ISOMETRY GROUPS AND LATTICES
OF NON-POSITIVELY CURVED SPACES

PIERRE-EMMANUEL CAPRACE* AND NICOLAS MONOD†

Abstract. We develop the structure theory of full isometry groups of locally compact non-positively curved metric spaces and their discrete subgroups. Classical results on Hadamard manifolds are extended to that broad geometric setting; properties that single out semi-simple groups and their lattices are highlighted. Amongst the discussed themes are de Rham decompositions, normal subgroup structure, characterising properties of symmetric spaces and Bruhat-Tits buildings, Borel density, Mostow rigidity, geometric superrigidity, abstract and geometric arithmeticity, and residual finiteness of lattices.

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1. Introduction

Non-positively curved metric spaces were introduced by A. D. Alexandrov [Ale57] and popularised by M. Gromov, who called them CAT(0) spaces. They provide a framework for the development of some form of generalised differential geometry, whose objects encompass Riemannian manifolds of non-positive sectional curvature as well as large families of singular spaces including Euclidean buildings and many other polyhedral complexes. Several aspects of classical and less classical themes may thereby be approached in a unified way. In order to describe a few of these, we shall define a CAT(0) lattice as a pair \((\Gamma, X)\) consisting of a proper CAT(0) space \(X\) with cocompact isometry group and a lattice subgroup \(\Gamma < \text{Is}(X)\), i.e. a discrete subgroup of finite invariant covolume (the compact-open topology makes \(\text{Is}(X)\) a locally compact second countable group which is thus canonically endowed with
Haar measures). We say that \((\Gamma, X)\) is **uniform** if \(\Gamma\) is cocompact in Is\((X)\) or, equivalently, if the quotient \(\Gamma\backslash X\) is compact; that case corresponds to \(\Gamma\) being a \(\text{CAT}(0)\) group in the usual terminology. Here are some examples.

- Let \(M\) be a compact Riemannian manifold of non-positive curvature and \(\Gamma\) its fundamental group. The universal covering \(\tilde{M}\) is a \(\text{CAT}(0)\) space on which \(\Gamma\) acts properly by deck transformations. The pair \((\Gamma, \tilde{M})\) is a uniform \(\text{CAT}(0)\) lattice.

- Many Gromov-hyperbolic groups \(\Gamma\) admit a properly discontinuous cocompact action on some \(\text{CAT}(-1)\) space \(X\) by isometries. In that case the pair \((\Gamma, X)\) is a uniform \(\text{CAT}(0)\) lattice. Amongst the examples arising in this way are hyperbolic Coxeter groups \([\text{Mou88}], C'(\frac{3}{2})\) and \(C'(\frac{3}{2})\)-\(T(4)\) small cancellation groups \([\text{Wis04}], 2\)-dimensional 7-systolic groups \([\text{JS06}]\). It is in fact a well known open problem of M. Gromov to construct an example of a Gromov-hyperbolic group which is *not* a \(\text{CAT}(0)\) group (see \([\text{Gro93}, 7.B]\); also Remark 2.3(2) in Chapter III.\(\Gamma\) of \([\text{BH99}]\)).

- An arithmetic group \(\Gamma\) may be realised as an irreducible lattice in a product \(G\) of semi-simple algebraic group over local fields. Moreover, any semi-simple group over a local field possesses a canonical continuous, proper and cocompact action on a symmetric space or a Bruhat–Tits building, according as the ground field is Archimedean or not. Thus \(G\) acts cocompactly on a product \(X\) of symmetric spaces and Bruhat–Tits buildings which, endowed with the \(L^2\)-metric, is a proper \(\text{CAT}(0)\) space. Thus the pair \((\Gamma, X)\) is a \(\text{CAT}(0)\) lattice, which we shall henceforth call an **arithmetic \(\text{CAT}(0)\)** lattice. More generally, any lattice in a semi-simple group provides an example of a \(\text{CAT}(0)\) lattice in a similar fashion.

- Extending some of the previous examples to a more general setting, tree lattices are \(\text{CAT}(0)\) lattices \((\Gamma, X)\), where \(X\) is a locally finite tree, see \([\text{BL01}]\). In \([\text{BM00b}]\), striking examples of finitely presented simple groups have been constructed as \(\text{CAT}(0)\) lattices whose underlying space is a product of two locally finite trees.

- A minimal adjoint Kac–Moody group \(\Gamma\) over a finite field, as defined by J. Tits \([\text{Tit87}]\), is endowed with two \(BN\)-pairs which yield strongly transitive \(\Gamma\)-actions on two buildings denoted \(X_+\) and \(X_-\) respectively. When the order of the ground field is large enough, the pair \((\Gamma, X_+ \times X_-)\) is a \(\text{CAT}(0)\) lattice.

Amongst these examples, the most important, and also the best understood, notably through the work of G. Margulis, consist undoubtedly of those arising from lattices in semi-simple groups over local fields. It is therefore natural to address two sets of questions.

(a) **What properties of these lattices are shared by all \(\text{CAT}(0)\) lattices?**

(b) **What properties characterise them within the class of \(\text{CAT}(0)\) lattices?**

In the same way as the study of arithmetic lattices requires a preliminary knowledge of their ambient semi-simple groups, tackling the above questions first necessitates a description of the basic properties of the overall geometry of proper \(\text{CAT}(0)\) spaces and their full isometry groups. Building upon earlier work which may be consulted in standard references including \([\text{BGSS85}], [\text{Bal93}]\) and \([\text{BH99}]\), the first part of the present article provides a detailed analysis of the algebraic structure of the full isometry group of a proper \(\text{CAT}(0)\) space, independently of the existence of lattices. The second part is devoted to \(\text{CAT}(0)\) lattices and centres around the above questions.

We now proceed to describe our main results; for many of them, the core of the text will contain a stronger, more precise and probably more cumbrous version.
Group decompositions. Let $X$ be a proper CAT(0) space. Our first aim is to obtain structural results on the locally compact group $\text{Is}(X)$ in a broad generality; we shall mostly ask that no point at infinity be fixed simultaneously by all isometries of $X$. This non-degeneracy assumption will be shown to hold automatically in the presence of lattices (Theorem 8.11 below).

**Theorem 1.1.** Let $X$ be a proper CAT(0) space with finite-dimensional Tits boundary. Assume that $\text{Is}(X)$ has no global fixed point in $\partial X$.

Then there is a canonical closed convex $\text{Is}(X)$-stable subset $X' \subseteq X$ such that $G = \text{Is}(X')$ has a finite index open characteristic subgroup $G^* < G$ which admits a canonical decomposition

$$G^* \cong S_1 \times \cdots \times S_p \times (\mathbb{R}^n \rtimes O(n)) \times D_1 \times \cdots \times D_q \quad (p, q, n \geq 0)$$

where $S_i$ are almost connected simple Lie groups with trivial centre and $D_j$ are totally disconnected irreducible groups with trivial amenable radical. Any product decomposition of $G^*$ is a regrouping of the factors in (1.i).

Moreover, all non-trivial normal, subnormal or ascending subgroups $N < D_j$ are still irreducible with trivial amenable radical and trivial centraliser in $D_j$. These properties also hold for lattices in $N$ and their normal, subnormal or ascending subgroups.

(A topological group is called **irreducible** if no finite index open subgroup splits non-trivially as a direct product of closed subgroups. The **amenable radical** of a locally compact group is the largest amenable normal subgroup; it is indeed a radical since the class of amenable locally compact groups is stable under group extensions.)

**Remarks 1.2.**

(i) The finite-dimensionality assumption holds automatically when $X$ has a cocompact group of isometries [Kle99, Theorem C]. It is also automatic for uniquely geodesic spaces, e.g. manifolds (Proposition 6.11).

(ii) The conclusion fails in various ways if $G$ fixes a point in $\partial X$.

(iii) The quotient $G/G^*$ is just a group of permutations of possibly isomorphic factors in the decomposition. In particular, $G = G^* \rtimes G/G^*$.

(iv) The canonical continuous homomorphism $\text{Is}(X) \to \text{Is}(X') = G$ is proper, but its image sometimes has infinite covolume.

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...όσα τις ἠν εἶχοι σφαίρας ἐγκώμια, αὐτὰ ταῦτα καὶ φαλάκρας ἐγκώμια διεξέρχεται.

Συνέσιος Κυρεναίος, Ἀδρανός ἐγκώμιον.1

**Minimality.** The definition of a CAT(0) space is flexible. For example, given a CAT(0) space $X$, there are many ways to deform it in order to construct another space $Y$, non-isometric to $X$, but with the property that $X$ and $Y$ have isomorphic isometry groups or/and identical boundaries. Amongst the simplest constructions, one can form (possibly warped) products with compact CAT(0) spaces or grow hair equivariantly along a discrete orbit. Much wilder (non-quasi-isometric) examples can be constructed for instance by considering

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1Synesius of Cyrene, Φαλάκρας ἐγκώμιον (known as Calistus encomium), end of Chapter 8 (at 72A in the page numbering from Denis Pétau’s 1633 edition). The Encomium was written around 402; we used the 1834 edition by J. G. Krabinger (Ch. G. Löfflund, Stuttgart). The above excerpt translates roughly to: *as much praise as is given to the spheres is due to the bald head too.*
warped products with the very vast family of CAT(0) spaces having no isometries and a unique point at infinity.

In order to address these issues, we introduce the following terminology.

An isometric action of a group $G$ on a CAT(0) space $X$ is said to be **minimal** if there is no non-empty $G$-invariant closed convex subset $X' \subsetneq X$; the space $X$ is itself called **minimal** if its full isometry group acts minimally. A CAT(0) space $X$ is called **boundary-minimal** if it possesses no closed convex subset $Y \subsetneq X$ such that $\partial Y = \partial X$. Here is how these notions relate to one another.

**Proposition 1.3.** Let $X$ be a proper CAT(0) space.

(i) Assume $\partial X$ finite-dimensional. If $X$ is minimal, then it is boundary-minimal.

(ii) Assume $\text{Is}(X)$ has full limit set. If $X$ is boundary-minimal, then it is minimal.

(iii) If $X$ is cocompact and geodesically complete, then it is both minimal and boundary-minimal.

(geodesic completeness means that every geodesic segment of can be extended to a bi-infinite geodesic line — which need not be unique. This assumption is satisfied by most classical examples, including [homology] manifolds and Euclidean buildings.)

In Theorem 1.1, the condition that $\text{Is}(X)$ has no global fixed point at infinity ensures the existence of a closed convex $\text{Is}(X)$-invariant subset $Y \subsetneq X$ on which $\text{Is}(X)$ acts minimally (see Proposition 3.1). The set of these minimal convex subsets possesses a canonical element, which is precisely the space $X'$ which appears in Theorem 1.1.

**De Rham decompositions.** It is known that product decompositions of isometry groups acting minimally and without global fixed point at infinity induce a splitting of the space (for cocompact Hadamard manifolds, this is the Lawson–Yau [LY72] and Gromoll–Wolff [GW71] theorem; in general and for more references, see [Mon06]). It is therefore natural that Theorem 1.1 is supplemented by a geometric statement.

**Addendum 1.4.** In Theorem 1.1, there is a canonical isometric decomposition

\[(1.ii) \quad X' \cong X_1 \times \cdots \times X_p \times \mathbb{R}^n \times Y_1 \times \cdots \times Y_q \]

where $G^*$ acts componentwise according to (1.i) and $G/G^*$ permutes any isometric factors. All $X_i$ and $Y_j$ are irreducible and minimal.

As it turns out, a geometric decomposition is the first of two independent steps in the proof of Theorem 1.1. In fact, we begin with an analogue of the de Rham decomposition [dR52] whose proof uses (a modification of) arguments from the generalised de Rham theorem of Foertsch–Lytchak [FL06]. In purely geometrical terms, we have the following statement.

**Theorem 1.5.** Let $X$ be a proper boundary-minimal CAT(0) space with $\partial X$ finite-dimensional. Then $X$ admits a canonical maximal isometric splitting

\[X \cong \mathbb{R}^n \times X_1 \times \cdots \times X_m \quad (n, m \geq 0)\]

with each $X_i$ irreducible and $\neq \mathbb{R}^0, \mathbb{R}^1$. Every isometry of $X$ preserves this decomposition upon permuting possibly isometric factors $X_i$. Moreover, if $X$ is minimal, so is every $X_i$.

To apply this theorem, it is desirable to know conditions ensuring boundary-minimality. In addition to the conditions provided by Proposition 1.3, we show that a canonical boundary-minimal subspace exists as soon as the boundary has circumradius $> \pi/2$ (Corollary 2.10).
In the second part of the proof of Theorem 1.1, we analyse the irreducible case where $X$ admits no isometric splitting, resulting in Theorem 1.6 to which we shall now turn. Combining these two steps, we then prove the unique decomposition of the groups, using also the splitting theorem from [Mon06].

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**Geometry of normal subgroups.** In É. Cartan’s correspondence between symmetric spaces and semi-simple Lie groups as well as in Bruhat–Tits theory, irreducible factors of the space correspond to simple groups. For general CAT(0) spaces and groups, simplicity fails of course very dramatically (free groups are perhaps the simplest, and yet most non-simple, CAT(0) groups). Nonetheless, we establish a geometric weakening of simplicity.

**Theorem 1.6.** Let $X \neq \mathbb{R}$ be an irreducible proper CAT(0) space with finite-dimensional Tits boundary and $G < \text{Is}(X)$ any subgroup whose action is minimal and does not have a global fixed point in $\partial X$.

Then every non-trivial normal subgroup $N < G$ still acts minimally and without fixed point in $\partial X$. Moreover, the amenable radical of $N$ and the centraliser $\mathcal{Z}(G)(N)$ are both trivial; $N$ does not split as a product.

This result can for instance be combined with the solution to Hilbert’s fifth problem in order to understand the connected component of the isometry group.

**Corollary 1.7.** $\text{Is}(X)$ is either totally disconnected or an almost connected simple Lie group with trivial centre.

The same holds for any closed subgroup acting minimally and without fixed point at infinity.

A more elementary application of Theorem 1.6 uses the fact that elements with a discrete conjugacy class have open centraliser.

**Corollary 1.8.** If $G$ is non-discrete, $N$ cannot be a finitely generated discrete subgroup.

A feature of Theorem 1.6 is that it can be iterated and thus applies to subnormal subgroups. Recall that more generally a subgroup $H < G$ is ascending if there is a (possibly transfinite) chain of normal subgroups starting with $H$ and abutting to $G$. Using limiting arguments, we bootstrap Theorem 1.6 and show:

**Theorem 1.9.** Let $N < G$ be any non-trivial ascending subgroup. Then all conclusions of Theorem 1.6 hold for $N$.

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**Actions of simple algebraic groups.** Both for the general theory and for the geometric superrigidity/arithmeticity statements of the second part of this paper, it is important to understand how algebraic groups act on CAT(0) spaces.

**Theorem 1.10.** Let $k$ be a local field and $G$ be an absolutely almost simple simply connected $k$-group. Let $X$ be a non-compact proper CAT(0) space on which $G = G(k)$ acts continuously and cocompactly by isometries.

Then there is a $G$-equivariant bijection $\partial X \cong \partial X_{\text{model}}$ which is a homeomorphism with respect to the cône topology and an isometry with respect to Tits’ metric, where $X_{\text{model}}$ is the symmetric space or Bruhat–Tits building associated with $G$.

If in addition $X$ is geodesically complete, then $X$ is isometric to $X_{\text{model}}$. 
A stronger and more detailed statement is provided below as Theorem 6.4; in particular, the cocompactness assumption can be relaxed, but we also show there by means of two examples that some assumptions remain necessary. (As a point of terminology, we do not choose a particular scaling factor on symmetric spaces or Bruhat–Tits buildings, so that the isometries of our statement could be homotheties on each irreducible factor.)

Cocompact spaces. As already mentioned, a natural situation where the finite-dimensionality assumption in Theorem 1.1 holds automatically is when $X$ has a cocompact group of isometries. This setting is particularly relevant in preparation to the study of CAT(0) groups and more generally CAT(0) lattices. When $\text{Is}(X)$ acts cocompactly, the space $X$ is a tubular neighbourhood of $X'$ and each factor in the decomposition (1.ii) of $X'$ admits a cocompact group of isometries. As we shall see, this yields more precise information on the action of both the connected and totally disconnected factors in the decomposition (1.i). The conclusions become even stronger if $X$ is assumed to be geodesically complete.

Theorem 1.11. Let $X$ be a proper CAT(0) space whose isometry group acts cocompactly without fixed point at infinity. Then all conclusions of Theorem 1.1 and Addendum 1.4 hold. Moreover:

If $X$ has extensible geodesics, then $X' = X$, each $X_i$ is isometric to the symmetric space of $G_i$ and each group $D_j$ acts by semi-simple isometries on $Y_j$.

If $X$ has uniquely extensible geodesics, then each group $D_j$ is discrete.

Remark 1.12. It is not true in general that a minimal CAT(0) space is geodesically complete, even if one assumes that the isometry group acts cocompactly and without global fixed point at infinity. Important examples of geodesically complete spaces are provided by Bruhat–Tits buildings and of course Hadamard manifolds, e.g. symmetric spaces. A complete CAT(0) space that is also a homology manifold has automatically extensible geodesics [BH99, II.5.12].

A further natural condition on non-discrete groups of isometries is that no open subgroup fixes a point at infinity; this holds notably for symmetric spaces and Bruhat–Tits buildings. Although we shall not impose this condition in our other investigations, we show that much structure can be derived from it. For instance, on the geometric side, it turns out that the de Rham factors are either symmetric spaces or come quite close to form a reasonable cellular complex, as is the case for Bruhat–Tits buildings.

Theorem 1.13. Let $X$ be a proper geodesically complete CAT(0) space such that some closed subgroup $G < \text{Is}(X)$ acts cocompactly. Suppose that no open subgroup of $G$ fixes a point at infinity. Then $X$ admits a canonical equivariant splitting

$$X \cong X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q$$

where each $X_i$ is a symmetric space (possibly Euclidean) and each $Y_j$ possesses a $G$-equivariant locally finite decomposition into compact convex cells.

