CANTOR SYSTEMS, PIECEWISE TRANSLATIONS AND SIMPLE AMENABLE GROUPS

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Abstract. We provide the first examples of finitely generated simple groups that are amenable (and infinite). To this end, we prove that topological full groups of minimal systems are amenable. This follows from a general existence result on invariant states for piecewise-translations of the integers. The states are obtained by constructing a suitable family of densities on the classical Bernoulli space.

1. Introduction

A *Cantor system* (T, C) is a homeomorphism T of the Cantor space C; it is called *minimal* if T admits no proper invariant closed subset. The *topological full group* $[[T]]$ of a Cantor system is the group of all homeomorphisms of C which are given piecewise by powers of T, each piece being open in C. This countable group is a complete invariant of flip-conjugacy for (T, C) by a result of Giordano–Putnam–Skau [GPS99, Cor. 4.4].

It turns out that this construction yields very interesting groups $[[T]]$. Indeed, Matui proved that the commutator subgroup of $[[T]]$ is simple for any minimal Cantor system, see Theorem 4.9 in [Mat06] and the remark preceding it (or [BM08, Thm. 3.4]). Moreover, he showed that this simple (infinite) group is finitely generated if and only if (T, C) is (conjugated to) a minimal subshift. This yielded a new uncountable family of non-isomorphic finitely generated simple groups since subshifts can be distinguished by their entropy; see [Mat06, p. 246] or Theorem 5.13 in [BM08].

Until now, no example of finitely generated simple group that is *amenable* (and infinite) was known. Grigorchuk–Medynets [GM] have proved that the topological full group $[[T]]$ of a minimal Cantor system (T, C) is locally approximable by finite groups in the Chabauty topology. They conjectured that $[[T]]$ is amenable; our first result confirms this conjecture.

Theorem A. *The topological full group of any minimal Cantor system is amenable.*

Surprisingly, this statement fails as soon as one allows two commuting homeomorphisms. Indeed, it is shown in [EM] that the topological full group of a minimal Cantor $\mathbf{Z}^2$-system can contain non-abelian free subgroups.

Combining Theorem A with the above-mentioned results from [GPS99, Mat06] we deduce:

Corollary B. *There exist finitely generated simple groups that are infinite amenable. In fact, there are $2^{\aleph_0}$ non-isomorphic such groups.* □

The next problem would be to find *finitely presented* examples (the groups considered above are never finitely presented [Mat06, Thm. 5.7]).

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In order to prove Theorem A, we reformulate the problem in terms of the group $W(\mathbf{Z})$ of *piecewise-translations* of the integers. More precisely, we denote by $W(\mathbf{Z})$ the group of all those permutations g of $\mathbf{Z}$ for which the quantity

$$|g|_{\mathrm{w}} := \sup \big\{ |g(j) - j| : j \in \mathbf{Z} \big\}$$

is finite. The topological full group of any minimal Cantor system (T, C) can be embedded into $W(\mathbf{Z})$ by identifying a T-orbit with $\mathbf{Z}$. However, $W(\mathbf{Z})$ also contains many other groups, including non-abelian free groups, see [vD90].

We shall introduce a model for random finite subsets of $\mathbf{Z}$ which has the following two properties: (i) the model is almost-invariant under shifts by piecewise-translations; (ii) a random finite set contains 0 with overwhelming probability. More precisely, Theorem A is proved using a general result about $W(\mathbf{Z})$ which has the following equivalent reformulation.

Theorem C. *The $W(\mathbf{Z})$-action on the collection of finite sets of integers admits an invariant mean which gives full weight to the collection of sets containing* 0.

Notice that for any given finite set $E \subseteq \mathbf{Z}$, a mean as in Theorem C will give full weight to the collection of sets containing E. In a subsequent paper [JS], Theorem C will be extended to a wider setting.

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2. Semi-densities on the Bernoulli shift

The technical core of our construction is a family of L^2-functions f_n on the classical Bernoulli space $\{0,1\}^{\mathbf{Z}}$. The relevance of these functions will be explained in Section 3.

For any $n \in \mathbf{N}$, we define

$$f_n \colon \{0,1\}^{\mathbf{Z}} \longrightarrow (0,1], \qquad f_n(x) = \exp\Bigg(-n \sum_{j\in\mathbf{Z}} x_j e^{-|j|/n} \Bigg),$$

where $x = \{x_j\}_{j\in\mathbf{Z}} \in \{0,1\}^{\mathbf{Z}}$. We consider f_n as an element of the Hilbert space $L^2(\{0,1\}^{\mathbf{Z}})$, where $\{0,1\}^{\mathbf{Z}}$ is endowed with the symmetric Bernoulli measure. The interest of the family f_n is that it satisfies the following two properties, each of which would be elementary to obtain separately.

