\bullet+1}, V)^G$ of alternating L^{∞} maps. Call B a *strong boundary* (for G) if the G-action is amenable and in addition every G-equivariant measurable map $B \times B \to V$ is constant (for all V as above). It follows then that $\mathrm{H}^2_{\mathrm{cb}}(G, V)$ is the space of cocycles in $L^{\infty}_{\mathrm{alt}}(B^3, V)^G$. It is shown in [1,4] that every compactly generated G virtually admits a strong boundary; Kaimanovich later generalized this to all σ-compact locally compact groups [2].

Theorem 1.9 ([1,4]; see also [2]). *Let $G = G_1 \times \cdots \times G_n$ be any locally compact σ-compact group and V be as above. Then there is a canonical isomorphism $\mathrm{H}^2_{\mathrm{cb}}(G, V) \cong \bigoplus \mathrm{H}_{\mathrm{cb}}(G_i, V^{G'_i})$.*

We shall give below a short proof – assuming the above functorial machinery (in [1], [4], Theorem 1.9 is deduced from a general Hochschild–Serre spectral sequence). Let now $\Gamma < G$ be an irreducible lattice and W be a separable dual isometric Banach Γ-module. Then one can define $W_i \subseteq W$ to be the largest (possibly trivial) Γ-submodule on which the Γ-action extends to a continuous G-action factoring through G_i. Using cohomological induction and strong boundaries, in [1,4] the following superrigidity formula for bounded cohomology is deduced from Theorem 1.9: $\mathrm{H}^2_{\mathrm{b}}(\Gamma, W) = \bigoplus \mathrm{H}^2_{\mathrm{cb}}(G_i, W_i)$.

2. Selected proofs

Let T be a simplicial tree; for the applications to Theorems 1.3 and 1.6, there is no loss of generality in assuming that T is countable (as a graph). Let $\overline{T} = T \sqcup \partial T$ be the usual ray bordification. Fix a positive integer n. We define a map $\alpha : \overline{T} \times \overline{T} \to \ell^{\infty}(E^n)$ as follows. Let $\zeta, \zeta' \in \overline{T}$. If the edges $e_1, \ldots, e_n$ constitute a geodesic path contained in the geodesic $[\zeta, \zeta']$, then we let $\alpha(\zeta, \zeta')(e_1, \ldots e_n)$ be ± 1 according to whether the path has the orientation induced by $[\zeta, \zeta']$ or the opposite orientation. In all other cases (in particular, if the e_i's do not constitute a geodesic path), we set $\alpha(\zeta, \zeta')(e_1, \ldots e_n) = 0$.

Define now $\omega : \overline{T}^3 \to \ell^{\infty}(E^n)$ by $\omega = d\alpha$. Observe that for $n = 1$ we have $\omega = 0$ and α is a well known 1-cocycle. But for general n, the situation is the following: whenever ζ, ζ', ζ'' are three distinct points of $\overline{T}$ such that each leg of the resulting tripod has length at least n, there are exactly $6(n-1)$ different n-tuples of edges in the support of $\omega(\zeta, \zeta', \zeta'')$; this can be immediately checked by observing that a path gets canceled if and only if it does not cross the centre of the tripod. It follows that ω yields by restriction a bounded measurable cocycle $(\partial T)^3 \to \ell^2(E^n)$ that is equivariant under the automorphisms of T and that does not vanish on any triple of distinct points in ∂T (when $n \geqslant 2$, which we assume from now on).