On the side of the group, we have the following result, for which we recall two definitions (which will be further discussed in the text). The quasi-centre consists of all elements with open centraliser and the socle of a group is the subgroup generated by the (possibly empty) family of all minimal non-trivial closed normal subgroups.
Theorem 1.14. In the setting of Theorem 1.13, assume that $X$ has no Euclidean factor. Then:

(i) Every compact subgroup of $G$ is contained in a maximal one; the maximal compact subgroups fall into finitely many conjugacy classes.

(ii) The quasi-centre of $G$ is trivial; in particular $G$ has no non-trivial discrete normal subgroup.

(iii) The socle of $G^*$ is a direct product of $p + q$ non-discrete characteristically simple groups.

(iv) $G$ possesses hyperbolic elements.

Smoothness. The rather common assumption of geodesic completeness has already occurred a few times above and is satisfied by most classical examples of CAT(0) spaces (an outstanding counter-example being the convex core of Fuchsian groups). One of the important properties of geodesically complete spaces is that isometric actions of totally disconnected groups are smooth in the following sense.

Theorem 1.15. Let $X$ be a geodesically complete proper CAT(0) space $X$ and $G < \text{Is}(G)$ a totally disconnected (closed) subgroup acting minimally.

The pointwise stabiliser in $G$ of every bounded set is open.

This property, which is familiar from classical examples, fails in general (Remark 11.10). It is an important ingredient for statements such as Theorems 1.13 and 1.14 above as well as for angle rigidity results regarding both the Alexandrov angle (Proposition 5.8) and the Tits angle (Proposition 6.15).

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A few cases of superrigidity. Combining the preceding general structure results with some of Margulis’ theorems, we obtain the following superrigidity statement.

Theorem 1.16. Let $X$ be a proper CAT(0) space whose isometry group acts cocompactly and without global fixed point at infinity. Let $\Gamma = \text{SL}_n(\mathbb{Z})$ with $n \geq 3$ and $G = \text{SL}_n(\mathbb{R})$.

For any isometric $\Gamma$-action on $X$ there is a non-empty $\Gamma$-invariant closed convex subset $Y \subseteq X$ on which the $\Gamma$-action extends uniquely to a continuous isometric action of $G$.

(The corresponding statement applies to all those lattices in semi-simple Lie groups that have virtually bounded generation by unipotents.)

Observe that the above theorem has no assumptions whatsoever on the action; cocompactness is an assumption on the given CAT(0) space. It can happen that $\Gamma$ fixes points in $\partial X$, but its action on $Y$ is without fixed points at infinity and minimal (as we shall establish in the proof).

The assumption on bounded generation holds conjecturally for all non-uniform irreducible lattices in higher rank semi-simple Lie groups (but always fails in rank one). It is known to

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2One can always artificially make a CAT(0) space geodesically complete by gluing rays, though it is not always possible to preserve properness (consider a compact but total set in an infinite-dimensional Hilbert space).
hold for arithmetic groups in split or quasi-split algebraic groups of a number field \( K \) of \( K \)-rank \( \geq 2 \) by \([\text{Tav90}]\), as well as in a few cases of isotropic but non-quasi-split groups \([\text{ER06}]\); see also \([\text{VM07}]\).

More generally, Theorem 1.16 holds for (S-)arithmetic groups provided the arithmetic subgroup (given by integers at infinite places) satisfies the above bounded generation property. For instance, the \( \text{SL}_n \) example is as follows:

**Theorem 1.17.** Let \( X \) be a proper \( \text{CAT}(0) \) space whose isometry group acts cocompactly and without global fixed point at infinity. Let \( m \) be an integer with distinct prime factors \( p_1, \ldots, p_k \) and set

\[
\Gamma = \text{SL}_n(\mathbb{Z}[(\frac{1}{m})]), \quad G = \text{SL}_n(\mathbb{R}) \times \text{SL}_n(\mathbb{Q}_{p_1}) \times \cdots \times \text{SL}_n(\mathbb{Q}_{p_k}),
\]

where \( n \geq 3 \). Then for any isometric \( \Gamma \)-action on \( X \) there is a non-empty \( \Gamma \)-invariant closed convex subset \( Y \subseteq X \) on which the \( \Gamma \)-action extends uniquely to a continuous isometric action of \( G \).

We point out that a fixed point property for similar groups acting on low-dimensional \( \text{CAT}(0) \) cell complexes was established by B. Farb \([\text{Far08a}]\).

Some of our general results also allow us to improve on the generality of the \( \text{CAT}(0) \) superrigidity theorem for irreducible lattices in arbitrary products of locally compact groups proved in \([\text{Mon06}]\). For actions on proper \( \text{CAT}(0) \) spaces, the results of loc. cit. establish an unrestricted superrigidity on the boundary but require, in order to deduce superrigidity on the space itself, the assumption that the action be reduced (or alternatively “indecomposable”).

We prove that, as soon as the boundary is finite-dimensional, any action without global fixed point at infinity is always reduced after suitably passing to subspaces and direct factors. It follows that the superrigidity theorem for arbitrary products holds in that generality, see Theorem 9.4 below.

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**A first characterisation of symmetric spaces and Euclidean buildings.** In symmetric spaces and Bruhat–Tits buildings, the stabilisers of points at infinity are exactly the parabolic subgroups; as such, they are cocompact. This cocompactness holds further for all Bass–Serre trees, namely bi-regular trees. Combining our results with results of B. Leeb \([\text{Lee00}]\) and A. Lytchak \([\text{Lyt05}]\), we establish a corresponding characterisation.

**Theorem 1.18.** Let \( X \) be a geodesically complete proper \( \text{CAT}(0) \) space. Suppose that the stabiliser of every point at infinity acts cocompactly on \( X \).

Then \( X \) is isometric to a product of symmetric spaces, Euclidean buildings and Bass–Serre trees.

The Euclidean buildings appearing in the preceding statement admit an automorphism group that is strongly transitive, i.e. acts transitively on pairs \((c, A)\) where \( c \) is a chamber and \( A \) an apartment containing \( c \). This property characterises the Bruhat–Tits buildings, except perhaps for some two-dimensional cases where this is a known open question.

The above characterisation is of a different nature and independent of the characterisations using lattices that will be presented below.
Geometric density of lattices. As a transition between the general theory and the study of CAT(0) lattices, we propose the following analogue of A. Borel’s density theorem [Bor60]. It also provides additional information about the totally disconnected groups $D_j$ occurring in Theorem 1.1. It can of course be gainfully combined with Theorem 1.6.

**Theorem 1.19.** Let $X$ be a proper CAT(0) space, $G$ a locally compact group acting continuously by isometries on $X$ and $\Gamma < G$ a lattice. Suppose that $X$ has no Euclidean factor.

If $G$ acts minimally without fixed point at infinity, so does $\Gamma$.

(This conclusion fails for spaces with a Euclidean factor. The theorem will be established more generally for closed subgroups with finite invariant covolume.)

As with classical Borel density, we shall use this density statement to derive statements about the centraliser, normaliser and radical of lattices in Section 7.

**Remark 1.20.** Theorem 1.19 applies to general proper CAT(0) spaces. Nonetheless, we point out that it implies in particular the classical Borel density theorem (see the end of Section 7).

A more elementary variant of the above theorem shows that a large class of non-amenable groups have rather restricted actions on proper CAT(0) spaces; as an application, one shows that any isometric action of R. Thompson’s group $F$ on any proper CAT(0) space $X$ has a fixed point in $X$, see Corollary 7.3.

Lattices: Euclidean factor, boundary, irreducibility and Mostow rigidity. Recall that the Flat Torus theorem, originating in the work of Gromoll–Wolf [GW71] and Lawson–Yau [LY72], associates Euclidean subspaces $\mathbb{R}^n$ to any subgroup $\mathbb{Z}^n$ of a CAT(0) group, see [BH99, §II.7]. (In the classical setting, when the CAT(0) group is given by a compact non-positively curved manifold, this amounts to the seemingly more symmetric statement that such a subgroup exists if and only if there is a flat torus in the manifold.)

The converse is a well known open problem stated by M. Gromov in [Gro93, §6.B3]; for manifolds see S.-T. Yau, problem 65 in [Yau82]). Point (i) in the following result is a (very partial) answer; in the special case of cocompact Riemannian manifolds, this was the main result of P. Eberlein’s article [Ebe83].

**Theorem 1.21.** Let $X$ be a proper CAT(0) space, $G < \text{Is}(X)$ a closed subgroup acting minimally and cocompactly on $X$ and $\Gamma < G$ a finitely generated lattice. Then:

(i) If the Euclidean factor of $X$ has dimension $n$, then $\Gamma$ possesses a finite index subgroup $\Gamma_0$ which splits as $\Gamma_0 \simeq \mathbb{Z}^n \times \Gamma'$. Moreover, the dimension of the Euclidean factor is characterised as the maximal rank of a free Abelian normal subgroup of $\Gamma$.

(ii) $G$ has no fixed point at infinity; the set of $\Gamma$-fixed points at infinity is contained in the (possibly empty) boundary of the Euclidean factor.

The interest of point (ii) above is clear in view of all the preceding statements assuming the absence of fixed points. In addition, it is already a first indication that the mere existence of a (finitely generated) lattice is a serious restriction on a proper CAT(0) space even within the class of cocompact minimal spaces. We recall that E. Heintze [Hei74] produced simply connected negatively curved Riemannian manifolds that are homogeneous (in particular, cocompact) but have a point at infinity fixed by all isometries.
Since a CAT(0) lattice consists of a group and a space, there are two natural notions of irreducibility: of the group or of the space. In the case of lattices in semi-simple groups, the two notions are known to coincide by a result of Margulis [Mar91, II.6.7]. We prove that this is the case for CAT(0) lattices as above.

**Theorem 1.22.** In the setting of Theorem 1.21, $\Gamma$ is irreducible as an abstract group if and only if for any finite index subgroup $\Gamma_1$ and any $\Gamma_1$-equivariant decomposition $X = X_1 \times X_2$ with $X_i$ non-compact, the projection of $\Gamma_1$ to both $\text{Is}(X_i)$ is non-discrete.

The combination of Theorem 1.22, Theorem 1.21 and of an appropriate form of superrigidity allow us to give a CAT(0) version of Mostow rigidity for reducible spaces (Section 9.C).

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**Geometric arithmeticity.** We now expose results giving perhaps unexpectedly strong conclusions for CAT(0) lattices — both for the group and for the space. These results were announced in [CM08] in the case of CAT(0) groups; the present setting of finitely generated lattices is more general since CAT(0) groups are finitely generated (cf. Lemma 8.3 below).

We recall that an isometry $g$ is parabolic if the translation length $\inf_{x \in X} d(gx, x)$ is not achieved. For general CAT(0) spaces, parabolic isometries are not well understood; in fact, ruling out their existence can sometimes be the essential difficulty in rigidity statements.

**Theorem 1.23.** Let $(\Gamma, X)$ be an irreducible finitely generated CAT(0) lattice with $X$ geodesically complete. Assume that $X$ possesses some parabolic isometry.

If $\Gamma$ is residually finite, then $X$ is a product of symmetric spaces and Bruhat–Tits buildings. In particular, $\Gamma$ is an arithmetic lattice unless $X$ is a real or complex hyperbolic space.

If $\Gamma$ is not residually finite, then $X$ still splits off a symmetric space factor. Moreover, the finite residual $\Gamma_D$ of $\Gamma$ is infinitely generated and $\Gamma/\Gamma_D$ is an arithmetic group.

(Recall that the finite residual of a group is the intersection of all finite index subgroups.)

We single out a purely geometric consequence.

**Corollary 1.24.** Let $(\Gamma, X)$ be a finitely generated CAT(0) lattice with $X$ geodesically complete.

Then $X$ possesses a parabolic isometry if and only if $X \cong M \times X'$, where $M$ is a symmetric space of non-compact type.

The above results fail without the assumption of geodesic completeness, see Section 11.C. Nevertheless, we still obtain an arithmeticity statement when the underlying space admits some parabolic isometry that is neutral, i.e. whose displacement length vanishes. Neutral parabolic isometries are even less understood, not even for their dynamical properties (which can be completely wild at least in Hilbert space [Ede64]); as for familiar examples, they are provided by unipotent elements in semi-simple algebraic groups.

**Theorem 1.25.** Let $(\Gamma, X)$ be an irreducible finitely generated CAT(0) lattice. If $X$ admits any neutral parabolic isometry, then either:

1. $\text{Is}(X)$ is a rank one simple Lie group with trivial centre; or:
2. $\Gamma$ has a normal subgroup $\Gamma_D$ such that $\Gamma/\Gamma_D$ is an arithmetic group. Moreover, $\Gamma_D$ is either finite or infinitely generated.

We turn to another type of statement of arithmeticity/geometric superrigidity. Having established an abstract arithmeticity theorem (presented below as Theorem 1.29), we can appeal to our geometric results and prove the following.
Theorem 1.26. Let $\Gamma, X$ be an irreducible finitely generated $\text{CAT}(0)$ lattice with $X$ geodesically complete. Assume that $\Gamma$ possesses some faithful finite-dimensional linear representation (in characteristic $\neq 2, 3$).

If $X$ is reducible, then $\Gamma$ is an arithmetic lattice and $X$ is a product of symmetric spaces and Bruhat-Tits buildings.

Section 11 contains more results of this nature but also demonstrates by a family of examples that some of the intricacies in the more detailed statements reflect indeed the existence of more exotic pairs $(\Gamma, X)$.

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Unique geodesic extension. Complete simply connected Riemannian manifolds of non-positive curvature, sometimes also called Hadamard manifolds, form a classical family of proper $\text{CAT}(0)$ spaces to which the preceding results may be applied. In fact, the natural class to consider in our context consists of those proper $\text{CAT}(0)$ spaces in which every geodesic segment extends uniquely to a bi-infinite geodesic line. Clearly, this class contains all Hadamard manifolds, but it presumably contains more examples. It is, however, somewhat restricted with respect to the main thrust of the present work since it does not allow for, say, simplicial complexes; accordingly, the conclusions of the theorem below are also more stringent.

Theorem 1.27. Let $X$ be a proper $\text{CAT}(0)$ space with uniquely extensible geodesics. Assume that $\text{Is}(X)$ acts cocompactly without fixed points at infinity.

(i) If $X$ is irreducible, then either $X$ is a symmetric space or $\text{Is}(X)$ is discrete.

(ii) If $\text{Is}(X)$ possesses a finitely generated non-uniform lattice $\Gamma$ which is irreducible as an abstract group, then $X$ is a symmetric space (without Euclidean factor).

(iii) Suppose that $\text{Is}(X)$ possesses a finitely generated lattice $\Gamma$ (if $\Gamma$ is uniform, this is equivalent to the condition that $\Gamma$ is a discrete cocompact group of isometries of $X$). If $\Gamma$ is irreducible (as an abstract group) and $X$ is reducible, then $X$ is a symmetric space (without Euclidean factor).

In the special case of Hadamard manifolds, statement (i) was known under the assumption that $\text{Is}(X)$ satisfies the duality condition (without assuming that $\text{Is}(X)$ acts cocompactly without fixed points at infinity). This is due to P. Eberlein (Proposition 4.8 in [Ebe82]). Likewise, statement (iii) for manifolds is Proposition 4.5 in [Ebe82].

More recently, Farb-Weinberger [FW06] investigated analogous questions for aspherical manifolds.

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Groups without geometry. In the course of this work, we are led to establish some purely group-theoretical results independently of any $\text{CAT}(0)$ considerations; here are a few examples.

Let us agree that a topological group is monolithic with monolith $L$ if the intersection $L$ of all non-trivial closed normal subgroups is itself non-trivial. If moreover the monolith is cocompact, then of course every (non-trivial) closed normal subgroup is cocompact, a property sometimes called “just non-compactness” in reference to just infinite groups. The case of discrete groups of the following statement was established by J. Wilson [Wil71].
Proposition 1.28. Let $G$ be a compactly generated non-compact locally compact group such that every non-trivial closed normal subgroup is cocompact.

Then either $G$ is monolithic or discrete and residually finite. In the former case, the monolith is either $\mathbb{R}^n$ or the direct product of finitely many isomorphic topologically simple groups.

We also study irreducible lattices in products of general topological groups in the abstract and establish the following arithmeticity statement.

Theorem 1.29. Let $\Gamma < G = G_1 \times \cdots \times G_n$ be an irreducible finitely generated lattice, where each $G_i$ is any locally compact group.

If $\Gamma$ admits a faithful Zariski-dense representation in a semi-simple group over some field of characteristic $\neq 2,3$, then the amenable radical $R$ of $G$ is compact and the quasi-centre $\text{QZ}(G)$ is virtually contained in $\Gamma \cdot R$. Furthermore, upon replacing $G$ by a finite index subgroup, the quotient $G/R$ splits as $G^+ \times \mathcal{Z}(G/R)$ where $G^+$ is a semi-simple algebraic group and the image of $\Gamma$ in $G^+$ is an arithmetic lattice.

In shorter terms, this theorem states that up to a compact extension, $G$ is the direct product of a semi-simple algebraic group by a (possibly trivial) discrete group, and that the image of $\Gamma$ in the non-discrete part is an arithmetic group. The assumption on the characteristic can be slightly weakened.

In the course of the proof, we characterise all irreducible finitely generated lattices in products of the form $G = S \times D$ where $S$ is a semi-simple Lie group and $D$ a totally disconnected group (Theorem 10.17). In particular, it turns out that $D$ must necessarily be locally profinite by analytic. The corresponding question for simple algebraic groups instead of Lie groups is also investigated (Theorem 10.19).

1.A. Notation. A metric space is proper if every closed ball is compact.

We refer to Bridson and Haefliger [BH99] for background on CAT(0) spaces. We recall that the comparison angle $\angle_p(x,y)$ determined by three points $p, x, y$ in any metric space is defined purely in terms of the corresponding three distances by looking at the corresponding Euclidean triangle. In other words, it is defined by

$$d^2(x, y) = d^2(p, x) + d^2(p, y) - 2d(p, y)d(p, y) \cos \angle_p(x, y).$$

The Alexandrov angle $\angle_p(x, y)$ in a CAT(0) space $X$ is the non-increasing limit of the comparison angle near $p$ along the geodesic segments $[p,x]$ and $[p,y]$, see [BH99, II.3.1]. In particular, $\angle_p(x, y) \leq \angle_p(x, y)$. Likewise, geodesic rays from $p$ determine the Alexandrov angle $\angle_p(\xi, \eta)$ for $\xi, \eta \in \partial X$. The Tits angle $\angle_T(\xi, \eta)$ is defined as the supremum of $\angle_p(\xi, \eta)$ over all $p \in X$ and has several useful characterisations given in Proposition II.9.8 of [BH99].