Theorem 2.1. *For any $g \in W(\mathbf{Z})$ we have $\langle g(f_n), f_n\rangle / \|f_n\|^2 \to 1$ as $n \to \infty$. Moreover, $\|f_n|_{x_0=0}\| / \|f_n\| \to 1$.*

The notation $f_n|_{x_0=0}$ represents the function f_n multiplied by the characteristic function of the cylinder set describing the elementary event $x_0 = 0$.

In preparation for the proof, we write

$$a_{n,j} = \exp(-ne^{-|j|/n}) \quad \text{for} \quad j \in \mathbf{Z}.$$

We shall often use implicitly the estimates

$$0 < a_{n,j} \le 1 \quad \text{and} \quad 0 < \frac{a_{n,j}^2}{1+a_{n,j}^2} \le a_{n,j}^2 \le a_{n,j}.$$

Since f_n is a product of the independent random variables $\exp\big(-nx_j e^{-|j|/n}\big)$, we have

$$\|f_n\|^2 = \prod_{j\in\mathbf{Z}} \left(\frac{1}{2} + \frac{1}{2}a_{n,j}^2\right).$$

Notice that $\exp\big(-nx_j e^{-|j|/n}\big)$ ranges in $(0,1]$ and that the above product converges unconditionally in the sense that the series of $\log\big(\frac{1}{2}+\frac{1}{2}a_{n,j}^2\big)$ converges absolutely (by a straightforward estimate). We can regroup factors and compute the ratio

$$\frac{\|f_n|_{x_0=0}\|^2}{\|f_n\|^2} = \frac{1}{1+a_{n,0}^2}$$

which thus converges to 1 as desired for the second statement of Theorem 2.1.

The proof of the first statement will be divided into two propositions. Define the function $F_n\colon W(\mathbf{Z}) \to \mathbf{R}$ by the absolutely convergent series:

$$F_n(g) = \sum_{j\in\mathbf{Z}} \frac{a_{n,j}^2}{1+a_{n,j}^2}\ e^{-|j|/n}\ \Big(|g(j)| - |j|\Big).$$

We begin with a conditional convergence:

Proposition 2.2. *For any $g \in W(\mathbf{Z})$ we have $\langle g(f_n), f_n\rangle / \|f_n\|^2 \to 1$ as $n\to\infty$ provided $F_n(g) \to 0$.*

The condition $F_n(g) \to 0$ is about a *signed* series for which the series of sums of absolute values does not converge to zero; it will be addressed by the following statement:

Proposition 2.3. *We have $\lim_{n\to\infty} F_n(g) = 0$ for every $g \in W(\mathbf{Z})$.*

We now undertake the proof of Proposition 2.2. Using again the product form of f_n, one obtains

$$\frac{\langle g^{-1}(f_n), f_n\rangle}{\|f_n\|^2} = \frac{\langle g(f_n), f_n\rangle}{\|f_n\|^2} = \prod_{j\in\mathbf{Z}} \frac{1+a_{n,j}a_{n,g(j)}}{1+a_{n,j}^2}.$$

Thus $\langle g(f_n), f_n\rangle / \|f_n\|^2 \to 1$ if and only if

$$\lim_{n\to\infty} \sum_{j\in\mathbf{Z}} \log \frac{1+a_{n,j}a_{n,g(j)}}{1+a_{n,j}^2} = 0. \tag{2.i}$$

Next, we point out the elementary fact that there is an absolute constant $C > 0$ (namely $C = 4\log 2 - 2$) such that

$$z - Cz^2 \le \log(1+z) \le z \qquad \forall\, z \ge -\frac{1}{2}. \tag{2.ii}$$

We can apply this inequality to each summand of the series in (2.i) by writing

$$z := \frac{1+a_{n,j}a_{n,g(j)}}{1+a_{n,j}^2} - 1 = \frac{a_{n,j}^2}{1+a_{n,j}^2}\left(\frac{a_{n,g(j)}}{a_{n,j}} - 1\right)$$

because $0 < a_{n,j} \leq 1$ for all n and j implies that we have

$$\frac{a_{n,j}^2}{1+a_{n,j}^2}\left(\frac{a_{n,g(j)}}{a_{n,j}}-1\right) \geq -\frac{a_{n,j}^2}{1+a_{n,j}^2} \geq -\frac{1}{2}.$$

Therefore, summing up the inequalities given by (2.ii), we conclude that Proposition 2.2 will follow once we prove the following two facts:

$$\sum_{j\in\mathbf{Z}}\left(\frac{a_{n,j}^2}{1+a_{n,j}^2}\right)^2\left(\frac{a_{n,g(j)}}{a_{n,j}}-1\right)^2 \to 0 \quad \forall\, g\in W(\mathbf{Z}), \tag{2.iii}$$

$$\sum_{j\in\mathbf{Z}}\frac{a_{n,j}^2}{1+a_{n,j}^2}\left(\frac{a_{n,g(j)}}{a_{n,j}}-1\right) \to 0 \quad \forall\, g\in W(\mathbf{Z}) \ \text{ provided } \ F_n(g)\to 0. \tag{2.iv}$$

Here is our first lemma.