Suppose now that Γ is a countable group with a non-elementary action on T. Let B be a strong boundary for Γ. If T were locally finite, a standard use of boundary theory would yield a measurable equivariant map $f : B \to \partial T$. In the general case, a difficulty arises because ∂T need not be compact. However, introducing a *weak* topology that does make $\overline{T}$ (though not ∂T) compact, we still obtain f as above (a basis for this topology is given by the closure in $\overline{T}$ of half-trees). The non-elementarity assumption implies that f cannot range essentially in triples of non-distinct points, so that $\omega \circ f^3$ is a non-zero cocycle $B^3 \to \ell^2(E^n)$. We conclude that $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \ell^2(E^n))$ is non-zero, proving the main claim of Theorem 1.3 when $n = 2$. In the particular case of a free product, or more generally when the action on E^n is proper, $\ell^2(E^n)$ is a (possibly infinite) multiple of a subrepresentation of $\ell^2(\Gamma)$. Using the realization of $\mathrm{H}^2_{\mathrm{b}}(\Gamma, -)$ by cocycles on a strong boundary, one shows that this implies $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \ell^2(\Gamma)) \neq 0$.

Assume now in addition that we are in the situation of Theorem 1.6. Then the superrigidity formula for bounded cohomology shows that for some i there is a non-zero Γ-invariant subspace $W_i \subseteq \ell^2(E \times E)$ such that the Γ-action on W_i extends continuously to a G-action factoring through G_i. Upon possibly passing to a Γ-invariant subtree of T, it follows (see [7], p. 45) that the Γ-action on T extends similarly, as claimed.

We now indicate a short proof of Theorem 1.9 (provided the functorial machinery). We may for simplicity assume that $G = G_1 \times G_2$. We denote by inf_i and res_i the maps induced in cohomology by the natural maps $G \to G_i$ and $G_i \to G$. Let us show that

$$\mathrm{inf}_1 + \mathrm{inf}_2 : \mathrm{H}^2_{\mathrm{cb}}\big(G_1, V^{G_2}\big) \oplus \mathrm{H}^2_{\mathrm{cb}}\big(G_2, V^{G_1}\big) \longrightarrow \mathrm{H}^2_{\mathrm{cb}}(G, V) \tag{$*$}$$

is an isomorphism (with inverse $\mathrm{res}_1 \oplus \mathrm{res}_2$). As in usual cohomology, inner automorphisms act trivially in bounded cohomology; it follows that the restriction $\mathrm{H}^\bullet_{\mathrm{cb}}(G, V) \to \mathrm{H}^\bullet_{\mathrm{cb}}(G_1, V)$ ranges in $\mathrm{H}^\bullet_{\mathrm{cb}}(G_1, V)^{G_2}$. Realizing the latter as a cocycle space on a strong boundary for G_1, and since the conjugation action of G_2 on G_1 is trivial, we see that $\mathrm{H}^2_{\mathrm{cb}}(G_1, V)^{G_2} = \mathrm{H}^2_{\mathrm{cb}}(G_1, V^{G_2})$ since there are no non-zero coboundaries. Given a class a in $\mathrm{H}^2_{\mathrm{cb}}(G, V)$, consider $b = a - \mathrm{inf}_1\mathrm{res}_1(a) - \mathrm{inf}_2\mathrm{res}_2(a)$. By functoriality, $\mathrm{res}_i(b) = 0$ for both i. Let B_i be strong boundaries for G_i; then $B = B_1 \times B_2$ is a strong boundary for G. Thus the class b can be realized by an element β of $L^\infty(B^3, V)^G$ and res_1 is induced by the inclusion map $L^\infty(B^\bullet, V)^G \to L^\infty(B^\bullet, V)^{G_i}$ since B is also G_i-amenable. Thus there is α_1 in $L^\infty(B^2, V)^{G_1}$ with $d\alpha_1 = \beta$. By Fubini, α_1 yields a map $B_2^2 \to L^\infty(B_1^2, V)^{G_1}$ so that α_1, and hence also β, cannot depend on the B_1 variables; by symmetry β is constant and hence trivial in cohomology. This shows that $(*)$ is surjective. Injectivity is apparent if one implements $(*)$ by the map

$$L^\infty_{\mathrm{alt}}\big(B_1^3, V^{G_2}\big)^{G_1} \oplus L^\infty_{\mathrm{alt}}\big(B_2^3, V^{G_1}\big)^{G_2} \longrightarrow L^\infty_{\mathrm{alt}}\big(B^3, V\big)^G$$

induced by the factor maps $B \to B_i$.

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