Recall that to any point at infinity $\xi \in \partial X$ is associated the Busemann function

$$B_\xi : X \times X \to \mathbb{R} : (x, y) \mapsto B_\xi(x)$$

defined by $B_\xi(x) = \lim_{t \to \infty} (d(q(t), y) - d(q(t), x))$, where $q : [0, \infty) \to X$ is any geodesic ray pointing towards $\xi$. The Busemann function does not depend on the choice of $q$ and
satisfies the following:

\begin{align*}
B_{\xi,x}(y) &= -B_{\xi,y}(x) \\
B_{\xi,x}(z) &= B_{\xi,x}(y) + B_{\xi,y}(z) \quad \text{(the "cocycle relation")} \\
B_{\xi,x}(y) &\leq d(x, y).
\end{align*}

Combining the definition of the Busemann function and of the comparison angle, we find that if \( r \) is the geodesic ray pointing towards \( \xi \) with \( r(0) = x \), then for any \( y \neq x \) we have

\[
\lim_{t \to \infty} \cos \angle x(r(t), y) = -\frac{B_{\xi,x}(y)}{d(x, y)} \quad \text{(the "asymptotic angle formula")}.
\]

By abuse of language, one refers to a Busemann function when it is more convenient to consider the convex 1-Lipschitz function \( b_\xi : X \to \mathbb{R} \) defined by \( B_{\xi,x} \) for some (usually implicit) choice of base-point \( x \in X \). We shall simply denote such a function by \( b_\xi \) in lower case; they all differ by a constant only in view of the cocycle relation.

The boundary at infinity \( \partial X \) is endowed with the cône topology [BH99, II.8.6] as well as with the (much finer) topology defined by the Tits angle. The former is often implicitly understood, but when referring to dimension or radius, the topology and distance defined by the Tits angle are considered (this is sometimes emphasised by referring to the "Tits boundary"). The later distance is not to be confused with the associated length metric called "Tits distance" in the literature; we will not need this concept (except in the discussions at the beginning of Section 6).

Recall that any complete CAT(0) space splits off a canonical maximal Hilbertian factor (Euclidean in the proper case studied here) and any isometry decomposes accordingly, see Theorem II.6.15(6) in [BH99].

Normalisers and centralisers in a group \( G \) are respectively denoted by \( N_G \) and \( Z_G \). When some group \( G \) acts on a set and \( x \) is a member of this set, the stabiliser of \( x \) in \( G \) is denoted by \( \text{Stab}_G(x) \) or by the shorthand \( G_x \). For the notation regarding algebraic groups, we follow the standard notation as in [Mar91].

Finally, we present two remarks that will never be used below but give some context on certain frequent assumptions.

The first remark is the following CAT(0) version of the Hopf–Rinow theorem: every geodesically complete locally compact CAT(0) space is proper. Surprisingly, we could not find this statement in the literature (though a different statement is often referred to as the Hopf–Rinow theorem, see [BH99, I.3.7]). As pointed out orally by A. Lytchak, the above result is readily established by following the strategy of proof of [BH99, I.3.7] and extending geodesics.

The second fact is that if a proper CAT(0) space is finite-dimensional (in the sense of [Kle99]), then so is its Tits boundary (generalising for instance Proposition 6.11 below). The argument (which will be found in [CL]) goes as follows. For any sphere \( S \) in the space \( X \), the "visual map" \( \partial X \to S \) is Tits-continuous; if it were injective, the result would follow. However, it becomes injective after replacing \( S \) with the ultraproduction of spheres of unbounded radius by the very definition of the boundary; the ultraproduct construction preserves the bound on the dimension, finishing the proof.
Part I. Structure theory of the isometry group

2. Convex subsets of the Tits boundary

2.A. Boundary subsets of small radius. Given a metric space $X$ and a subset $Z \subseteq X$, one defines the circumradius of $Z$ in $X$ as

$$\inf_{x \in X} \sup_{z \in Z} d(x, z).$$

A point $x$ realising the infimum is called a circumcentre of $Z$ in $X$. The intrinsic circumradius of $Z$ is its circumradius in $Z$ itself; one defines similarly an intrinsic circumcentre. It is called canonical if it is fixed by every isometry of $X$ which stabilises $Z$. We shall make frequent use of the following construction of circumcentres, due to A. Balser and A. Lytchak [BL05, Proposition 1.4]:

**Proposition 2.1.** Let $X$ be a complete CAT(1) space and $Y \subseteq X$ be a finite-dimensional closed convex subset. If $Y$ has intrinsic circumradius $\leq \pi/2$, then the set $C(Y)$ of intrinsic circumcentres of $Y$ has a unique circumcentre, which is therefore a canonical (intrinsic) circumcentre of $Y$.  

Let now $X$ be a proper CAT(0) space.

**Proposition 2.2.** Let $X \supset X_1 \supset \ldots$ be a nested sequence of non-empty closed convex subsets of $X$ such that $\bigcap_n X_n$ is empty. Then the intersection $\bigcap_n \partial X_n$ is a non-empty closed convex subset of $\partial X$ of intrinsic circumradius at most $\pi/2$.

In particular, if the Tits boundary is finite-dimensional, then $\bigcap_n \partial X_n$ has a canonical intrinsic circumcentre.

**Proof.** Pick any $x \in X$ and let $x_n$ be its projection to $X_n$. The assumption $\bigcap_n X_n = \emptyset$ implies that $x_n$ goes to infinity. Upon extracting, we can assume that it converges to some point $\xi \in \partial X$; observe that $\xi \in \bigcap_n \partial X_n$. We claim that any $\eta \in \bigcap_n \partial X_n$ satisfies $\angle_T(\xi, \eta) \leq \pi/2$. The proposition then follows because (i) the boundary of any closed convex set is closed and $\pi$-convex [BH99, II.9.13] and (ii) each $\partial X_n$ is non-empty since otherwise $X_n$ would be bounded, contradicting $\bigcap_n X_n = \emptyset$. When $\partial X$ has finite dimension, there is a canonical intrinsic circumcentre by Proposition 2.1.

For the claim, observe that there exists a sequence of points $y_n \in X_n$ converging to $\eta$. It suffices to prove that the comparison angle $\angle_x(x_n, y_n)$ is bounded by $\pi/2$ for all $n$, see [BH99, II.9.16]. This follows from

$$\angle_{x_n}(x, y_n) \geq \angle_{x}(x, y_n) \geq \pi/2,$$

where the second inequality holds by the properties of the projection on a convex set [BH99, II.2.4(3)].

The combination of the preceding two propositions has the following consequence, which improves the results established by Fujiwara, Nagano and Shioya (Theorems 1.1 and 1.3 in [FNS06]).

**Corollary 2.3.** Let $g$ be a parabolic isometry of $X$. The following assertions hold:

(i) The fixed point set of $g$ in $\partial X$ has intrinsic circumradius at most $\pi/2$.

(ii) If $\partial X$ finite-dimensional, then the centraliser $\mathcal{Z}(g)$ has a canonical global fixed point in $\partial X$.

(iii) For any subgroup $H < \text{Is}(X)$ containing $g$, the (possibly empty) fixed point set of $H$ in $\partial X$ has circumradius at most $\pi/2$.  

Here is another immediate consequence.

**Corollary 2.4.** Let $G$ be a topological group with a continuous action by isometries on $X$ without global fixed point. Suppose that $G$ is the union of an increasing sequence of compact subgroups and that $\partial X$ is finite-dimensional. Then there is a canonical $G$-fixed point in $\partial X$, fixed by all isometries normalising $G$.

**Proof.** Consider the sequence of fixed point sets $X^{K_n}$ of the compact subgroups $K_n$. Its intersection is empty by assumption and thus Proposition 2.2 applies. □

Finally, we record the following elementary fact, which may also be deduced by means of Proposition 2.2:

**Lemma 2.5.** Let $\xi \in \partial X$. Given any closed horoball $B$ centred at $\xi$, the boundary $\partial B$ coincides with the ball of Tits radius $\pi/2$ centred at $\xi$ in $\partial X$.

**Proof.** Any two horoballs centred at the same point at infinity lie at bounded Hausdorff distance from one another. Therefore, they have the same boundary at infinity. In particular, the boundary $\partial B$ of the given horoball coincides with the intersection of the boundaries of all horoballs centred at $\xi$. By Proposition 2.2, this is of circumradius at most $\pi/2$; in fact the proof of that proposition shows precisely that the set is contained in the ball of radius at most $\pi/2$ around $\xi$.

Conversely, let $\eta \in \partial X$ be a point which does not belong to $\partial B$. We claim that $\angle_T(\xi, \eta) \geq \pi/2$. This shows that every point of $\partial X$ at Tits distance less than $\pi/2$ from $\xi$ belongs to $\partial B$. Since the latter is closed, it follows that $\partial B$ contains the closed ball of Tits radius $\pi/2$.

We turn to the claim. Let $b_\xi$ be a Busemann function centred at $\xi$. Since every geodesic ray pointing towards $\eta$ escapes every horoball centred at $\xi$, there exists a ray $\varrho : [0, \infty) \to X$ pointing to $\eta$ such that $b_\xi(\varrho(0)) = 0$ and $b_\xi(\varrho(t)) > 0$ for all $t > 0$ (actually, this increases to infinity by convexity). Let $c : [0, \infty) \to X$ be the geodesic ray emanating from $\varrho(0)$ and pointing to $\xi$. We have $\angle_T(\xi, \eta) = \lim_{t \to \infty} Z_{\varrho(0)}(\varrho(t), c(s))$, see [BH99, II.9.8]. Therefore the claim follows from the asymptotic angle formula (Section 1.A) by taking $y = c(s)$ with $s$ large enough. □

### 2.B. Subspaces with boundary of large radius

As before, let $X$ be a proper CAT(0) space. The following result improves Proposition 2.2 in [Lee00]:

**Proposition 2.6.** Let $Y \subseteq X$ be a closed convex subset such that $\partial Y$ has intrinsic circumradius $> \pi/2$. Then there exists a closed convex subset $Z \subseteq X$ with $\partial Z = \partial Y$ which is minimal for these properties. Moreover, the union $Z_0$ of all such minimal subspaces is closed, convex and splits as a product $Z_0 \cong Z \times Z'$.

**Proof.** If no minimal such $Z$ existed, there would be a chain of such subsets with empty intersection. The distance to a base-point must then go to infinity and thus the chain contains a countable sequence to which we apply Proposition 2.2, contradicting the assumption on the circumradius.

Let $Z'$ denote the set of all such minimal sets and $Z_0 = \bigcup Z'$ be its union. As in [Lee00, p. 10] one observes that for any $Z_1, Z_2 \in Z'$, the distance $z \mapsto d(z, Z_2)$ is constant on $Z_1$ and that the nearest point projection $p_{Z_2}$ restricted to $Z_1$ defines an isometry $Z_1 \to Z_2$. By the Sandwich Lemma [BH99, II.2.12], this implies that $Z_0$ is convex and that the map $Z' \times Z' \to R_+ : (Z_1, Z_2) \mapsto d(Z_1, Z_2)$ is a geodesic metric on $Z'$. As in [Mon06, Section 4.3], this yields a bijection $\alpha : Z_0 \to Z \times Z' : x \mapsto (p_{Z}(x), Z_x)$, where $Z_x$ is the unique element of $Z'$ containing $x$. The product of metric spaces $Z \times Z'$ is given the product metric. In order
to establish that $\alpha$ is an isometry, it remains as in [Mon06, Proposition 38], to trivialise “holonomy”; it the current setting, this is achieved by Lemma 2.7, which thus concludes the proof of Proposition 2.6. (Notice that $Z_0$ is indeed closed since otherwise we could extend $\alpha^{-1}$ to the completion of $Z \times Z'$.)

\textbf{Lemma 2.7.} For all $Z_1, Z_2, Z_3 \in Z'$, we have $p_{Z_1} \circ p_{Z_2} \circ p_{Z_3}|_{Z_1} = \text{Id}_{Z_1}$.

\textbf{Proof of Lemma 2.7.} Let $\vartheta : Z_1 \to Z_1$ be the isometry defined by $p_{Z_1} \circ p_{Z_2} \circ p_{Z_3}|_{Z_1}$ and let $f$ be its displacement function. Then $f : Z_1 \to \mathbb{R}$ is a non-negative convex function which is bounded above by $d(Z_1, Z_2) + d(Z_2, Z_3) + d(Z_3, Z_1)$. In particular, the restriction of $f$ to any geodesic ray in $Z_1$ is non-increasing. Therefore, a sublevel set of $f$ is a closed convex subset $Z$ of $Z_1$ with full boundary, namely $\partial Z = \partial Z_1$. By definition, the subspace $Z_1$ is minimal with respect to the property that $\partial Z_1 = \partial Y$ and hence we deduce $Z = Z_1$. It follows that the convex function $f$ is constant. In other words, the isometry $\vartheta$ is a Clifford translation. If it is not trivial, then $Z_1$ would contain a $\vartheta$-stable geodesic line on which $\vartheta$ acts by translation. But by [BH99, Lemma II.2.15], the restriction of $\vartheta$ to any geodesic line is the identity. Therefore $\vartheta$ is trivial, as desired. □

Let $\Gamma$ be a group acting on $X$ by isometries. Following [Mon06, Definition 5], we say that the $\Gamma$-action is \textit{reduced} if there is no unbounded closed convex subset $Y \subset X$ such that $gY$ is at finite Hausdorff distance from $Y$ for all $g \in \Gamma$.

\textbf{Corollary 2.8.} Let $X$ be a proper irreducible CAT(0) space with finite-dimensional Tits boundary, and $\Gamma < \text{Is}(X)$ be a subgroup acting minimally without fixed point at infinity. Then the $\Gamma$-action is reduced.

\textbf{Proof.} Suppose for a contradiction that the $\Gamma$-action on $X$ is not reduced. Then there exists an unbounded closed convex subset $Y \subset X$ such that $gY$ is at finite Hausdorff distance from $Y$ for all $g \in \Gamma$. In particular $\partial Y$ is $\Gamma$-invariant. By Proposition 2.1, it must have intrinsic circumradius $> \pi/2$. Proposition 2.6 therefore yields a canonical closed convex subset $Z_0 = Z \times Z'$ with $\partial(Z \times \{z'\}) = \partial Y$ for all $z' \in Z'$; clearly $Z_0$ is $\Gamma$-invariant and hence we have $Z_0 = X$ by minimality. Since $X$ is irreducible by assumption, we deduce $X = Z$ and hence $X = Y$, as desired. □

2.C. \textbf{Minimal actions and boundary-minimal spaces.} Boundary-minimality and minimality, as defined in the Introduction, are two possible ways for a CAT(0) space to be “non-degenerate”, as illustrated by the following.

\textbf{Lemma 2.9.} Let $X$ be a complete CAT(0) space.

\begin{enumerate}[(i)]
\item A group $G < \text{Is}(X)$ acts minimally if and only if any continuous convex $G$-invariant function on $X$ is constant.
\item If $X$ is boundary-minimal then any bounded convex function on $X$ is constant.
\end{enumerate}

\textbf{Proof.} Necessity in the first assertion follows immediately by considering sub-level sets (see [Mon06, Lemma 37]). Sufficiency is due to the fact that the distance to a closed convex set is a convex continuous function [BH99, II.2.5]. The second assertion was established in the proof of Lemma 2.7. □

Proposition 2.6 has the following important consequence:

\textbf{Corollary 2.10.} Let $X$ be a proper CAT(0) space. If $\partial X$ has circumradius $> \pi/2$, then $X$ possesses a canonical closed convex subspace $Y \subset X$ such that $Y$ is boundary-minimal and $\partial Y = \partial X$. 
Proof. Let $Z_0 = Z \times Z'$ be the product decomposition provided by Proposition 2.6. The group $\text{Is}(X)$ permutes the elements of $Z'$ and hence acts by isometries on $Z'$. Under the present hypotheses, the space $Z'$ is bounded since $\partial Z = \partial X$. Therefore it has a circumcentre $z'$, and the fibre $Y = Z \times \{z'\}$ is thus $\text{Is}(X)$-invariant. \hfill $\square$

**Proposition 2.11.** Let $X$ be a proper CAT(0) space which is minimal. Assume that $\partial X$ has finite dimension. Then $\partial X$ has circumradius $> \pi/2$ (unless $X$ is reduced to a point). In particular, $X$ is boundary-minimal.

The proof of Proposition 2.11 requires some preliminaries. Given a point at infinity $\xi$, consider the Busemann function $B_\xi$; the cocycle property (recalled in Section 1.A) implies in particular that for any isometry $g \in \text{Is}(X)$ fixing $\xi$ and any $x \in X$ the real number $B_{\xi,x}(g.x)$ is independent on the choice of $x$ and yields a canonical homomorphism

$$\beta_\xi : \text{Is}(X)_\xi \rightarrow \mathbb{R} : g \mapsto B_{\xi,x}(g.x)$$

called the **Busemann character** centred at $\xi$.

Given an isometry $g$, it follows by the CAT(0) property that $\inf_{n \geq 0} d(g^n x, x)/n$ coincides with the translation length of $g$ independently of $x$. We call an isometry **ballistic** when this number is positive. An important fact about a ballistic isometry $g$ of any complete CAT(0) space $X$ is that for any $x \in X$ the sequence $\{g^n x\}_{n \geq 0}$ converges to a point $\eta_x \in \partial X$ independent of $x$; $\eta_x$ is called the (canonical) **attracting fixed point** of $g$ in $\partial X$. Moreover, this convergence holds also in angle, which means that $\lim Z_x(g^n x, r(t))$ vanishes as $n, t \rightarrow \infty$ when $r : \mathbb{R}_+ \rightarrow X$ is any ray pointing to $\eta_x$. This is a (very) special case of the results in [KM99].