Lemma 2.4. *For all n we have*

$$\sum_{j\in\mathbf{Z}} a_{n,j}e^{-|j|/n} \leq 3 \quad \textit{and} \quad \sum_{j\in\mathbf{Z}} a_{n,j}^2 e^{-2|j|/n} \leq \frac{1}{n}.$$

It is based on the following elementary comparison argument.

Lemma 2.5. *Let $t_0 \geq 0$ and let $\varphi\colon \mathbf{R}_{\geq 0}\to\mathbf{R}_{\geq 0}$ be a function which is increasing on $[0,t_0]$ and decreasing on $[t_0,\infty)$. Then*

$$\sum_{j\geq 0}\varphi(j) \leq \varphi(t_0) + \int_0^\infty \varphi(t)\,dt. \qquad \square$$

Proof of Lemma 2.4. For the first series, we consider the function φ defined by $\varphi(t) = \exp(-ne^{-t/n})e^{-t/n}$. One verifies that it satisfies the condition of Lemma 2.5 for $t_0 = n\log n$. Therefore we can estimate

$$\sum_{j\in\mathbf{Z}} a_{n,j}e^{-|j|/n} < 2\sum_{j\geq 0}\varphi(j) \leq 2e^{-1}/n + 2\int_0^\infty \exp(-ne^{-t/n})e^{-t/n}\,dt.$$

The change of variable $s = e^{-t/n}$ shows that the integral is $\int_0^1 ne^{-ns}\,ds = 1-e^{-n}$ and thus in particular the series is bounded by $2(e^{-1}+1) < 3$. For the second series, consider $\varphi(t) = \exp(-2ne^{-t/n})e^{-2t/n}$, again with $t_0 = n\log n$. Lemma 2.5 yields

$$\sum_{j\in\mathbf{Z}} a_{n,j}^2 e^{-2|j|/n} < 2\sum_{j\geq 0}\varphi(j) \leq 2(ne)^{-2} + 2\int_0^\infty \exp(-2ne^{-t/n})e^{-2t/n}\,dt.$$

The change of variable $s = e^{-t/n}$ shows that the integral is

$$\int_0^1 ne^{-2ns}s\,ds = \frac{1-(1+2n)e^{-2n}}{4n} < \frac{1}{4n}$$

and thus in particular the series is bounded by $2(ne)^{-2} + 1/(2n) < 1/n$. $\square$

Lemma 2.6. *For any $g \in W(\mathbf{Z})$ there are constants C_g, C'_g and C''_g which depend only on $|g|_{\mathrm{w}}$ such that for all n and j we have:*

$$\frac{a_{n,g(j)}}{a_{n,j}} = \exp\left(e^{-\frac{|j|}{n}}(|g(j)| - |j| + \eta(g,j,n))\right), \text{ where } |\eta(g,j,n)| \leq C_g/n. \tag{2.v}$$

$$\frac{a_{n,g(j)}}{a_{n,j}} - 1 = e^{-\frac{|j|}{n}}(|g(j)| - |j|) + \eta(g,n,j)e^{-\frac{|j|}{n}} + \vartheta(g,n,j), \tag{2.vi}$$

$$\text{where } |\vartheta(g,n,j)| \leq C'_g e^{-2\frac{|j|}{n}}.$$

$$\left|\frac{a_{n,g(j)}}{a_{n,j}} - 1\right| \leq C''_g e^{-\frac{|j|}{n}}. \tag{2.vii}$$

Proof. Note that the conclusion (2.vii) is an easy consequence of (2.v) and (2.vi). From the definition of $a_{n,j}$ we have

$$\frac{a_{n,g(j)}}{a_{n,j}} = \exp\left(e^{-\frac{|j|}{n}} n \left(1 - e^{\frac{|j|-|g(j)|}{n}}\right)\right).$$