**Lemma 2.12.** Let $\xi \in X$ and $g \in \text{Is}(X)_\xi$ be an isometry which is not annihilated by the Busemann character centred at $\xi$. Then $g$ is ballistic. Furthermore, if $\beta_\xi(g) > 0$ then $\angle_T(\xi, \eta_x) > \pi/2$.

Proof. We have $\beta_\xi(g) = B_{\xi,x}(g.x) \leq d(x, g.x)$ for all $x \in X$. Thus $g$ is ballistic as soon as $\beta_\xi(g)$ is non-zero.

Assume $\beta_\xi(g) > 0$ and suppose for a contradiction that $\angle_T(\xi, \eta_x) \leq \pi/2$. Choose $x \in X$ and let $g, \sigma$ be the rays issuing from $x$ and pointing towards $\xi$ and $\eta_x$ respectively. Recall from [BH99, II.9.8] that $\angle_T(\xi, \eta_x) = \lim \angle(\eta_x(g(s)), \sigma(s))$. The convergence in direction of $g^n x$ implies that this angle is also given by $\lim Z_x(g(t), g^n x)$. Since $\beta_\xi(g) > 0$ we can fix $n$ large enough to have

$$\cos \liminf_{t \rightarrow \infty} Z_x(g(t), g^n x) > -\frac{\beta_\xi(g)}{d(gx, x)}.$$

We now apply the asymptotic angle formula from Section 1.A with $y = g^n x$ and deduce that the left hand side is $-\beta_\xi(g^n x)/d(g^n x, x)$. Since $\beta_\xi(g^n x) = n\beta_\xi(gx)$ and $d(g^n x, x) \leq nd(gx, x)$, we have a contradiction. \hfill $\square$

**Proof of Proposition 2.11.** We can assume that $\partial X$ is non-empty since otherwise $X$ is a point by minimality. Suppose for a contradiction that its circumradius is $\leq \pi/2$. Then $\text{Is}(X)$ possesses a global fixed point $\xi \in \partial X$ and $\xi$ is a circumcentre of $\partial X$, see Proposition 2.1. Lemma 2.12 implies that $\text{Is}(X) = \text{Is}(X)_\xi$ is annihilated by the Busemann character centred at $\xi$. Thus $\text{Is}(X)$ stabilises every horoball, contradicting minimality. \hfill $\square$

We shall use repeatedly the following elementary fact.
Lemma 2.13. Let $G$ be a group with an isometric action on a proper geodesically complete CAT(0) space $X$. If $G$ acts cocompactly or more generally has full limit set, then the action is minimal. (This holds more generally when $\Delta G = \partial X$ in the sense of Section 3.B below.)

Proof. Let $Y \subseteq X$ be a non-empty closed convex invariant subset, choose $y \in Y$ and suppose for a contradiction that there is $x \not\in Y$. Let $r : \mathbb{R}_+ \to X$ be a geodesic ray starting at $y$ and going through $x$. By convexity [BH99, II.2.5(1)], the function $d(r(t), Y)$ tends to infinity and thus $r(\infty) \not\in \partial Y$. This is absurd since $\Delta G \subseteq \partial Y$. □

Proof of Proposition 1.3. (i) See Proposition 2.11.

(ii) Since $\text{Is}(X)$ has full limit set, any $\text{Is}(X)$-invariant subspace has full boundary. Minimality follows, since boundary-minimality ensures that $X$ possesses no proper subspace with full boundary.

(iii) $X$ is minimal by Lemma 2.13, hence boundary-minimal by (i), since any cocompact space has finite-dimensional boundary by [Kle99, Theorem C]. □

3. Minimal invariant subspaces for subgroups

3.A. Existence of a minimal invariant subspace. For the record, we recall the following elementary dichotomy; a refinement will be given in Theorem 3.3 below:

Proposition 3.1. Let $G$ be a group acting by isometries on a proper CAT(0) space $X$. Then either $G$ has a global fixed point at infinity, or any filtering family of non-empty closed convex $G$-invariant subsets has non-empty intersection.

(Recall that a family of sets is filtering if it is directed by containment $\supseteq$.)

Proof. (Remark 36 in [Mon06].) Suppose $\mathcal{Y}$ is such a family, choose $x \in X$ and let $x_Y$ be its projection on each $Y \in \mathcal{Y}$. If the net $\{x_Y\}_{Y \in \mathcal{Y}}$ is bounded, then $\bigcap_{Y \in \mathcal{Y}} Y$ is non-empty. Otherwise it goes to infinity and any accumulation point in $\partial X$ is $G$-fixed in view of $d(gx_Y, x_Y) \leq d(gx, x)$. □

3.B. Dichotomy. Let $G$ be a group acting by isometries on a complete CAT(0) space $X$.

Lemma 3.2. Given any two $x, y \in X$, the convex closures of the respective $G$-orbits of $x$ and $y$ in $X$ have the same boundary in $\partial X$.

Proof. Let $Y$ be the convex closure of the $G$-orbit of $x$. In particular $Y$ is the minimal closed convex $G$-invariant subset containing $x$. Given any closed convex $G$-invariant subset $Z$, let $r = d(x, Z)$. Recall that the tubular closed neighbourhood $N_r(Z)$ is convex [BH99, II.2.5(1)]. Since it is also $G$-invariant and contains $x$, the minimality of $Y$ implies $Y \subseteq N_r(Z)$. □

This yields a canonical closed convex $G$-invariant subset of the boundary $\partial X$, which we denote by $\Delta G$. It contains the limit set $\Delta G$ but is sometimes larger.

Combining what we established thus far with the splitting arguments from [Mon06], we obtain a dichotomy:

Theorem 3.3. Let $G$ be a group acting by isometries on a complete CAT(0) space $X$ and $H \leq G$ any subgroup.

If $H$ admits no minimal non-empty closed convex invariant subset and $X$ is proper, then:

(A.i) $\Delta H$ is a non-empty closed convex subset of $\partial X$ of intrinsic circumradius at most $\pi/2$.

(A.ii) If $\partial X$ is finite-dimensional, then the normaliser $N_G(H)$ of $H$ in $G$ has a global fixed point in $\partial X$. 

If \( H \) admits a minimal non-empty closed convex invariant subset \( Y \subseteq X \), then:

(B.i) The union \( Z \) of all such subsets is a closed convex \( \mathcal{N}_G(H) \)-invariant subset.

(B.ii) \( Z \) splits \( H \)-equivariantly and isometrically as a product \( Z \simeq Y \times C \), where \( C \) is a complete \( \text{CAT}(0) \) space which admits a canonical \( \mathcal{N}_G(H)/H \)-action by isometries.

(B.iii) If the \( H \)-action on \( X \) is non-evanescent, then \( C \) is bounded and there is a canonical minimal non-empty closed convex \( H \)-invariant subset which is \( \mathcal{N}_G(H) \)-stable.

(When \( X \) is proper, the non-evanescent condition of (iii) simply means that \( H \) has no fixed point in \( \partial X \); see [Mon06].)

**Proof.** In view of Lemma 3.2, the set \( \Delta H \) is contained in the boundary of any non-empty closed convex \( H \)-invariant set and is \( \mathcal{N}_G(H) \)-invariant. Thus the assertions (A.i) and (A.ii) follow from Proposition 2.2, noticing that in a proper space \( \Delta H \) is non-empty unless \( H \) has bounded orbits, in which case it fixes a point, providing a minimal subspace. For (B.i), (B.ii) and (B.iii), see Remarks 39 in [Mon06]. \( \Box \)

### 3.C. Normal subgroups.

**Proof of Theorem 1.6.** We adopt the notation and assumptions of the theorem. By (A.ii), \( N \) admits a minimal non-empty closed convex invariant subset \( Y \subseteq X \). This set is unbounded, since otherwise \( N \) fixes a point and thus by \( G \)-minimality \( X^N = X \), hence \( N = 1 \). Since \( X \) is irreducible, points (B.i) and (B.ii) show \( Y = X \) and thus \( N \) acts indeed minimally.

Since the displacement function of any \( g \in \mathcal{Z}_G(N) \) is a convex \( N \)-invariant function, it is constant by minimality. Hence \( g \) is a Clifford translation and must be trivial since otherwise \( X \) splits off a Euclidean factor, see [BH99, II.6.15].

The derived subgroup \( N' = [N,N] \) is also normal in \( G \) and therefore acts minimally by the previous discussion, noticing that \( N' \) is non-trivial since otherwise \( N \subseteq \mathcal{Z}_G(N) \). If \( N \) fixed a point at infinity, \( N' \) would preserve all corresponding horoballs, contradicting minimality.

Having established that \( N \) acts minimally and without fixed point at infinity, we can apply the splitting theorem (Corollary 10 in [Mon06]) and deduce from the irreducibility of \( X \) that \( N \) does not split.

Finally, let \( R \lhd N \) be the amenable radical and observe that it is normal in \( G \). The theorem of Adams–Ballmann [AB98a] states that \( R \) either (i) fixes a point at infinity or (ii) preserves a Euclidean flat in \( X \). (Although their result is stated for amenable groups without mentioning any topology, the proof applies indeed to every topological group that preserves a probability measure whenever it acts continuously on a compact metrisable space.) If \( R \) is non-trivial, we know already from the above discussion that (i) is impossible and that \( R \) acts minimally; it follows that \( X \) is a flat. By irreducibility and since \( X \neq R \), this forces \( X \) to be a point, contradicting \( R \neq 1 \). \( \Box \)

Corollary 1.7 will be proved in Section 4.B. For Corollary 1.8, it suffices to observe that the centraliser of any element of a discrete normal subgroup is open. Next, we recall the following definition.

A subgroup \( N \) of a group \( G \) is **ascending** if there is a family of subgroups \( N_\alpha < G \) indexed by the ordinals and such that \( N_0 = N \), \( N_\alpha \lhd N_{\alpha+1} \), \( N_\alpha = \bigcup_{\beta < \alpha} N_\beta \) if \( \alpha \) is a limit ordinal and \( N_\alpha = G \) for \( \alpha \) large enough. The smallest such ordinal is the **order**.

**Proposition 3.4.** Consider a group acting minimally by isometries on a proper \( \text{CAT}(0) \) space. Then any ascending subgroup without global fixed point at infinity still acts minimally.
Proof. We argue by transfinite induction on the order \( \vartheta \) of ascending subgroups \( N < G \), the case \( \vartheta = 0 \) being trivial. Let \( X \) be a space as in the statement. By Proposition 3.1, each \( N_\alpha \) has a minimal set. If \( \vartheta = \vartheta' + 1 \), it follows from (B.iii) that \( N_{\vartheta'} \) acts minimally and we are done by induction hypothesis. Assume now that \( \vartheta \) is a limit ordinal. For all \( \alpha \), we denote as in (B.i) by \( Z_\alpha \subseteq X \) the union of all \( N_\alpha \)-minimal sets. The induction hypothesis implies that for all \( \alpha \leq \beta < \vartheta \), any \( N_\beta \)-minimal set is \( N_\alpha \)-minimal. Thus, if \( Z_0 = Y_0 \times C_0 \) is a splitting as in (B.ii) with a \( N_{\vartheta'} \)-minimal set \( Y_0 \), we have a nested family of decompositions \( Z_\alpha = Y_0 \times C_\alpha \) for a nested family of closed convex subspaces \( C_\alpha \) of the compact \( \text{CAT}(0) \) space \( C_0 \), indexed by \( \alpha < \vartheta \). Thus, for any \( c \in \bigcap_{\alpha < \vartheta} C_\alpha \), the space \( Y_0 \times \{c\} \) is \( G \)-invariant and hence \( Y_0 = X \) indeed. \( \square \)

Remark 3.5. Proposition 3.4 holds more generally for complete \( \text{CAT}(0) \) spaces if \( N \) is non-evanescent. Indeed Proposition 3.1 hold in that generality (Remark 36 in [Mon06]) and \( C \) remains compact in a weaker topology (Theorem 14 in [Mon06]).

Proof of Theorem 1.9. In view of Theorem 1.6, it suffices to prove that any non-trivial ascending subgroup \( N < G \) in that statement still acts minimally and without global fixed point at infinity. We argue by induction on the order \( \vartheta \) and we can assume that \( \vartheta \) is a limit ordinal by Theorem 1.6. Then \( \bigcap_{\alpha < \vartheta} (\partial X)^{N_\alpha} \) is empty and thus by compactness there is some \( \alpha < \vartheta \) such that \( (\partial X)^{N_\alpha} \) is empty. Now \( N_\alpha \) acts minimally on \( X \) by Proposition 3.4 and thus we conclude using the induction hypothesis. \( \square \)

4. Algebraic and geometric product decompositions

4.A. Preliminary decomposition of the space. We shall prepare our spaces by means of a geometric decomposition. For any geodesic metric space with finite affine rank, Foertsch–Lyttchak [FL06] established a canonical decomposition generalising the classical theorem of de Rham [dR52]. However, such a statement fails to be true for \( \text{CAT}(0) \) spaces that are merely proper, due notably to compact factors that can be infinite products. Nevertheless, using asymptotic \( \text{CAT}(0) \) geometry and Section 2.A, we can adapt the arguments from [FL06] and obtain:

**Theorem 4.1.** Let \( X \) be a proper \( \text{CAT}(0) \) space with \( \partial X \) finite-dimensional and of circumradius \( > \pi/2 \). Then there is a canonical closed convex subset \( Z \subseteq X \) with \( \partial Z = \partial X \), invariant under all isometries, and admitting a canonical maximal isometric splitting

\[
(4.1) \quad Z \cong \mathbb{R}^n \times Z_1 \times \cdots \times Z_m \quad (n, m \geq 0)
\]

with each \( Z_i \) irreducible and \( \neq \mathbb{R}^0, \mathbb{R}^1 \). Every isometry of \( Z \) preserves this decomposition upon permuting possibly isometric factors \( Z_i \).

**Remark 4.2.** It is well known that in the above situation the splitting (4.1) induces a decomposition

\[
\text{Is}(Z) = \text{Is}(\mathbb{R}^n) \times \left( \left( \text{Is}(Z_1) \times \cdots \times \text{Is}(Z_m) \right) \rtimes F \right),
\]

where \( F \) is the permutation group of \( \{1, \ldots, d\} \) permuting possible isometric factors amongst the \( Y_j \). Indeed, this follows from the statement that isometries preserve the splitting upon permutation of factors, see e.g. Proposition I.5.3(4) in [BH99]. Of course, this does not *a priori* mean that we have a unique, nor even canonical, splitting in the category of groups; this shall however be established for Theorem 1.1.

The hypotheses of Theorem 4.1 are satisfied in some naturally occurring situations:

**Corollary 4.3.** Let \( X \) be a proper \( \text{CAT}(0) \) space with finite-dimensional boundary.
(i) If \( \text{Is}(X) \) has no fixed point at infinity, then \( X \) possesses a subspace \( Z \) satisfying all the conclusions of Theorem 4.1.

(ii) If \( \text{Is}(X) \) acts minimally, then \( X \) admits a canonical splitting as in 4.i.

Proof of Corollary 4.3. By Proposition 2.1, if \( \text{Is}(X) \) has no fixed point at infinity, then \( \partial X \) has circumradius \( > \pi/2 \). By Proposition 2.11, the same conclusion holds if \( \text{Is}(X) \) acts minimally.

Proof of Theorems 1.5 and 4.1. For Theorem 4.1, we let \( Z \subseteq X \) be the canonical boundary-minimal subset with \( \partial Z = \partial X \) provided by Corollary 2.10; we shall not use the circumradius assumption any more. For Theorem 1.5, we let \( Z = X \). The remainder of the argument is common for both statements.

Recalling that in complete generality all isometries preserving the Euclidean factor decomposition [BH99, II.6.15], we can assume that \( Z \) has no Euclidean factor and shall obtain the decomposition (4.i) with \( n = 0 \).

Since \( Z \) is minimal amongst closed convex subsets with \( \partial Z = \partial X \), it has no non-trivial compact factor. On the other hand, any proper geodesic metric space admits some maximal product decomposition into non-compact factors. In conclusion, \( Z \) admits some maximal splitting \( Z = Z_1 \times \cdots \times Z_m \) with each \( Z_i \) irreducible and \( \neq \mathbb{R}^0, \mathbb{R}^1 \). (This can fail in presence of compact factors).

It remains to prove that any other such decomposition \( Z = Z'_1 \times \cdots \times Z'_m \), coincides with the first one after possibly permuting the factors (in particular, \( m' = m \)). We now borrow from the argumentation in [FL06], indicating the steps and the necessary changes. It is assumed that the reader has a copy of [FL06] at hand but keeps in mind that our spaces might lack the finite affine rank condition assumed in that paper. We shall replace the notion of affine subspaces with a large-scale particular case: a cône shall be any subspace isometric to a closed convex cône in some Euclidean space. This includes the particular cases of a point, a ray or a full Euclidean space.

Whenever a space \( Y \) has some product decomposition and \( Y' \) is a factor, write \( Y_y \subseteq Y \) for the corresponding fibre \( Y'_y \equiv Y' \) through \( y \in Y \). The following is an analogue of Corollary 1.2 in [FL06].

Lemma 4.4. Let \( Y \) be a proper CAT(0) space with finite-dimensional boundary and without compact factors. Suppose given two decompositions \( Y = Y_1 \times Y_2 = S_1 \times S_2 \) with all four \((Y_i)_y \cap (S_j)_y\) reduced to \( \{y\} \) for some \( y \in Y \). Then \( Y \) is a Euclidean space.

Proof of Lemma 4.4. Any \( y \in Y \) is contained in a maximal cône based at \( y \) since \( \partial Y \) has finite dimension; by abuse of language we call such cônes maximal. The arguments of Sections 3 and 4 in [FL06] show that any maximal cône is rectangular, which means that it inherit a product structure from any product decomposition of the ambient CAT(0) space. Specifically, it suffices to observe that the product of two cônes is a cône and that the projection of a cône along a product decomposition of CAT(0) spaces remains a cône. (In fact, the “equality of slopes” of Section 4.2 in [FL06], namely the fact that parallel geodesic segments in a CAT(0) space have identical slopes in product decompositions, is a general fact for CAT(0) spaces. It follows from the convexity of the metric, see for instance [Mon06, Proposition 49] for a more general statement.) The deduction of the statement of Lemma 4.4 from the rectangularity of maximal cônes following [FL06] is particularly short since all proper CAT(0) Banach spaces are Euclidean.

Lemma 4.5. For a given \( z \in Z \) and any product decomposition \( Z = S \times S' \), the intersection \((Z_i)_z \cap S_z\) is either \( \{z\} \) or \((Z_i)_z\).
Proof of Lemma 4.5. Write $P^S : Z \to S$ and $P^Z_i : Z \to Z_i$ for the projections and set $F_z = S_z \cap (Z_i)_z$. Following [FL06], define $T \subseteq Z$ by $T = P^S(F_z) \times S'$. We contend that $P^{Z_i}(T)$ has full boundary in $Z_i$.

Indeed, given any point in $\partial (Z_i)_z$, we represent it by a ray $r$ originating from $z$. We can choose a maximal $\text{cone}$ in $P$ is rectangular, and therefore the proof of Lemma 5.2 in [FL06] shows that $P^{Z_i}(r)$ lies in $P^{Z_i}(T)$, justifying our contention.

We observe that $Z_i$ inherits from $Z$ the property that it has no closed convex proper subset of full boundary. In conclusion, since $P^{Z_i}(T)$ is a convex set, it is dense in $Z$. However, according to Lemma 5.1 in [FL06], it splits as $P^{Z_i}(T) = P^{Z_i}(F_z) \times P^{Z_i}(S')$. Upon possibly replacing $P^{Z_i}(S')$ by its completion (whilst $P^{Z_i}(F_z)$ is already closed in $Z_i$ since $P^{Z_i}$ is isometric on $(Z_i)_z$), we obtain a splitting of the closure of $P^{Z_i}(T)$, and hence of $Z_i$. This completes the proof of the lemma since $Z_i$ is irreducible. \hfill $\Box$

Now the main argument runs by induction over $m \geq 2$. Lemma 4.5 identifies by induction $Z_i$ with some $Z_i'$. Indeed, Lemma 4.4 excludes that all pairwise intersections reduce to a point since $Z$ has no Euclidean factor. \hfill $\Box$

4.B. Proof of Theorem 1.1 and Addendum 1.4. The following consequence of the solution to Hilbert’s fifth problem belongs to the mathematical lore.

Theorem 4.6. Let $G$ be a locally compact group with trivial amenable radical. Then $G$ possesses a canonical finite index open normal subgroup $G^\dagger$ such that $G^\dagger = L \times D$, where $L$ is a connected semi-simple Lie group with trivial centre and no compact factors, and $D$ is totally disconnected.

Proof. This follows from the Gleason–Montgomery–Zippin solution to Hilbert’s fifth problem and the fact that connected semi-simple Lie groups have finite outer automorphism groups. More details may be found for example in [Mon01, §11.3]. \hfill $\Box$

Combining Theorem 4.6 with Theorem 1.6, we find the statement given as Corollary 1.7 in the Introduction.

Theorem 4.7. Let $X \neq R$ be an irreducible proper CAT(0) space with finite-dimensional Tits boundary and $G < Is(X)$ any closed subgroup whose action is minimal and does not have a global fixed point in $\partial X$.

Then $G$ is either totally disconnected or an almost connected simple Lie group with trivial centre.

Proof. By Theorem 1.6, $G$ has trivial amenable radical. Let $G^\dagger$ be as in Theorem 4.6. Applying Theorem 1.6 to this normal subgroup of $G$, deduce that we have either $G^\dagger = L$ with $L$ simple or $G^\dagger = D$. \hfill $\Box$

We can now complete the proof of Theorem 1.1 and Addendum 1.4 and we adopt their notation. Since $Is(X)$ has no global fixed point at infinity, there is a canonical minimal non-empty closed convex $Is(X)$-invariant subset $X' \subseteq X$ (Remarks 39 in [Mon06]). We apply Corollary 4.3 to $Z = X'$ and Remark 4.2 to $G = Is(Z)$, setting

$$G^* = Is(R^n) \times Is(Z_1) \times \cdots \times Is(Z_m).$$

All the claimed properties of the resulting factor groups are established in Theorem 1.6, Theorem 1.9 and Theorem 1.19 (the proof of which is completely independent from the present considerations). Finally, the claim that any product decomposition of $G^*$ is a regrouping of the factors in (1.i) is established as follows. Notice that the $G^*$-action on $Z$ is
still minimal and without fixed point at infinity (this is almost by definition but alternatively also follows from Theorem 1.6). Therefore, given any product decomposition of $G^*$, we can apply the splitting theorem (Corollary 10 in [Mon06]) and obtain a corresponding splitting of $Z$. Now the uniqueness of the decomposition of the space $Z$ (away from the Euclidean factor) implies that the given decomposition of $G^*$ is a regrouping of the factors occurring in Remark 4.2.

4.C. CAT(0) spaces without Euclidean factor. For the sake of future references, we record the following consequence of the results obtained thus far:

**Corollary 4.8.** Let $X$ be a proper CAT(0) space with finite-dimensional boundary and no Euclidean factor, such that $G = \text{Is}(X)$ acts minimally without fixed point at infinity. Then $G$ has trivial amenable radical and any subgroup of $G$ acting minimally on $X$ has trivial centraliser. Furthermore, given a non-trivial normal subgroup $N \leq G$, any $N$-minimal $N$-invariant closed subspace of $X$ is a regrouping of factors in the decomposition of Addendum 1.4. In particular, if each irreducible factor of $G$ is non-discrete, then $G$ has no non-trivial finitely generated closed normal subgroup.

**Proof.** The triviality of the amenable radical comes from the corresponding statement in irreducible factors of $X$, see Theorem 1.6. By the second paragraph of the proof of Theorem 1.6, any subgroup of $G$ acting minimally has trivial centraliser. The fact that minimal invariant subspaces for normal subgroups are fibres in the product decomposition (1.ii) follows since any product decomposition of $X$ is a regrouping of factors in (1.ii) and since any normal subgroup of $G$ yields such a product decomposition by Theorem 3.3(B.i) and (B.ii). Assume finally that each irreducible factor in (1.i) is non-discrete and let $N < G$ be a finitely generated closed normal subgroup. Then $N$ is discrete by Baire’s category theorem, and $N$ acts minimally on a fibre, say $Y$, of the space decomposition (1.ii). Therefore, the projection of $N$ to $\text{Is}(Y)$ has trivial centraliser, unless $N$ is trivial. Since $N$ is discrete, normal and finitely generated, its centraliser is open. Since $\text{Is}(Y)$ is non-discrete by assumption, we deduce that $N$ is trivial, as desired. □

5. Totally disconnected group actions

5.A. Smoothness. When considering actions of totally disconnected groups, a desirable property is smoothness, namely that points have open stabilisers. This condition is important in representation theory, but also in our geometric context, see point (ii) of Corollary 5.3 below and [Cap07].

In general, this condition does not hold, even for actions that are cocompact, minimal and without fixed point at infinity. An example will be constructed in Section 11.C below. However, we establish it under a rather common additional hypothesis. Recall that a metric space $X$ is called geodesically complete (or said to have extensible geodesics) if every geodesic segment of positive length may be extended to a locally isometric embedding of the whole real line. The following contains Theorem 1.15 from the Introduction.

**Theorem 5.1.** Let $G$ be a totally disconnected locally compact group with a minimal, continuous and proper action by isometries on a proper CAT(0) space $X$.

If $X$ is geodesically complete, then the action is smooth. In fact, the pointwise stabiliser of every bounded set is open.

**Remark 5.2.** In particular, the stabiliser of a point acts as a finite group of isometries on any given ball around this point in the setting of Theorem 5.1.
Corollary 5.3. Let $X$ be a proper $\text{CAT}(0)$ space and $G$ be a totally disconnected locally compact group acting continuously properly on $X$ by isometries. Then:

(i) If the $G$-action is cocompact, then every element of zero translation length is elliptic.
(ii) If the $G$-action is cocompact and every point $x \in X$ has an open stabiliser, then the $G$-action is semi-simple.
(iii) If the $G$-action is cocompact and $X$ is geodesically complete, then the $G$-action is semi-simple.

Proof of Corollary 5.3. Points (i) and (ii) follow readily from Theorem 5.1, see [Cap07, Corollary 3.3].

(iii) In view of Lemma 2.13, this follows from Theorem 5.1 and (ii).

The following is a key fact for Theorem 5.1:

Lemma 5.4. Let $X$ be a geodesically complete proper $\text{CAT}(0)$ space. Let $(C_n)_{n \geq 0}$ be an increasing sequence of closed convex subsets whose union $C = \bigcup_n C_n$ is dense in $X$.

Then every bounded subset of $X$ is contained in some $C_n$; in particular, $C = X$.

Proof. Suppose for a contradiction that for some $r > 0$ and $x \in X$ the $r$-ball around $x$ contains an element $x_n$ not in $C_n$ for each $n$. We shall construct inductively a sequence $\{c_k\}_{k \geq 1}$ of pairwise $r$-disjoint elements in $C$ with $d(x, c_k) \leq 2r + 2$, contradicting the properness of $X$.

If $c_1, \ldots, c_{k-1}$ have been constructed, choose $n$ large enough to that $C_n$ contains them all and $d(x, C_n) \leq 1$. Consider the (non-trivial) geodesic segment from $x_n$ to its nearest point projection $\overline{x_n}$ on $C_n$; by geodesic completeness, it is contained in a geodesic line and we choose $y$ at distance $r + 1$ from $C_n$ on this line. Notice that $x_n \in [\overline{x_n}, y]$ and hence $d(y, x) \leq 2r + 1$. Moreover, $d(y, c_i) \geq r + 1$ for all $i < k$. Since $C$ is dense, we can choose $c_k$ close enough to $y$ to ensure $d(c_k, x) \leq 2r + 2$ and $d(c_k, c_i) \geq r$ for all $i < k$, completing the induction step.

End of proof of Theorem 5.1. The subset $C \subseteq X$ consisting of those points $x \in X$ such that the stabiliser $G_x$ is open is clearly convex and $G$-stable. By [Bou71, III §4 No 6], the group $G$ contains a compact open subgroup and hence $C$ is non-empty. Thus $C$ is dense by minimality of the action. Since $\text{Is}(X)$ is second countable, we can choose a descending chain $Q_n < G$ of compact open subgroups whose intersection acts trivially on $X$. Therefore, $C$ may be written as the union of an ascending family of closed convex subsets $C_n \subseteq X$, where $C_n$ is the fixed point set of $Q_n$. Now the statement of the theorem follows from Lemma 5.4.

5.B. Locally finite equivariant partitions and cellular decompositions. Let $X$ be a locally finite cell complex and $G$ be its group of cellular automorphisms, endowed with the topology of pointwise convergence on bounded subsets. Then $G$ is a totally disconnected locally compact group and every bounded subset of $X$ has an open pointwise stabiliser in $G$. One of the interest of Theorem 5.1 is that it allows for a partial converse to the latter statement:

Proposition 5.5. Let $X$ be a proper $\text{CAT}(0)$ space and $G$ be a totally disconnected locally compact group acting continuously properly on $X$ by isometries. Assume that the pointwise stabiliser of every bounded subset of $X$ is open in $G$. Then we have the following:

(i) $X$ admits a canonical locally finite $G$-equivariant partition.
(ii) Denoting by $\sigma(x)$ the piece supporting the point $x \in X$ in that partition, we have $\text{Stab}_G(\sigma(x)) = \mathcal{N}_G(G_x)$ and $\mathcal{N}_G(G_x)/G_x$ acts freely on $\sigma(x)$.

(iii) If $G\backslash X$ is compact, then so is $\text{Stab}_G(\sigma(x))\backslash \sigma(x)$ for all $x \in X$.

**Proof.** Consider the equivalence relation on $X$ defined by

$$x \sim y \iff G_x = G_y.$$ 

This yields a canonical $G$-invariant partition of $X$. We need to show that it is locally finite. Assume for a contradiction that there exists a converging sequence $\{x_n\}_{n \geq 0}$ such that the subgroups $G_{x_n}$ are pairwise distinct. Let $x = \lim_n x_n$.

We claim that $G_{x_n} < G_x$ for all sufficiently large $n$. Indeed, upon extracting there would otherwise exist a sequence $g_n \in G_{x_n}$ such that $g_n x \neq x$ for all $n$. Upon a further extraction, we may assume that $g_n$ converges to some $g \in G$. By construction $g$ fixes $x$. Since $G_x$ is open by hypothesis, this implies that $g_n$ fixes $x$ for sufficiently large $n$, a contradiction. This proves the claim.

By hypothesis the pointwise stabiliser of any ball centred at $x$ is open. Thus $G_x$ possesses a compact open subgroup $U$ which fixes every $x_n$. This implies that we have the inclusion $U < G_{x_n} < G_x$ for all $n$. Since the index of $U$ in $G_x$ is finite, there are only finitely many subgroups of $G_x$ containing $U$. This final contradiction finishes the proof of (i).

(ii) Straightforward in view of the definitions.

(iii) Suppose for a contradiction that $H\backslash \sigma(x)$ is not compact, where $H = \text{Stab}_G(\sigma(x))$. Let then $y_n \in \sigma(x)$ be a sequence such that $d(y_n,H.x) > n$. Let now $g_n \in G$ be such that $\{g_n y_n\}$ is bounded, say of diameter $C$. By (i), the set $\{g_n G_{y_n}g_n^{-1}\}$ is thus finite. Upon extracting, we shall assume that it is constant. Now, for all $n < k$, the element $g_n^{-1}y_k$ normalises $G_{y_k} = G_x$ and maps $y_k$ to a point at distance $\leq C$ from $y_n$. In view of (ii), this is absurd.

**Remark 5.6.** The partition of $X$ constructed above is non-trivial whenever $G$ does not act freely. This is for example the case whenever $G$ is non-discrete and acts faithfully.

The pieces in the above partition are generally neither bounded (even if $G\backslash X$ is compact), nor convex, nor even connected. However, if one assumes that the space admits a sufficiently large amount of symmetry, then one obtains a partition which deserves to be viewed as an equivariant cellular decomposition.

**Corollary 5.7.** Let $X$ be a proper CAT(0) space and $G$ be a totally disconnected locally compact group acting continuously properly on $X$ by isometries. Assume that the pointwise stabiliser of every bounded subset of $X$ is open in $G$, and that no open subgroup of $G$ fixes a point at infinity. Then $X$ admits a canonical locally finite $G$-equivariant decomposition into compact convex pieces.

**Proof.** For each $x \in X$, let $\tau(x)$ be the fixed-point-set of $G_x$. Then $\tau(x)$ is clearly convex; it is compact by hypothesis. Furthermore the map $x \mapsto \tau(x)$ is $G$-equivariant. The fact that the collection $\{\tau(x) \mid x \in X\}$ is locally finite follows from Proposition 5.5.

5.C. **Alexandrov angle rigidity.** A further consequence of Theorem 5.1 is a phenomenon of angle rigidity. Given an elliptic isometry $g$ of complete a CAT(0) space $X$ and a point $x \in X$, we denote by $c_{g,x}$ the projection of $x$ on the closed convex set of $g$-fixed points.

**Proposition 5.8.** Let $G$ be a totally disconnected locally compact group with a continuous and proper cocompact action by isometries on a geodesically complete proper CAT(0) space $X$. Then there is $\varepsilon > 0$ such that for any elliptic $g \in G$ and any $x \in X$ with $gx \neq x$ we have $\angle_{c_{g,x}}(gx,x) \geq \varepsilon$. 
Let and state the following.

Proof. First we observe that this bound on the Alexandrov angle is really a local property at \( c_{g,x} \) of the germ of the geodesic \([c_{g,x}, x]\) since for any \( y \in [c_{g,x}, x] \) we have \( c_{g,y} = c_{g,x} \).

Next, we claim that for any \( n \in \mathbb{N} \), any isometry of order \( \leq n \) of any complete CAT(0) space \( B \) satisfies \( \angle_{c_{g,x}}(gx, x) \geq 1/n \) for all \( x \in B \) that are not \( g \)-fixed. Indeed, it follows from the definition of Alexandrov angles (see [BH99, II.3.1]) that for any \( y \in [c_{g,x}, x] \) we have

\[
d(gy, y) \leq d(c_{g,x}, y) \angle_{c_{g,x}}(gx, x).
\]

Therefore, if \( \angle_{c_{g,x}}(gx, x) < 1/n \), the entire \( g \)-orbit of \( y \) would be contained in a ball around \( y \) not containing \( c_{g,x} = c_{g,y} \). This is absurd since the circumcentre of this orbit is a \( g \)-fixed point.

In order to prove the proposition, we now suppose for a contradiction that there are sequences \( \{g_n\} \) of elliptic elements in \( G \) and \( \{x_n\} \) in \( X \) with \( g_n x_n \neq x_n \) and \( \angle_{c_n}(g_n x_n, x_n) \to 0 \), where \( c_n = c_{g_n, x_n} \). Since the \( G \)-action is cocompact, there is (upon extracting) a sequence \( \{h_n\} \) in \( G \) such that \( h_n c_n \) converges to some \( c \in X \). Upon conjugating \( g_n \) by \( h_n \), replacing \( x_n \) by \( h_n x_n \) and \( c_n \) by \( h_n c_n \), we can assume \( c_n \to c \) without losing any of the conditions on \( g_n, x_n \), and \( c_n \), including the relation \( c_n = c_{g_n, x_n} \).

Since \( d(g_n c, c) \leq 2d(c_n, c) \), we can further extract and assume that \( \{g_n\} \) converges to some limit \( g \in G \); notice also that \( g \) fixes \( c \). By Lemma 2.13, the action is minimal and hence Theorem 5.1 applies. Therefore, we can assume that all \( g_n \) coincide with \( g \) on some ball \( B \) around \( c \) and in particular preserve \( B \). Using Remark 5.2, this provides a contradiction. \( \square \)

A first consequence is an analogue of a result that E. Swenson proved for discrete groups (Theorem 11 in [Swe99]).