Then using the Taylor series we have

$$n\left(1 - e^{\frac{|j|-|g(j)|}{n}}\right) = |g(j)| - |j| + \eta(g,j,n),$$

wherein

$$\eta(g,j,n) := -\sum_{k\geq 2} \frac{(|j| - |g(j)|)^k}{k! n^{k-1}}.$$

Now

$$|\eta(g,j,n)| \leq \frac{1}{n}\sum_{k\geq 2}\frac{|g|_{\mathrm{w}}^k}{k!} \leq \frac{e^{|g|_{\mathrm{w}}}}{n}$$

which proves (2.v). Continuing to expand (2.v), we have

$$\begin{aligned}\frac{a_{n,g(j)}}{a_{n,j}} - 1 &= \exp\left(e^{-\frac{|j|}{n}}(|g(j)| - |j| + \eta(g,j,n))\right) - 1 \\ &= e^{-\frac{|j|}{n}}(|g(j)| - |j|) + e^{-\frac{|j|}{n}}\eta(g,j,n) + \vartheta(g,j,n)\end{aligned}$$

wherein

$$\vartheta(g,j,n) := \sum_{k\geq 2}\frac{1}{k!}e^{-\frac{k|j|}{n}}\left(|g(j)| - |j| + \eta(g,j,n)\right)^k.$$

Thus we have

$$|\vartheta(g,j,n)| \leq e^{-\frac{2|j|}{n}}\sum_{k\geq 2}\frac{1}{k!}\Big||g(j)| - |j| + \eta(g,j,n)\Big|^k \leq e^{-\frac{2|j|}{n}}\exp\left(|g|_{\mathrm{w}} + \frac{C_g}{n}\right) \leq e^{-\frac{2|j|}{n}}C'_g,$$

as required for (2.vi). □

End of the proof of Proposition 2.2. Recall that we have reduced the proof to showing (2.iii) and (2.iv). By Lemma 2.6(2.vii) and Lemma 2.4 we have

$$\sum_{j\in\mathbf{Z}}\left(\frac{a_{n,j}^2}{1 + a_{n,j}^2}\right)^2\left(\frac{a_{n,g(j)}}{a_{n,j}} - 1\right)^2 \leq C''^2_g\sum_{j\in\mathbf{Z}}a_{n,j}^4 e^{-2\frac{|j|}{n}} \leq C''^2_g\sum_{j\in\mathbf{Z}}a_{n,j}^2 e^{-2\frac{|j|}{n}} \leq C''^2_g/n,$$

which implies the convergence (2.iii). For (2.iv), keep the notations of Lemma 2.6. By point (2.vi) of that lemma, we have

$$\sum_{j\in\mathbf{Z}} \frac{a_{n,j}^2}{1+a_{n,j}^2}\left(\frac{a_{n,g(j)}}{a_{n,j}}-1\right) = \sum_{j\in\mathbf{Z}} \frac{a_{n,j}^2}{1+a_{n,j}^2} e^{-\frac{|j|}{n}}\Big(|g(j)|-|j|\Big) + \sum_{j\in\mathbf{Z}} \frac{a_{n,j}^2}{1+a_{n,j}^2} e^{-\frac{|j|}{n}}\eta(g,j,n) + \sum_{j\in\mathbf{Z}} \frac{a_{n,j}^2}{1+a_{n,j}^2}\vartheta(g,j,n)$$

and we recall that the first of the three terms is $F_n(g)$, which is assumed to go to zero. For the second term, since $|\eta(g,j,n)|\le C_g/n$, Lemma 2.4 gives

$$\left|\sum_{j\in\mathbf{Z}} \frac{a_{n,j}^2}{1+a_{n,j}^2} e^{-\frac{|j|}{n}}\eta(g,j,n)\right| \le \frac{C_g}{n}\sum_{j\in\mathbf{Z}} a_{n,j} e^{-\frac{|j|}{n}} \le \frac{3C_g}{n}.$$

For the last term, since $|\vartheta(g,j,n)|\le C'_g e^{-2\frac{|j|}{n}}$, Lemma 2.4 implies

$$\left|\sum_{j\in\mathbf{Z}} \frac{a_{n,j}^2}{1+a_{n,j}^2}\vartheta(g,j,n)\right| \le C'_g\sum_{j\in\mathbf{Z}} a_{n,j}^2 e^{-2\frac{|j|}{n}} \le \frac{C'_g}{n}.$$