Corollary 5.9. Let \( G \) be a totally disconnected locally compact group with a continuous and proper cocompact action by isometries on a geodesically complete proper CAT(0) space \( X \) not reduced to a point.

Then \( G \) contains hyperbolic elements (thus in particular elements of infinite order).

Beyond the totally disconnected case, we can appeal to Theorem 1.1 and Addendum 1.4 and state the following.

Corollary 5.10. Let \( G \) be any locally compact group with a continuous and proper cocompact action by isometries on a geodesically complete proper CAT(0) space \( X \) not reduced to a point.

Then \( G \) contains elements of infinite order; if moreover \( (\partial X)^G = \emptyset \), then \( G \) contains hyperbolic elements.

Proof of Corollary 5.9. Proposition 5.8 allows us use the argument form [Swe99]: We can choose a geodesic ray \( r : \mathbb{R}_+ \to X \), an increasing sequence \( \{t_i\} \) going to infinity in \( \mathbb{R}_+ \) and \( \{g_i\} \) in \( G \) such that the function \( t \to g_i r(t + t_i) \) converges uniformly on bounded intervals (to a geodesic line). For \( i < j \) large enough, the angle \( \angle_{h(r(t_i))}(r(t_i), h^2(r(t_i))) \) defined with \( h = g_i^{-1} g_j \) is arbitrarily close to \( \pi \). In order to prove that \( h \) is hyperbolic, it suffices to show that this angle will eventually equal \( \pi \). Suppose this does not happen; by Corollary 5.3(iii), we can assume that \( h \) is elliptic. We set \( x = r(t_i) \) and \( c = e_h x \). Considering the congruent triangles \((c, x, hx)\) and \((c, hx, h^2 x)\), we find that \( \angle_c(x, hx) \) is arbitrarily small. This is in contradiction with Proposition 5.8. \( \square \)
Proof of Corollary 5.10. If the connected component $G^o$ is non-trivial, then it contains elements of infinite order; if it is trivial, we can apply Corollary 5.9.

Assume now $(\partial X)^G = \emptyset$. Then Theorem 1.1 and Addendum 1.4 apply. Therefore, we obtain hyperbolic elements either from Corollary 5.9 or from the fact that any non-compact semi-simple group contains elements that are algebraically hyperbolic, combined with the fact that the latter act as hyperbolic isometries. That fact is established in Theorem 6.4(i) below, the proof of which is independent of Corollary 5.10. □

5.D. On minimal normal subgroups of t.d. locally compact groups. We begin with a general fact (compare Lemma 1.4.1 in [BM00a]):

**Proposition 5.11.** Let $G$ be a compactly generated totally disconnected locally compact group without non-trivial compact normal subgroups. Then any filtering family of non-discrete closed normal subgroups has non-trivial (thus non-compact) intersection.

**Proof.** Let $\mathfrak{g}$ be a Schreier graph for $G$. We recall that it consists in choosing any open compact subgroup $U < G$ (which exists by [Bou71, III §4 No 6]), defining the vertex set of $\mathfrak{g}$ as $G/U$ and drawing edges according to a compact generating set which is a union of double cosets modulo $U$; see [Mon01, §11.3]. Since $G$ has no non-trivial compact normal subgroup, the continuous $G$-action on $\mathfrak{g}$ is faithful. Let $v_0$ be a vertex of $\mathfrak{g}$ and denote by $v_0^\perp$ the set of neighbouring vertices. Since $G$ is vertex-transitive on $\mathfrak{g}$, it follows that for any normal subgroup $N \lhd G$, the $N_{v_0}$-action on $v_0^\perp$ defines a finite permutation group $F_N < \text{Sym}(v_0^\perp)$ which, as an abstract permutation group, is independent of the choice of $v_0$. Therefore, if $N$ is non-discrete, this permutation group $F_N$ has to be non-trivial since $U$ is open and $\mathfrak{g}$ connected. Now a filtering family $\mathcal{F}$ of non-discrete normal subgroups yields a filtering family of non-trivial finite subgroups of $\text{Sym}(v_0^\perp)$. Thus the intersection of these finite groups is non-trivial. Let $g$ be a non-trivial element in this intersection. For any $N \in \mathcal{F}$, let $N_g$ be the inverse image of $\{g\}$ in $N_{v_0}$. Thus $N_g$ is a non-empty compact subset of $N$ for each $N \in \mathcal{F}$. Since the family $\mathcal{F}$ is filtering, so are $\{N_{v_0} \mid N \in \mathcal{F}\}$ and $\{N_g \mid N \in \mathcal{F}\}$. The result follows, since a filtering family of non-empty closed subsets of the compact set $G_{v_0}$ has a non-empty intersection. □

Evidently open normal subgroups form a filtering family; we can thus deduce:

**Corollary 5.12.** Let $G$ be a compactly generated locally compact group without any non-trivial compact normal subgroup. If $G$ is residually discrete, then it is discrete. □

For convenience, we shall say that a topological group is **monolithic** with **monolith** $L$ if the intersection $L$ of all non-trivial closed normal subgroups is itself non-trivial. A group is called **quasi-simple** if it possesses a cocompact normal subgroup which is topologically simple and contained in every non-trivial closed normal subgroup of $G$; in other words, a quasi-simple group is a monolithic group whose monolith is cocompact and topologically simple. The following statement (containing Proposition 1.28 from the Introduction) clarifies the relation between these notions and a very natural weaker condition. The corresponding statement in the discrete case is due to J. Wilson [Wil71].

**Proposition 5.13.** Let $G$ be a compactly generated non-compact locally compact group such that every non-trivial closed normal subgroup is cocompact. Then one of the following holds.

(i) $G$ is monolithic; its monolith is the direct product of finitely many isomorphic topologically simple groups.

(ii) $G$ is monolithic with monolith $L \cong \mathbb{R}^n$. Moreover $G/L$ is isomorphic to a compact irreducible subgroup of $\text{GL}_n(\mathbb{R})$. In particular $G$ is an almost connected Lie group.
(iii) $G$ is discrete and residually finite.

Notice that a compactly generated locally compact group with every proper quotient compact may be monolithic without being quasi-simple. Examples of this nature are indeed provided by the standard wreath product of a topologically simple group by a finite transitive permutation group (see [Wil71, Construction 1]).

Before undertaking the proof of the proposition, we recall that the quasi-centre of a locally compact group $G$ is the subset $\mathcal{Z}(G)$ consisting of all those elements possessing an open centraliser. Clearly $\mathcal{Z}(G)$ is a (topologically) characteristic subgroup of $G$. Since any element with a discrete conjugacy class possesses an open centraliser, it follows that the quasi-centre contains all discrete normal subgroups of $G$.

Another relevant notion is the following. We say that a subgroup $H$ of a topological group $G$ is (topologically) locally finite if every finite subset of $H$ is contained in a compact subgroup of $G$. Any locally compact group possesses a maximal normal locally finite subgroup which is closed and called the LF-radical; another important fact is that any compact subset of a locally compact topologically locally finite group is contained in a compact subgroup. We refer to [Pla65] and [Cap07, §2] for details.

It is well known that the LF-radical is compact for connected groups:

**Lemma 5.14.** Every connected locally compact group admits a maximal compact normal subgroup. Moreover, the corresponding quotient is a connected Lie group.

**Proof.** The solution to Hilbert’s fifth problem [MZ55, Theorem 4.6] provides a compact normal subgroup such that the quotient is a Lie group; now the statement follows from the corresponding fact for connected Lie groups. □

**Proof of Proposition 5.13.** Since $G$ is non-compact, our assumption implies that it has no non-trivial compact normal subgroup. We start with a preliminary observation: For any closed normal subgroup $H \triangleleft G$, the kernel of the representation $G \to \text{Out}(H)$ is a (not necessarily direct) product $H \cdot \mathcal{Z}_G(H)$. Thus, according to whether the normal subgroup $\mathcal{Z}_G(H) \triangleleft G$ is trivial or cocompact, we see that $G/H$ injects into $\text{Out}(H)$ or the Abelian group $\mathcal{Z}(H)$ is cocompact in $G$, being an intersection of two cocompact normal subgroups.

We now begin by treating the case where $G$ is totally disconnected. Let $L$ be the intersection of all non-trivial closed normal subgroups. We distinguish two cases.

If $L$ is trivial, then Proposition 5.11 shows that $G$ admits a non-trivial discrete normal subgroup $\Gamma$; notice that $\Gamma$ is infinite. Now $\Gamma$ is a cocompact lattice in $G$ and is thus finitely generated [Mar91, I.0.40]. In particular Aut($\Gamma$) is countable and thus the image of the compact group $G/\Gamma$ in Out($\Gamma$) is finite since $G$ acts continuously on $\Gamma$. By the preliminary observation, we deduce that either $G/\Gamma$ is finite, in which case (iii) holds, or $\Gamma$ is virtually Abelian. In the latter case, the Abelian group $\mathcal{Z}(\Gamma)$ is cocompact in $G$, and $G$ has finite image in Out($\mathcal{Z}(\Gamma)$) which now is AUT($\mathcal{Z}(\Gamma)$). Thus $G$ virtually centralises $\mathcal{Z}(\Gamma)$. In other words $G$ is discrete and virtually Abelian and we are again in case (iii).

Assume now that $L$ is not trivial. Then it is cocompact whence compactly generated since $G$ is so. Notice that by definition $L$ is characteristically simple. Furthermore $L$ has no non-trivial compact normal subgroup. Indeed, otherwise $L$ would possess a non-trivial LF-radical and hence $L$ would be topologically locally finite. Since $L$ is compactly generated, this would imply by [Cap07, Lemma 2.3] that $L$ is compact, which is absurd.

We distinguish two cases.
On the one hand, assume that the quasi-centre $\mathcal{Z}(L)$ is non-trivial. Then it is dense in $L$. Since $L$ is compactly generated, the arguments of the proof of Theorem 4.8 in [BEW08] show that $L$ possesses a compact open normal subgroup. For the sake of completeness, we briefly describe them here. Since $L$ is compactly generated, it admits a generating set consisting of a finite set $\{g_1, \ldots, g_n\}$ of elements together with a compact open subgroup $U < L$. Since $L = \mathcal{Z}(L) \cdot U$ by density of $\mathcal{Z}(L)$, we may assume that each $g_i$ belongs to $\mathcal{Z}(L)$. The subgroup $\bigcap_{i=1}^n \mathcal{Z}(g_i) < U$ is open and hence contains a finite index open subgroup $V$ which is normalised by $U$. Thus $V$ is a compact open normal subgroup of $L$.

Now, since $L$ has no non-trivial compact normal subgroup, we deduce that $L$ is discrete. The preliminary observation shows that $L$ is of finite index in $G$, whence $G$ is discrete as well. In this case, we are done by the result of J. Wilson [Wil71].

On the other hand, assume that the quasi-centre $\mathcal{Z}(L)$ is trivial. In particular $L$ possesses no non-trivial discrete normal subgroup. Proposition 5.11 combined with Zorn’s lemma then shows that the set $\mathcal{M}$ of non-trivial minimal closed normal subgroups of $L$ is non-empty. Since $L$ has no non-trivial compact normal subgroup, no element of $\mathcal{M}$ is compact. Some of the following arguments are inspired by the proof of Proposition 1.5.1 in [BM00a].

For each subset $\mathcal{E} \subseteq \mathcal{M}$ we set $M_{\mathcal{E}} = \langle M \mid M \in \mathcal{E} \rangle$. We claim that if $\mathcal{E}$ is a proper subset of $\mathcal{M}$, then $\overline{M}_{\mathcal{E}}$ is a proper subgroup of $L$. Indeed, for all $M \in \mathcal{E}$ and $M' \in \mathcal{M} \setminus \mathcal{E}$ we have $[M, M'] \subseteq M \cap M' = 1$. Thus $M'$ centralises $\overline{M}_{\mathcal{E}} = L$. In particular, if $\mathcal{M} \setminus \mathcal{E}$ is non-empty then $L$ has a non-trivial centre, which is absurd since $\mathcal{Z}(L) = 1$ by assumption. Thus the claim holds indeed.

Clearly $G$ acts on $\mathcal{M}$ by conjugation. Let $\mathcal{E}$ denote a $G$-orbit in $\mathcal{M}$. Since $M_{\mathcal{E}}$ is normal in $G$, it is dense in $L$. The preceding claim thus shows that $\mathcal{E} = \mathcal{M}$. In other words $G$ acts transitively on $\mathcal{M}$.

Consider the family

$$\mathcal{F} = \{\overline{M}_{\mathcal{M}\setminus F} \mid F \subseteq \mathcal{M} \text{ is finite}\}$$

of closed normal subgroups of $L$. Clearly this family is filtering. Furthermore the above claim shows that $\overline{M}_{\mathcal{M}\setminus F}$ is properly contained in $L$ whenever $F$ is non-empty. Thus $\bigcap \mathcal{F}$ is a normal subgroup of $G$ which is properly contained in $L$. In particular $\bigcap \mathcal{F}$ is trivial and Proposition 5.11 therefore implies that $\mathcal{M}$ is finite.

It now follows that $L \cong \prod_{M \in \mathcal{M}} M$. In particular, any normal subgroup of $M$ is in fact normalised by $L$. Since $M$ is a minimal normal subgroup of $L$, it follows that $M$ is topologically simple and (i) holds.

Now we turn to the case where $G$ is not totally disconnected, hence its identity component $G^0$ is cocompact. Since the compact group of Lemma 5.14 is characteristic, it is trivial and hence $G^0$ is a connected Lie group. Since its soluble radical is characteristic, it is trivial or cocompact.

In the first case, $G^0$ is semi-simple without compact factors. Since its isotypic factors are characteristic, there is only one isotypic factor and we conclude that (i) holds.

If on the other hand the radical of $G^0$ is cocompact, we deduce that $G$ admits a normal cocompact connected soluble subgroup $R \triangleleft G$. Since the commutator subgroup of $R$ is characteristic and since $R$ has no non-trivial compact subgroup, we have $R \cong \mathbb{R}^n$. The kernel of the homomorphism $G \to \text{Out}(R) = \text{Aut}(R)$ is a cocompact normal subgroup $N$ containing containing $R$ in its centre, and such that $N/R$ is compact. In particular $N$ is a compactly generated locally compact group in which every conjugacy class is relatively compact. By [Usa63], it follows that $N$ is a compact extension of an Abelian group, which is the direct product of $R$ with a free Abelian group of finite rank. Since $N$ has trivial
LF-radical, it follows that \( N \) is Abelian. Since \( R \) is cocompact in \( N \) it finally follows that \( N = R \). Using again the map \( G \to \text{Aut}(\mathbb{R}^n) \cong \text{GL}_n(\mathbb{R}) \), we deduce that \( G/R \) is isomorphic to a compact subgroup of \( \text{GL}_n(\mathbb{R}) \), which has to be irreducible otherwise \( R \) would contain a non-cocompact subgroup normalised by \( G \).

We record one further consequence of Proposition 5.11, concerning locally compact groups admitting various quasi-simple quotients. It will be needed below and presents some independent interest.

**Corollary 5.15.** Let \( G \) be a compactly generated locally compact group and \( \{ N_v \mid v \in \Sigma \} \) be a collection of pairwise distinct closed normal subgroups of \( G \) such that for each \( v \in \Sigma \), the quotient \( H_v = G/N_v \) is quasi-simple, non-discrete and non-compact. If \( \bigcap_{v \in \Sigma} N_v = 1 \) then \( \Sigma \) is finite and \( G \) has a characteristic closed cocompact subgroup which splits as a finite direct product of \(|\Sigma|\) topologically simple groups.

**Proof.** We claim that \( G \) has no non-trivial compact normal subgroup. Let indeed \( Q \triangleleft G \) be a compact normal subgroup of \( G \). By the assumptions made on \( H_v \), the image of \( Q \) in \( H_v = G/N_v \) is trivial for each \( v \in \Sigma \). Thus \( Q \subseteq \bigcap_{v \in \Sigma} N_v \) and hence \( Q \) is trivial.

By assumption each quotient \( H_v \) is monolithic with simple monolith, which we denote by \( S_v \). We claim that \( S_v \) is non-discrete. Indeed, if \( S_v \) were discrete, then it would be finitely generated since \( G \) is compactly generated (see [Mar91, I.0.40]). In particular \( \text{Aut}(S_v) \) is countable. Therefore the map \( G \to \text{Aut}(S_v) \) is not injective. Its kernel is a normal subgroup of \( G \) containing \( N_v \) but projecting to a non-trivial normal subgroup of \( H_v \) meeting \( S_v \) trivially, which is absurd. This shows that \( S_v \) is indeed non-discrete for each \( v \in \Sigma \).

Let now \( M_v < G \) denote the pre-image of \( S_v \) in \( G \); thus \( M_v \) is the intersection of all closed normal subgroups of \( G \) containing \( N_v \) properly. For \( v \neq v' \in \Sigma \), the normal subgroup \( M_v \cap M_{v'} \triangleleft G \) is cocompact. Its projection to \( G/N_v \) therefore contains the monolith \( S_v \). In other words \( M_v < M_{v'} \). By symmetry \( M_{v'} < M_v \) and we deduce that \( M_v \) is independent of \( v \in \Sigma \); we denote by \( M \) the common value.

Let us now deal with the special case when \( G \) is totally disconnected. For each finite subset \( P \subseteq \Sigma \), set \( N_P = \bigcap_{v \in P} N_v \). An elementary argument (compare the Goursat lemma) shows that \( M/N_P \cong \prod_{v \in P} S_v \) since each \( S_v \) is simple. In particular, arguing with \( P \cup \{v\} \) we deduce that, for each \( v \notin P \), the restriction to \( N_P \) of the map \( M \to S_v \) is still surjective. In particular \( N_P \) is uncountable, whence non-discrete. Proposition 5.11 therefore shows that if \( \Sigma \) is infinite, then the intersection \( \bigcap_{v \in \Sigma} N_v \) is non-trivial. This yields the desired conclusion in the totally disconnected case.

We now turn to the general case and denote by \( G^0 \) the identity component of \( G \). The initial claim shows that the characteristic subgroup of \( G^0 \) of Lemma 5.14 is trivial and hence \( G^0 \) is a connected Lie group.