This completes the proof of (2.iv) and therefore of the proposition. □

In order to apply Proposition 2.2, we need to control F_n as stated in Proposition 2.3. Let thus $g\in W(\mathbf{Z})$ be given; writing $b_0=|g(0)|$ and

$$b_j = \Big(|g(j)|-|j|\Big) + \Big(|g(-j)|-|-j|\Big) \quad \text{for} \quad j>0,$$

we have

$$F_n(g) = \sum_{j=0}^{\infty} \frac{a_{n,j}^2}{1+a_{n,j}^2} e^{-j/n} b_j$$

since $a_{n,j}=a_{n,-j}$. Define functions B and ψ on $\mathbf{R}_{\ge 0}$ by

$$B(t) = \sum_{0\le j\le t} b_j, \qquad \psi(t) = \frac{\exp(-2ne^{-t/n})}{1+\exp(-2ne^{-t/n})}e^{-t/n}.$$

Then the Abel summation formula gives

$$\sum_{j=0}^{N}\psi(j)b_j = \psi(N)B(N) - \int_0^N B(t)\,d\psi(t). \qquad (\forall\, N\in\mathbf{N}) \tag{2.viii}$$

Lemma 2.7. *We have* $\big|B(u)\big|\le 4|g|_{\mathrm{w}}^2$ *for all* $u\ge 0$.

Proof. We claim that $-2|g|_{\mathrm{w}}^2 \le B(u) \le 4|g|_{\mathrm{w}}^2$ holds for all $u>|g|_{\mathrm{w}}$. For simplicity, write $c:=|g|_{\mathrm{w}}$ and $J_u:=\{j : |j|\le u\}$. Thus $B(u)=\sum_{j\in g(J_u)}|j| - \sum_{j\in J_u}|j|$. Since $J_{u-c}\subseteq g(J_u)$, we have

$$B(u) = \sum_{j\in g(J_u)}|j| - \sum_{j\in J_u}|j| = \sum_{j\in g(J_u)\setminus J_{u-c}}|j| - \sum_{j\in J_u\setminus J_{u-c}}|j|. \tag{2.ix}$$

Now note first that since $J_{u-c} \subseteq g(J_u)$, the number of elements in the set $g(J_u) \setminus J_{u-c}$ is equal to the number of elements in $J_u \setminus J_{u-c}$, which is $2c$. Also, for any $j \in g(J_u) \setminus J_{u-c}$ we have $u - c < |j| \leq u + c$, and for any $j \in J_u \setminus J_{u-c}$, $u - c < |j| \leq u$. Hence (2.ix) implies

$$-2c^2 = 2c(u-c) - 2cu \leq B(u) \leq 2c(u+c) - 2c(u-c) = 4c^2,$$

as claimed.

It remains to show $\big|B(u)\big| \leq 4c^2$ for $u \leq c$, and we can assume $c \geq 1$ since otherwise g is trivial and $B = 0$. Now

$$\big|B(u)\big| \leq \sum_{-u \leq j \leq u} \big|g(j) - j\big| \leq (2c+1)c \leq 3c^2,$$

finishing the proof. □

End of the proof of Proposition 2.3. Since $B(N)$ is bounded by Lemma 2.7 and since $\lim_{N\to\infty} \psi(N)$ vanishes, the equality (2.viii) gives $F_n(g) = -\int_0^\infty B(t)\, d\psi(t)$. After computing explicitly the derivative ψ', this rewrites as

$$F_n(g) = \frac{1}{n}\int_0^\infty B(t)\psi(t)dt - \int_0^\infty B(t)\frac{2\exp(-2ne^{-t/n})e^{-2t/n}}{(1+\exp(-2ne^{-t/n}))^2}dt.$$

Using Lemma 2.7 and $0 < \psi(t) \leq \exp(-ne^{-t/n})e^{-t/n}$, the first integral is bounded by

$$\left|\frac{1}{n}\int_0^\infty B(t)\psi(t)dt\right| \leq \frac{1}{n}4|g|_{\mathrm{w}}^2 \int_0^\infty \exp(-ne^{-t/n})e^{-t/n}dt = \frac{1}{n}4|g|_{\mathrm{w}}^2(1-e^{-n}),$$

which goes to zero. Similarly, the second integral is bounded by

$$\left|\int_0^\infty B(t)\frac{2\exp(-2ne^{-t/n})}{(1+\exp(-2ne^{-t/n}))^2}e^{-2t/n}dt\right| \leq 8|g|_{\mathrm{w}}^2 \int_0^\infty \exp(-2ne^{-t/n})e^{-2t/n}dt < \frac{2|g|_{\mathrm{w}}^2}{n},$$

the last inequality having already been observed in the proof of Lemma 2.4. □

Taken together, Proposition 2.3 and Proposition 2.2 finish the proof of Theorem 2.1 since we already observed $\|f_n|_{x_0=0}\|/\|f_n\| \to 1$.

3. Actions on sets of finite subsets

Let G be a group acting on a set X. The collection $\mathscr{P}_{\mathrm{f}}(X)$ of finite subsets of X is an abelian G-group for the operation $\triangle$ of symmetric difference. The resulting semidirect product $\mathscr{P}_{\mathrm{f}}(X) \rtimes G$, which can be thought of as the "lamplighter" restricted wreath product associated to the G-action on X, has itself a natural "affine" action on $\mathscr{P}_{\mathrm{f}}(X)$, where the latter set can be considered as the coset space $(\mathscr{P}_{\mathrm{f}}(X) \rtimes G)/G$.

It will be convenient to identify the Pontryagin dual of the (discrete) group $\mathscr{P}_{\mathrm{f}}(X)$ with the generalised Bernoulli G-shift $\{0,1\}^X$, the duality pairing being given for $E \in \mathscr{P}_{\mathrm{f}}(X)$ and $\omega = \{\omega_x\}_{x\in X} \in \{0,1\}^X$ by the character $\exp(i\pi \sum_{x\in E}\omega_x) \in \{\pm 1\} \subseteq \mathbf{C}^*$. The normalised Haar measure corresponds to the symmetric Bernoulli measure on $\{0,1\}^X$.

Lemma 3.1. *Assume that G acts transitively on X and choose $x_0 \in X$. The following assertions are equivalent.*

(i) There is a net $\{f_n\}$ of G-almost invariant vectors in $L^2(\{0,1\}^X)$ such that the ratio $\|f_n|_{\omega_{x_0}=0}\|/\|f_n\|$ converges to 1.

(ii) The $\mathscr{P}_{\mathrm{f}}(X) \rtimes G$-action on $\mathscr{P}_{\mathrm{f}}(X)$ admits an invariant mean.

(iii) The G-action on $\mathscr{P}_{\rm f}(X)$ admits an invariant mean giving weight $1/2$ to the collection of sets containing x_0.
(iv) The G-action on $\mathscr{P}_{\rm f}(X)$ admits an invariant mean giving full weight to the collection of sets containing x_0.

Again, $f_n|_{\omega_{x_0}=0}$ denotes the function f_n multiplied by the characteristic function of the cylinder set describing the elementary event $\omega_{x_0}=0$. The net $\{f_n\}$ can of course be chosen to be a sequence when G (and hence X) is countable.

Proof of Lemma 3.1. Recall the well-known *Reiter criterion*: a group action on a set S admits an invariant mean if and only if the corresponding representation $\ell^p(S)$ almost has invariant vectors for some or equivalently for all $1 \le p < \infty$. We shall use the fact (based on Mazur's lemma) that almost invariant probability measures on S are obtained as convex combinations of a net approximating an invariant mean on S in the weak-* topology given by duality with $\ell^\infty(S)$. All this is classical and can be found e.g. in [Pat88].

(i)$\Longrightarrow$(ii). The Fourier transform $\widehat{f_n}$ provides G-almost invariant vectors in $\ell^2(\mathscr{P}_{\rm f}(X))$. Moreover, $\|f_n|_{\omega_{x_0}=0}\|$ is the norm of the image of $\widehat{f_n}$ projected to the subspace of vectors in $\ell^2(\mathscr{P}_{\rm f}(X))$ that are invariant under $\{x_0\}$ viewed as group element in $\mathscr{P}_{\rm f}(X)$. Thus $\widehat{f_n}$ is $\{x_0\}$-almost invariant. Since the G-action is transitive, it follows that $\widehat{f_n}$ is $\mathscr{P}_{\rm f}(X)$-almost invariant as $n \to \infty$.

(ii)$\Longrightarrow$(iii). Given a mean as in (ii), the condition on x_0 follows from the invariance under the element $\{x_0\}$ of $\mathscr{P}_{\rm f}(X)$.

(iii)$\Longrightarrow$(iv). It suffices to show that for each $k \in \mathbf{N}$ there are G-almost-invariant probability measures on $\mathscr{P}_{\rm f}(X)$ such that the collection of sets containing x_0 has probability at least $1-2^{-k}$. By (iii), we have G-almost-invariant probability measures such that the collection of sets containing x_0 has probability $1/2$. Indeed, these probability measures arise as convex combinations of a net approximating an invariant mean in the weak-* topology, and our restriction about x_0 is preserved under convex combinations. If we take the union of k independently chosen such finite sets, we obtain a distribution as required.