Arguing as in the initial claim, \( G \) has no non-trivial soluble normal subgroup, and hence the radical of \( G^0 \) is trivial. Thus \( G^0 \) is a connected semi-simple Lie group with trivial centre and no compact factor. It follows (compare Theorem 4.6) that \( G \) has a closed characteristic subgroup of finite index which splits as a direct product of the form \( G^0 \times D \). Projecting each \( N_v \) to the group of components \( G/G^0 \) we obtain a family of normal subgroups intersecting trivially, each yielding a quasi-simple quotient. We deduce from what has already been proved that the totally disconnected group \( G/G^0 \) has a cocompact closed characteristic subgroup which splits as a finite direct product of non-discrete topologically simple groups.

\( \square \)
5.E. **Algebraic structure.** Given a topological group $G$, we define its **socle** $\text{soc}(G)$ as the subgroup generated by all minimal non-trivial closed normal subgroups of $G$. Notice that $G$ might have no minimal non-trivial closed normal subgroup, in which case its socle is trivial.

**Proposition 5.16.** Let $X$ be a proper CAT(0) space without Euclidean factor and $G < \text{Is}(X)$ be a closed subgroup acting minimally cocompactly without fixed point at infinity. If $G$ has trivial quasi-centre, then $\text{soc}(G^*)$ is direct product of non-trivial characteristically simple groups, where $r$ is the number of irreducible factors of $X$ and $G^*$ is the canonical finite index open normal subgroup acting trivially on the set of factors of $X$.

**Proof.** We first observe that $G^*$ has no non-trivial discrete normal subgroup. Indeed, such a subgroup has finitely many $G$-conjugates, which implies that each of its elements has discrete $G$-conjugacy class and hence belongs to $\mathcal{Z}(G)$, which was assumed trivial.

Let now $\{N_i\}$ be a chain of non-trivial closed normal subgroups of $G^*$. If $N_i$ is totally disconnected for some $i$, then the intersection $\bigcap_i N_i$ is non-trivial by Proposition 5.11. Otherwise $N_i^o$ is non-trivial and normal in $(G^*)^o$ for each $i$, and the intersection $\bigcap_i N_i$ is non-trivial by Theorem 1.1 (since the latter describes in particular the possible normal connected subgroups of $G^*$). In all cases, Zorn’s lemma implies that the ordered set of non-trivial closed normal subgroups of $G^*$ possesses minimal elements.

Given two minimal closed normal subgroups $M, M'$, the intersection $M \cap M'$ is thus trivial and, hence, so is $[M, M']$. Thus minimal closed normal subgroups of $G^*$ centralise one another. We deduce from Corollary 4.8 that the number of minimal closed normal subgroups is at most $r$.

Consider now an irreducible totally disconnected factor $H$ of $G^*$. We claim that the collection of non-trivial closed normal subgroups of $H$ forms a filtering family. Indeed, given two such normal subgroup $N_1, N_2$, then $N_1 \cap N_2$ is again a closed normal subgroup of $H$. It is is trivial, then the commutator $[N_1, N_2]$ is trivial and, hence, the centraliser of $N_1$ in $H$ is non-trivial, contradicting Theorem 1.6. This confirms the claim. Thus the intersection of all non-trivial closed normal subgroups of $H$ is non-trivial by Proposition 5.11. Clearly this intersection is the socle of $H$; it is clear we have just established that it is contained in every non-trivial closed normal subgroup of $H$. In particular $\text{soc}(H)$ is characteristically simple. The desired result follows, since $\text{soc}(H)$ is clearly a minimal closed normal subgroup of $G^*$.

**Theorem 5.17.** Let $X$ be a proper irreducible geodesically complete CAT(0) space. Let $G < \text{Is}(X)$ be a closed totally disconnected subgroup acting cocompactly, in such a way that no open subgroup fixes a point at infinity. Then we have the following:

(i) Every compact subgroup of $G$ is contained in a maximal one; the maximal compact subgroups fall into finitely many conjugacy classes.

(ii) $\mathcal{Z}(G) = 1$.

(iii) $\text{soc}(G)$ is a non-discrete characteristically simple group.

**Proof.** (i) By Lemma 2.13, the action is minimal and hence Theorem 5.1 applies. In particular, we can apply Corollary 5.7 and consider the resulting equivariant decomposition. Let $Q < G$ be a compact subgroup and $x$ be a $Q$-fixed point. If $G_x$ is not contained in a maximal compact subgroup of $G$, then there is an infinite sequence $(x_n)_{n \geq 0}$ such that $x_0 = x$ and $G_{x_n} \subseteq G_{x_{n+1}}$. By Corollary 5.7, the sequence $x_n$ leaves every bounded subset. Since the fixed points $X^{G_{x_n}}$ form a nested sequence, it follows that $X^{G_x}$ is unbounded. In particular its visual boundary $\partial(X^{G_x})$ is non-empty and the open subgroup $G_x$ has a fixed point at infinity. This contradicts the hypotheses, and the claim is proved. Notice that
a similar argument shows that for each $x \in X$, there are finitely many maximal compact subgroups $Q_i < G$ containing $G_x$.

The fact that $G$ possesses finitely many conjugacy classes of maximal compact subgroups now follows from the compactness of $G \setminus X$.

(ii) We claim that $\mathcal{P}(G)$ is topologically locally finite. The desired result follows since it is then amenable but $G$ has trivial amenable radical by Theorem 1.6. Let $S \subseteq \mathcal{P}(G)$ be a finite subset. Then $G$ possesses a compact open subgroup $U$ centralising $S$. By hypothesis the fixed point set of $U$ is compact. Since $\langle S \rangle$ stabilises $X_U$, it follows that $\langle S \rangle$ is compact, whence the claim.

(iii) Follows from (ii) and Proposition 5.16.

6. Cocompact CAT(0) spaces

6.A. Fixed points at infinity. We begin with a simple observation. We recall that two points at infinity are opposite if they are the two endpoints of a geodesic line. We denote by $\xi^\text{op}$ the set of points opposite to $\xi$. Recall from [Bal95, Theorem 4.11(i)] that, if $X$ is proper, then two points $\xi, \eta \in \partial X$ at Tits distance $> \pi$ are necessarily opposite. (Recall that Tits distance is by definition the length metric associated to the Tits angle.) However, it is not true in general that two points at Tits distance $\pi$ are opposite.

**Proposition 6.1.** Let $X$ be a proper CAT(0) space and $H < \text{Is}(X)$ a closed subgroup acting cocompactly. If $H$ fixes a point $\xi$ at infinity, then $\xi^\text{op} \neq \emptyset$ and $H$ acts transitively on $\xi^\text{op}$.

**Proof.** First we claim that there is a geodesic line $\sigma : \mathbb{R} \to X$ with $\sigma(\infty) = \xi$. Indeed, let $r: \mathbb{R}_+ \to X'$ be a ray pointing to $\xi$ and $\{g_n\}$ a sequence in $H$ such that $g_n r(n)$ remains bounded. The Arzelà-Ascoli theorem implies that $g_n r(\mathbb{R}_+)$ subconverges to a geodesic line in $X$. Since $\xi$ is fixed by all $g_n$, this line has an endpoint at $\xi$.

Let now $\sigma' : \mathbb{R} \to X$ be any other geodesic with $\sigma'(\infty) = \xi$ and choose a sequence $\{h_n\}_{n \in \mathbb{N}}$ in $H$ such that $d(h_n \sigma(-n), \sigma'(-n))$ remains bounded. By convexity and since all $h_n$ fix $\xi$, $d(h_n \sigma(t), \sigma'(t))$ is bounded for all $t$ and thus subconverges (uniformly for $t$ in bounded intervals). On the one hand, it implies that $\{h_n\}$ has an accumulation point $h$. On the other hand, it follows that $h \sigma(-\infty) = \sigma'(-\infty)$. \hfill \QED

Recall that any complete CAT(0) space $X$ admits a canonical splitting $X = X' \times V$ preserved by all isometries, where $V$ is a (maximal) Hilbert space called the Euclidean factor of $X$, see [BH99, II.6.15(6)]. Furthermore, there is a canonical embedding $X' \subseteq X'' \times V'$, where $V'$ is a Hilbert space generated by all directions in $X'$ pointing to “flat points” at infinity, namely points for which the Busemann functions are affine on $X'$; moreover, every isometry of $X'$ extends uniquely to an isometry of $X'' \times V'$ which preserves that splitting. This is a result of Adams–Ballmann [AB98a, Theorem 1.6], who call $V'$ the pseudo-Euclidean factor (one could also propose “Euclidean pseudo-factor”).

**Corollary 6.2.** Let $X$ be a proper CAT(0) space with a cocompact group of isometries. Then the pseudo-Euclidean factor of $X$ is trivial.

**Proof.** In view of the above discussion, $X'$ is also a proper CAT(0) space with a cocompact group of isometries. The set of flat points in $\partial X'$ admits a canonical (intrinsic) circumcentre $\xi$ by Lemma 1.7 in [AB98a]. In particular, $\xi$ is fixed by all isometries and therefore, by Proposition 6.1, it has an opposite point, which is impossible for a flat point unless it lies already in the Euclidean factor (see [AB98a]). \hfill \QED
**Proposition 6.3.** Let $G$ be a group acting cocompactly by isometries on a proper CAT(0) space $X$ without Euclidean factor and assume that the stabiliser of every point at infinity acts minimally on $X$. Then $G$ has no fixed point at infinity.

**Proof.** If $G$ has a global fixed point $\xi$, then the stabiliser $G_\eta$ of an opposite point $\eta \in \xi^{\text{op}}$ (which exists by Proposition 6.1) preserves the union $Y \subseteq X$ of all geodesic lines connecting $\xi$ to $\eta$. By [BH99, II.2.14], this space is convex and splits as $Y = \mathbb{R} \times Y_0$. Since $G_\eta$ acts minimally, we deduce $Y = X$ which provides a Euclidean factor. □

**6.B. Actions of simple algebraic groups.** Let $X$ be a CAT(0) space and $G$ be an algebraic group defined over the field $k$. An isometric action of $G(k)$ on $X$ is called algebraic if every (algebraically) semi-simple element $g \in G(k)$ acts as a semi-simple isometry.

**Theorem 6.4.** Let $k$ be a local field and $G$ be an absolutely almost simple simply connected $k$-group. Let $X$ be a non-compact proper CAT(0) space on which $G = G(k)$ acts continuously by isometries.

Assume either: (a) the action is cocompact; or: (b) it has full limit set, is minimal and $\partial X$ is finite-dimensional. Then:

(i) The $G$-action is algebraic.

(ii) There is a $G$-equivariant bijection $\partial X \cong \partial X_{\text{model}}$ which is a homeomorphism with respect to the cone topology and an isometry with respect to Tits’ metric, where $X_{\text{model}}$ is the symmetric space or Bruhat–Tits building associated with $G$.

(iii) If $X$ is geodesically complete, then $X$ is isometric to $X_{\text{model}}$.

(iv) For any semi-simple $k$-subgroup $L < G$, there non-empty closed convex subspace $Y \subseteq X$ minimal for $L = L(k)$; moreover, there is no $L$-fixed point in $\partial Y$.

**Remarks 6.5.**

(i) Notice that there is no assumption on the $k$-rank of $G$ in this result.

(ii) We recall for (b) that minimality follows from full limit set in the geodesically complete case (Lemma 2.13).

(iii) Recall that two CAT(0) spaces with the same cocompact isometry group need not have homeomorphic boundaries [CK00].

Before proceeding to the proof, we give two examples showing that the assumptions made in Theorem 6.4 are necessary.

**Example 6.6.** Without the assumption of geodesic completeness, it is not true in general that, in the setting of the theorem, the space $X$ contains a closed convex $G$-invariant subspace which is isometric to $X_{\text{model}}$. A simple example of this situation may obtained as follows. Consider the case where $k$ is non-Archimedean and $G$ has $k$-rank one. Let $0 < r < 1/2$ and let $X$ be the space obtained by replacing the $r$-ball centred at each vertex in the tree $X_{\text{model}}$ by an isometric copy of a given Euclidean $n$-simplex, where $n + 1$ is valence of the vertex. In this way, one obtains a CAT(0) space which is still endowed with an isometric $G$-action that is cocompact and minimal, but clearly $X$ is not isometric to $X_{\text{model}}$.

We do not know whether such a construction may also be performed in the Archimedean case (see Problem 12.2 below).

**Example 6.7.** Under the assumptions (b), minimality is needed. Indeed, we claim that for any CAT(0) space $X_0$ there is a canonical CAT(-1) space $X$ (in particular $X$ is a CAT(0) space) together with a canonical map $i : \text{Is}(X_0) \hookrightarrow \text{Is}(X)$ with the following properties: The boundary $\partial X$ is reduced to a single point; $X$ non-compact; $X$ is proper if and only if $X_0$ is so; the map $i$ is an isomorphism of topological groups onto its image. This claim justifies
that minimality is needed since we can apply it to the case where \(X_0\) is the symmetric space or Bruhat–Tits building associated to \(G(k)\). (In that case the action has indeed full limit set, a cheap feat as the isometry group is non-compact and the boundary rather incapacious.)

To prove the claim, consider the parabolic cone \(Y\) associated to \(X_0\). This is the metric space with underlying set \(X_0 \times \mathbb{R}_+\) where the distance is defined as follows: given two points \((x,t)\) and \((x',t')\) of \(Y\), identify the interval \([x,x']\subseteq X_0\) with an interval of corresponding length in \(\mathbb{R}\) and measure the length from the resulting points \((x,t)\) and \((x',t')\) in the upper half-plane model for the hyperbolic plane. This is a particular case of the synthetic version ([Che99], [AB98b]) of the Bishop–O’Neill “warped products” [BO69] and its properties are described in [BGP92], [AB04, 1.2(2A)] and [HLS00, §2]. In particular, \(Y\) is \(\text{CAT}(−1)\).

We now let \(\xi \in \partial Y\) be the point at infinity corresponding to \(t \to \infty\) and define \(X \subseteq Y\) to be an associated horoball; for definiteness, set \(X = X_0 \times \{1, \infty\}\). We now have \(\partial X = \{\xi\}\) by the \(\text{CAT}(−1)\) property or alternatively by the explicit description of geodesic rays (e.g. 2(iv) in [HLS00]). The remaining properties follow readily.

**Proof of Theorem 6.4.** We start with a few preliminary observations. Finite-dimensionality of the boundary always holds since it is automatic in the cocompact case. Since \(X\) is non-compact, the action is non-trivial, because it has full limit set. It is well known that every non-trivial continuous homomorphism of \(G\) to a locally compact second countable group is proper [BM96, Lemma 5.3]. Thus the \(G\)-action on \(X\) is proper.

We claim that the stabiliser of any point \(\xi \in \partial X\) contains the unipotent radical of some proper parabolic subgroup of \(G\). Indeed, fix a polar decomposition \(G = \text{KAK}\). Let \(x_0 \in X\) be a \(\text{K}\)-fixed point. Choose a sequence \(\{g_n\}_{n \geq 0}\) of elements of \(G\) such that \(g_nx_0\) converges to \(\xi\). Write \(g_n = k_n.a_n.k'_{n}\) with \(k_n,k'_{n}\in \text{K}\) and \(a_n\in \text{A}\). We may furthermore assume, upon replacing \(\{g_n\}\) by a subsequence, that \(\{k_n\}\) converges to some \(k\in \text{K}\), that \(\{a_n.x_0\}\) converges in \(X\cup \partial X\) and that \(\{a_n.p\}\) converges in \(X_{\text{model}}\cup \partial X_{\text{model}}\), where \(p\in X_{\text{model}}\) is some base point. Let \(\eta = \lim_{n\to \infty} a_n.x_0\) and observe \(\eta = k^{-1}.\xi\). Furthermore, the stabiliser of \(\eta\) contains the group

\[
U = \{g \in G \mid \lim_{n\to \infty} a_n^{-1} g a_n = 1\}.
\]

The convergence in direction of \(\{a_n\}\) in \(\text{A}\) implies that \(U\) contains the unipotent radical \(U_Q\) of the parabolic subgroup \(Q < G\) corresponding to \(\lim_{n\to \infty} a_n.p\in \partial X_{\text{model}}\). (In fact, the arguments for Lemma 2.4 in [Pra77] probably show \(U = U_Q\); this follows a posteriori from (ii) below.) Therefore, the stabiliser of \(\xi = k.\eta\) in \(G\) contains the unipotent radical of \(kQk^{-1}\), proving the claim.

Notice that we have seen in passing that any point at infinity lies in the limit set of some torus; in the above notation, \(\xi\) is in the limit set of \(kAk^{-1}\).

(i) Every element of \(G\) which is algebraically elliptic acts with a fixed point in \(X\), since it generates a relatively compact subgroup. We need to show that every non-trivial element of a maximal split torus \(T < G\) acts as a semi-simple isometry. Assume for a contradiction that some element \(t \in T\) acts as a parabolic isometry. Since \(X\) has finite-dimensional boundary and we can apply Corollary 2.3(ii). It follows that the Abelian group \(T\) has a canonical fixed point at infinity \(\xi\) fixed by the normaliser \(\mathcal{N}_G\)\((T)\). By the preceding paragraph, we know furthermore that the stabiliser of \(\xi\) in \(G\) also contains the unipotent radical of some parabolic subgroup of \(G\). Recall that \(G\) is generated by \(\mathcal{N}_G\)\((T)\) together with any such unipotent radical; this follows from the fact that \(\mathcal{N}_G\)\((T)\) has no fixed point at infinity in \(X_{\text{model}}\) and that \(G\) is generated by the unipotent radicals of any two distinct parabolic subgroups. Therefore \(\xi\) is fixed by the entire group \(G\). Since \(G\) has trivial Abelianisation, its image under the Busemann character centred at \(\xi\) vanishes, thereby showing that \(G\) must
stabilise every horoball centred at $\xi$. This is absurd both in the minimal and the cocompact case.