(iv)$\Longrightarrow$(i). The assumption implies that there are G-almost-invariant probability measures μ on $\mathscr{P}_{\rm f}(X)$ such that the collection of sets containing x_0 has probability 1, using the same convexity argument as in (iii)$\Rightarrow$(iv). We can assume that each μ is supported on a collection of sets of fixed cardinal $n(\mu) \in \mathbf{N}$. We define a function f_μ on $\{0,1\}^X$ as follows. Given $E \in \mathscr{P}_{\rm f}(X)$, consider the cylinder set $C_E \subseteq \{0,1\}^X$ consisting of all ω such that $\omega_x = 0$ for all $x \in E$. We set $f_\mu = 2^{n(\mu)} \sum_{E \in \mathscr{P}_{\rm f}(X)} \mu(\{E\}) 1_{C_E}$, where 1_{C_E} is the characteristic function of C_E. Then f_μ is supported on $\{\omega_{x_0}=0\}$, has L^1-norm one and satisfies $\|gf_\mu - f_\mu\|_1 \le \|g\mu - \mu\|_1$ for all $g \in G$. Therefore, the function $f_\mu^{1/2}$ is as required by (i) as μ becomes increasingly invariant since $\|gf_\mu^{1/2} - f_\mu^{1/2}\| \le \|gf_\mu - f_\mu\|_1^{1/2}$. □

Proof of Theorem C. The sequence $\{f_n\}$ constructed in Section 2 satisfies the criterion (i) of Lemma 3.1 in view of Theorem 2.1. Therefore, the criterion (iv) provides the desired conclusion. □

The following is well-known.

Lemma 3.2. *Let H be a group acting on a set Y with an invariant mean. If the stabiliser in H of every $y \in Y$ is an amenable group, then H is amenable.*

Proof. The amenability of stabilisers implies that there is an H-map $Y \to \mathscr{M}(H)$ to the (convex compact) space $\mathscr{M}(H)$ of means on H (by choosing for each H-orbit in Y the orbital map associated to a mean fixed by the corresponding stabiliser). The push-forward of an invariant mean on Y is an invariant mean on $\mathscr{M}(H)$. Its barycenter is an invariant mean on H. (An alternative argument giving explicit Følner sets can be found in the proof of Lemma 4.5 in [GM07].) □

The next proposition will leverage the fact that $\mathbf{N} \triangle g(\mathbf{N})$ is finite for all $g \in W(\mathbf{Z})$.

Proposition 3.3. *Let $G < W(\mathbf{Z})$ be a subgroup such that the stabiliser in G of $E \triangle \mathbf{N}$ is amenable whenever $E \in \mathscr{P}_{\mathrm{f}}(\mathbf{Z})$. Then G is amenable.*

Proof. As noted in the proof of Theorem C, the $W(\mathbf{Z})$-action on $\mathbf{Z}$ satisfies the equivalent conditions of Lemma 3.1 thanks to Theorem 2.1. In particular, there is a $\mathscr{P}_{\mathrm{f}}(\mathbf{Z}) \rtimes G$-invariant mean on $\mathscr{P}_{\mathrm{f}}(\mathbf{Z})$. Thus, in view of Lemma 3.2, it suffices to find an embedding $\iota\colon G \to \mathscr{P}_{\mathrm{f}}(\mathbf{Z}) \rtimes G$ in such a way that the stabiliser in $\iota(G)$ of any finite set E is the stabiliser in G of $E \triangle \mathbf{N}$. The homomorphism defined by $\iota(g) = \big(\mathbf{N} \triangle g(\mathbf{N}), g\big)$ has the required properties. □

4. From Cantor systems to piecewise translations

It is known that the stabiliser of a forward orbit in the topological full group of a minimal Cantor system is locally finite. This follows from the (much more detailed) description of this stabilizer given in Section 5 of [Put89], where this group is realized as subquotient of unitaries of an AF-algebra (in the notation of [Put89], the stabiliser of the forward orbit of a point y is $\Gamma_{\{y\}}$).

In the following two lemmas, we shall give an elementary proof (without C^*-algebras) of the corresponding fact in the setting of the group $W(\mathbf{Z})$. A forward orbit then corresponds to $\mathbf{N} \subseteq \mathbf{Z}$ and the case of finite set differences $E \triangle \mathbf{N}$ is a minor extension.

A subgroup G of $W(\mathbf{Z})$ has the *ubiquitous pattern property* if for every finite set $F \subseteq G$ and every $n \in \mathbf{N}$ there exists a constant $k = k(n, F)$ such that for every $j \in \mathbf{Z}$ there exists $t \in \mathbf{Z}$ such that $[t-n, t+n] \subseteq [j-k, j+k]$ and such that for every $i \in [-n, n]$ and every $g \in F$ we have $g(i+t) = g(i) + t$.

Informally: the partial action of F on $[-n, n]$ can be found, suitably translated, within any interval of length $2k+1$.

Lemma 4.1. *Let $G < W(\mathbf{Z})$ be a subgroup with the ubiquitous pattern property. Then the stabiliser of $E \triangle \mathbf{N}$ in G is locally finite for every $E \in \mathscr{P}_{\mathrm{f}}(\mathbf{Z})$.*

Proof. Let $E \in \mathscr{P}_{\mathrm{f}}(\mathbf{Z})$ and F be a finite set of elements of the stabiliser of $E \triangle \mathbf{N}$ in G. In order to prove that the set F generates a finite group it is sufficient to show that $\mathbf{Z}$ is a disjoint union of finite sets B_i of uniformly bounded cardinality such that each of this sets is invariant under the action of F, since this will realize the group generated by F as a subgroup of a power of a finite group. We will achieve this by taking the B_i to be the ubiquitous translated copies of the "phase transition" region of $E \triangle \mathbf{N}$, suitably identifying the "top part" of $E \triangle \mathbf{N}$ with the "bottom part" of the complement of the next translated copy.

Let $c = \max\{|e| : e \in E\}$ (with $c = 0$ if $E = \varnothing$) and let $m = \max\{|g|_{\mathrm{w}} : g \in F\}$. Let $k = k(c+m+1, F)$ be the constant from the definition of the ubiquitous pattern property. Denote $E_0 = E \triangle \mathbf{N} \cap [-c-m-1, c+m+1]$. Decompose $\mathbf{Z}$ as disjoint union of consecutive intervals I_i ($i \in \mathbf{Z}$) of length $2k+1$ such that $[-c-m-1, c+m+1] \subseteq I_0$. Then, by the ubiquitous pattern property, for each interval I_i there exists a set $E_i \subseteq I_i$ (a translate of E_0) such that the action of F on E_i corresponds to the action of F on E_0. Let

$$B_i = \Big(E_i \cup [\max(E_i)+1, \max(E_{i+1})]\Big) \setminus E_{i+1}.$$

By construction, we have $\mathbf{Z} = \bigsqcup B_i$. The choice of m ensures that each B_i is F-invariant because F preserves $E \triangle \mathbf{N}$. Finally, since $B_i \subseteq I_i \cup I_{i+1}$, we have $|B_i| \leq 4k+2$ for all i. □

Let T be a homeomorphism of a Cantor space C and choose a point $p \in C$. If T has no finite orbits, then we can define a map

$$\pi_p \colon [[T]] \longrightarrow W(\mathbf{Z})$$

by the requirement

$$g(T^j p) = T^{\pi_p(g)(j)} p, \qquad (g \in [[T]], j \in \mathbf{Z}).$$

The map π_p is a group homomorphism and is injective if the orbit of p is dense.

Lemma 4.2. *If T is minimal, then the image $\pi_p([[T]])$ of the injective homomorphism π_p has the ubiquitous pattern property.*

Proof. Let $F \subset [[T]]$ be a finite set and let $n \in \mathbf{N}$. By definition of $[[T]]$, there is a finite clopen partition $\mathscr{D}$ of C such that each $g \in F$ is a power of T when restricted to any element of $\mathscr{D}$. Thus there is an open neighborhood V of p such that for all $i \in [-n, n]$ the set $T^i V$ is contained in some $D \in \mathscr{D}$. By minimality of T, the non-empty open T-invariant set $\bigcup_{q \geq 1} \bigcup_{|r| \leq q} T^r V$ is C. By compactness, there is $q \in \mathbf{N}$ such that $C = \bigcup_{|r| \leq q} T^r V$.

Set $k = k(n, F) = q + n$. For all $j \in \mathbf{Z}$ we have $C = T^{-j} C = T^{-(j+q)} V \cup \ldots \cup T^{-(j-q)} V$ and hence there an integer $t \in [j-q, j+q]$ such that $p \in T^{-t} V$. In particular, $[t-n, t+n] \subseteq [j-k, j+k]$. Now $T^t p \in V$ and thus both $T^i p$ and $T^{i+t} p$ are in $T^i V$ for all i. Therefore, when $i \in [-n, n]$, every $g \in F$ acts on $T^i p$ and on $T^{i+t} p$ as the same power of T. This is exactly the ubiquitous pattern property under the π_p-equivariant identification of the T-orbit of p with $\mathbf{Z}$. □

Proof of Theorem A. By Lemma 4.2, the (injective image of the) topological full group $[[T]]$ has the ubiquitous pattern property. Therefore, Lemma 4.1 shows that the stabiliser of $E \triangle \mathbf{N}$ in G is amenable for every $E \in \mathscr{P}_{\mathrm{f}}(\mathbf{Z})$. Now Proposition 3.3 implies that $[[T]]$ is amenable. □

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