(ii) Let $T < G$ be a maximal split torus. Let $F_{\text{model}} \subseteq X_{\text{model}}$ be the (maximal) flat stabilised by $T$. In view of (i) and the properness of the $T$-action, we know that $T$ also stabilises a flat $F \subseteq X$ with $\dim F = \dim T$, see [BH99, II.7.1]. Choose a base point $p_0 \in F_{\text{model}}$ in such a way that its stabiliser $K := G_{p_0}$ is a maximal compact subgroup of $G$. The union of all $T$-invariant flats which are parallel to $F$ is $\mathcal{N}_G(T)$-invariant. Therefore, upon replacing $F$ by a parallel flat, we may and shall assume that $F$ contains a point $x_0$ which is stabilised by $N_K := \mathcal{N}_G(T) \cap K$. Note that, since $\mathcal{N}_G(T) = \langle N_K \cup T \rangle$, the flat $F$ is $\mathcal{N}_G(T)$-invariant. Therefore, there is a well defined $\mathcal{N}_G(T)$-equivariant map $\alpha$ of the $\mathcal{N}_G(T)$-orbit of $p_0$ to $F$, defined by $\alpha(g.p_0) = g.x_0$ for all $g \in \mathcal{N}_G(T)$.

We claim that, up to a scaling factor, the map $\alpha$ is isometric. In order to establish this, remark that the Weyl group $W := \mathcal{N}_G(T)/\mathcal{Z}_G(T)$ acts on $F$, since $W = N_K/T_K$, where $T_K := \mathcal{Z}_G(T) \cap K$ acts trivially on $F$. The group $N_K$ normalises the coroot lattice $\Lambda < T$. Furthermore $N_K.\Lambda$ acts on $F_{\text{model}}$ as an affine Weyl group since $N_K.\Lambda/T_K \cong W.\Lambda$. Moreover, since any reflection in $W$ centralises an Abelian subgroup of corank 1 in $\Lambda$, it follows that $N_K.\Lambda$ acts on $F$ as a discrete reflection group. But a given affine Weyl group has a unique (up to scaling factor) discrete cocompact action as a reflection group on Euclidean spaces, as follows from [Bou68, Ch. VI, §2, Proposition 8]. Therefore the restriction of $\alpha$ to $\Lambda.x_0$ is a homothety. Since $\Lambda$ is a uniform lattice in $T$, the claim follows.

At this point, it follows that $\alpha$ induces an $\mathcal{N}_G(T)$-equivariant map $\partial \alpha : \partial F_{\text{model}} \to \partial F$, which is isometric with respect to Tits’ distance. Since $\mathcal{N}_G(T)$ is the stabiliser of $\partial F_{\text{model}}$ in $G$, this map extends to a well defined $G$-equivariant map $\partial X_{\text{model}} \to \partial X$, which we denote again by $\partial \alpha$. Since any two points of $\partial X_{\text{model}}$ are contained in a common maximal sphere (i.e. an apartment), and since $G$ acts transitively on these spheres, the map $\partial \alpha$ is isometric, because so is its restriction to the sphere $\partial F_{\text{model}}$. Note that $\partial \alpha$ is surjective; indeed, this follows from the last preliminary observation, which, combined with (i), shows in particular that $\partial X = K.\partial F$.

It remains to prove that $\partial \alpha$ is a homeomorphism with respect to the cône topology. Since $\partial X_{\text{model}}$ is compact, it is enough to show that $\partial \alpha$ is continuous. Now any convergent sequence in $\partial X_{\text{model}}$ may be written as $\{k_n.\xi_n\}_{n \geq 0}$, where $\{k_n\}_{n \geq 0}$ (resp. $\{\xi_n\}_{n \geq 0}$) is a convergent sequence of elements of $K$ (resp. $\partial F_{\text{model}}$). On the sphere $\partial F_{\text{model}}$, the cône topology coincides with the one induced by Tits’ metric. Therefore, the equivariance of the Tits’ isometry $\partial \alpha$ shows that $\{\partial \alpha(k_n.\xi_n)\}_{n \geq 0}$ is a convergent sequence in $\partial X$, as was to be proved.

(iii) In the higher rank case, assertion (iii) follows from (ii) and the main result of [Lee00]. However, the full strength of loc. cit. is really not needed here, since the main difficulty there is precisely the absence of any group action, which is part of the hypotheses in our setting. For example, when the ground field $k$ is the field of real numbers, the arguments may be dramatically shortened as follows; they are valid without any rank assumption.

Given any $\xi \in \partial X$, the unipotent radical of the parabolic subgroup $G_{\xi}$ acts sharply transitively on the boundary points opposite to $\xi$. In view of this and of the properness of the $G$-action, the arguments of [Lee00, Proposition 4.27] show that geodesic lines in $X$ do not branch; in other words $X$ has uniquely extendible geodesics. From this, it follows that the group $N_\xi := \mathcal{N}_G(T) \cap K$ considered in the proof of (ii) has a unique fixed point in $X$, since otherwise it would fix pointwise a geodesic line, and hence, by (ii), opposite points in
∂$X_{\text{model}}$. The fact that this is impossible is purely a statement on the classical symmetric space $X_{\text{model}}$; we give a proof for the reader’s convenience:

Let $F_{\text{model}}$ be the flat corresponding to $T$ and $p_0 \in F_{\text{model}}$ be the $K$-fixed point. If $N_K$ fixed a point $\xi \in \partial X_{\text{model}}$, then the ray $[p_0, \xi]$ would be pointwise fixed and, hence, the group $N_K$ would fix a non-zero vector in the tangent space of $X_{\text{model}}$ at $p_0$. A Cartan decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of $G$ yields an isomorphism between the isotropy representation of $N_K$ on $T_{p_0}X_{\text{model}}$ and the representation of the Weyl group $\mathcal{W}$ on $\mathfrak{p}$. An easy explicit computation shows that the latter representation has no non-zero fixed vector.

Since $N_K$ has a unique fixed point, the latter is stabilised by the entire group $K$. Hence $K$ fixes a point lying on a flat $F$ stabilised by $T$. From the $KAK$-decomposition, it follows that the $G$-orbit of this fixed point is convex. Since the $G$-action on $X$ is minimal by geodesic completeness (Lemma 2.13), we deduce that $G$ is transitive on $X$. In particular $X$ is covered by flats which are $G$-conjugate to $F$, and the existence of a $G$-equivariant homeomorphism $X_{\text{model}} \to X$ follows from the existence of a $\mathcal{N}_G(T)$-equivariant homeomorphism $F_{\text{model}} \to F$, which has been established above. It remains only to choose the right scale on $X_{\text{model}}$ to make it an isometry.

In the non-Archimedean case, we consider only the rank one case, referring to [Lee00] for higher rank. Let $K$ be a maximal compact subgroup of $G$ and $x_0 \in X$ be a $K$-fixed point. By (ii), the group $K$ acts transitively on $\partial X$. Since $X$ is geodesically complete, it follows that the $K$-translates of any ray emanating from $x_0$ cover $X$ entirely. On the other hand every point in $X$ has an open stabiliser by Theorem 5.1, any point in $X$ has a finite $K$-orbit. This implies that the space of directions at each point $p \in X$ is finite. In other words $X$ is 1-dimensional. Since $X$ is CAT(0) and locally compact, it follows that $X$ is a locally finite metric tree. As we have just seen, the group $K$ acts transitively on the geodesic segments of a given length emanating from $x_0$. One deduces that $G$ is transitive on the edges of $X$. In particular all edges of $X$ have the same length, which we can assume to be as in $X_{\text{model}}$, $G$ has at most two orbits of vertices, and $X$ is either regular or bi-regular. The valence of any vertex $p$ equals

$$\min_{g \mathcal{N}_G(G_p)} [G_p : G_p \cap gG_pg^{-1}],$$

and coincides therefore with the valence of $X_{\text{model}}$. It finally follows that $X$ and $X_{\text{model}}$ are isometric, as was to be proved.

(iv) Let $P$ be a $k$-parabolic subgroup of $G$ that is minimal amongst those containing $L$. We may assume $P \neq G$ since otherwise $L$ has no fixed point in $\partial X$ and the conclusion holds in view of Proposition 3.1. It follows that $L$ centralises a $k$-split torus $T$ of positive dimension $d$. It follows from (i) that $T = T(k)$ acts by hyperbolic isometries, and thus there is a $T$-invariant closed convex subset $Z \subseteq X$ of the form $Z = Z_1 \times \mathbb{R}^d$ such that the $T$-action is trivial on the $Z_1$-factor; this follows from Theorem II.6.8 in [BH99] and the properness of the action. Moreover, $L$ preserves $Z$ and its decomposition $Z = Z_1 \times \mathbb{R}^d$, acting by translations on the $\mathbb{R}^d$ factor (loc. cit.). Since $L$ is semi-simple, this translation action is trivial and thus $L$ preserves any $Z_1$ fibre, say for instance $Z_0 := Z_1 \times \{0\} \subseteq Z$. For both the existence of a minimal set $Y$ and the condition $(\partial Y)^L = \emptyset$, it suffices to show that $L$ has no fixed point in $\partial Z_0$ (Proposition 3.1).

We claim that $\partial Z_0$ is Tits-isometric to the spherical building of the Lévi subgroup $\mathcal{Z}_G(T)$. Indeed, we know from (ii) that $\partial X$ is equivariantly isometric to $\partial X_{\text{model}}$ and the building of $\mathcal{Z}_G(T)$ is characterised as the points at distance $\pi/2$ from the boundary of the $T$-invariant flat in $\partial X_{\text{model}}$. 
On the other hand, \( L \) has maximal semi-simple rank in \( Z_G(T) \) by the choice of \( P \) and therefore cannot be contained in a proper parabolic subgroup of \( Z_G(T) \). This shows that \( L \) has no fixed point in \( \partial Z_0 \) and completes the proof.

6.C. No branching geodesics. Recall that in a geodesic metric space \( X \), the space of directions \( \Sigma_x \) at a point \( x \) is the completion of the space \( \tilde{\Sigma}_x \) of geodesic germs equipped with the Alexandrov angle metric at \( x \). If \( X \) has \emph{uniquely} extensible geodesics, then \( \tilde{\Sigma}_x = \Sigma_x \).

The following is a result of V. Berestovskii [Bert02] (we read it in [Ber, §3]; it also follows from A. Lytchak’s arguments in [Lyt05, §4]).

**Theorem 6.8.** Let \( X \) be a proper CAT(0) space with uniquely extensible geodesics and \( x \in X \). Then \( \tilde{\Sigma}_x = \Sigma_x \) is isometric to a Euclidean sphere.

We use this result to establish the following.

**Proposition 6.9.** Let \( X \) be a proper CAT(0) space with uniquely extensible geodesics. Then any totally disconnected closed subgroup \( D < \text{Is}(X) \) is discrete.

**Proof.** There is some compact open subgroup \( Q < D \), see [Bou71, III §4 No 6]. Let \( x \) be a \( Q \)-fixed point. The isometry group of \( \Sigma_x \) is a compact Lie group by Theorem 6.8 and thus the image of the profinite group \( Q \) in it is finite. Let thus \( K < Q \) be the kernel of this representation, which is open. Denote by \( S(x,r) \) the \( r \)-sphere around \( x \). The \( Q \)-equivariant “visual” map \( S(x,r) \to \Sigma_x \) is a bijection by unique extensibility. It follows that \( K \) is trivial.

We are now ready for:

**End of proof of Theorem 1.11.** Since the action is cocompact, it is minimal by Lemma 2.13. The fact that extensibility of geodesics is inherited by direct factors of the space follows from the characterisation of geodesics in products, see [BH99, I.5.3(3)]. Each factor \( X_i \) is thus a symmetric space in view of Theorem 6.4(iii). By virtue of Corollary 5.3(iii), the totally disconnected factors \( D_j \) act by semi-simple isometries.

Assume now that \( X \) has uniquely extensible geodesics. For the same reason as before, this property is inherited by each direct factor of the space. Thus each \( D_j \) is discrete by Proposition 6.9.

**Theorem 6.10.** Let \( X \) be a proper irreducible CAT(0) space with uniquely extensible geodesics. If \( X \) admits a non-discrete group of isometries with full limit set but no global fixed point at infinity, then \( X \) is a symmetric space.

The condition on fixed points at infinity is necessary in view of E. Heintze’s examples [Hei74] of negatively curved homogeneous manifolds which are not symmetric spaces. In fact these spaces consist of certain simply connected soluble Lie groups endowed with a left-invariant negatively curved Riemannian metric.

**Proposition 6.11.** Let \( X \) be a proper CAT(0) space with uniquely extensible geodesics. Then \( \partial X \) has finite dimension.

**Proof.** Let \( x \in X \) and recall that by Berestovskii’s result quoted in Theorem 6.8 above, \( \Sigma_x \) is isometric to a Euclidean sphere. By definition of the Tits angle, the “visual” map \( \partial X \to \Sigma_x \) associating to a geodesic ray its germ at \( x \) is Tits-continuous (in fact, \( L \)-Lipschitz). It is furthermore injective (actually, bijective) by unique extensibility. Therefore, the topological dimension of any \emph{compact} subset of \( \partial X \) is bounded by the dimension of the sphere \( \Sigma_x \).
claim follows now from Kleiner’s characterisation of the dimension of spaces with curvature bounded above in terms of the topological dimension of compact subsets (Theorem A in [Kle99]).

Proof of Theorem 6.10. By Lemma 2.13, the action of \( G := \text{Is}(X) \) is minimal. In view of Proposition 6.11, the boundary \( \partial X \) is finite-dimensional. Thus we can apply Theorem 1.1 and Addendum 1.4. Since \( X \) is irreducible and non-discrete, Proposition 6.9 implies that \( G \) is an almost connected simple Lie group (unless \( X = \mathbb{R} \), in which case \( X \) is indeed a symmetric space). We conclude by Theorem 6.4.

6.D. No open stabiliser at infinity. The following statement sums up some of the preceding considerations:

Corollary 6.12. Let \( X \) be a proper geodesically complete \( \text{CAT}(0) \) space without Euclidean factor such that some closed subgroup \( G < \text{Is}(X) \) acts cocompactly. Suppose that no open subgroup of \( G \) fixes a point at infinity. Then we have the following:

(i) \( X \) admits a canonical equivariant splitting

\[ X \cong X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q \]

where each \( X_i \) is a symmetric space and each \( Y_j \) possesses a \( G \)-equivariant locally finite decomposition into compact convex cells.

(ii) \( G \) possesses hyperbolic elements.

(iii) Every compact subgroup of \( G \) is contained in a maximal one; the maximal compact subgroups fall into finitely many conjugacy classes.

(iv) \( \mathcal{Z}(G) = 1 \); in particular \( G \) has no non-trivial discrete normal subgroup.

(v) \( \text{soc}(G^*) \) is a direct product of \( p + q \) non-discrete characteristically simple groups.

Proof. (i) Follows from Theorem 1.11 and Corollary 5.7.

(ii) Clear from Corollary 5.10.

(iii) and (iv) Immediate from (i) and Theorem 5.17(i) and (ii).

(v) Follows from (i), (iv) and Proposition 5.16.

Proof of Theorems 1.13 and 1.14. It suffices to observe that the above proof of (i) is immune to Euclidean factors (as is the statement of (i) itself actually).

6.E. Cocompact stabilisers at infinity. We undertake the proof of Theorem 1.18 which describes isometrically any geodesically complete proper \( \text{CAT}(0) \) space such that the stabiliser of every point at infinity acts cocompactly.

Remark 6.13. (i) The formulation of Theorem 1.18 allows for symmetric spaces of Euclidean type. (ii) A Bass–Serre tree is a tree admitting an edge-transitive automorphism group; in particular, it is regular or bi-regular (the regular case being a special case of Euclidean buildings).

Lemma 6.14. Let \( X \) be a proper \( \text{CAT}(0) \) space such that the stabiliser of every point at infinity acts cocompactly on \( X \). For any \( \xi \in \partial X \), the set of \( \eta \in \partial X \) with \( \angle_T(\xi, \eta) = \pi \) is contained in a single orbit under \( \text{Is}(X) \).

Proof. Write \( G = \text{Is}(X) \). In view of Proposition 6.1 applied to \( G_\xi \), it suffices to prove that the \( G \)-orbit of any such \( \eta \) contains a point opposite to \( \xi \). By definition of the Tits angle, there is a sequence \( \{x_n\} \) in \( X \) such that \( \angle_{x_n}(\xi, \eta) \) tends to \( \pi \). Since \( G_\xi \) acts cocompactly, it contains a sequence \( \{g_n\} \) such that, upon extracting, \( g_n x_n \) converges to some \( x \in X \) and \( g_n \eta \) to some \( \eta' \in \partial X \). The angle semi-continuity arguments given in the proof of
Proposition II.9.5(3) in [BH99] show that $\angle_x(\xi, \eta') = \pi$, recalling that all $g_n$ fix $\xi$. This means that there is a geodesic $\sigma : \mathbb{R} \to X$ through $x$ with $\sigma(-\infty) = \xi$ and $\sigma(\infty) = \eta'$. On the other hand, since $G_\eta$ is cocompact in $G$, the $G$-orbit of $\eta$ is closed in the cone topology. This means that there is $g \in G$ with $\eta' = g\eta$, as was to be shown. \hfill \Box

We shall need another form of angle rigidity (compare Proposition 5.8), this time for Tits angles.

**Proposition 6.15.** Let $X$ be a geodesically complete proper CAT(0) space, $G < \text{Is}(X)$ a closed totally disconnected subgroup and $\xi \in \partial X$. If the stabiliser $G_\xi$ acts cocompactly on $X$, then the $G$-orbit of $\xi$ is discrete in the Tits topology.

**Proof.** Suppose for a contradiction that there is a sequence $\{g_n\}$ such that $g_n\xi \neq \xi$ for all $n$ but $\angle_T(g_n\xi, \xi)$ tends to zero. Since $G_\xi$ is cocompact, we can assume that $g_n$ converges in $G$; since the Tits topology is finer than the cone topology for which the $G$-action is continuous, the limit of $g_n$ must fix $\xi$ and we can therefore assume $g_n \to 1$. Let $B \subseteq X$ be an open ball large enough so that $G_\xi.B = X$. Since by Lemma 2.13 we can apply Theorem 5.1, there is no loss of generality in assuming that each $g_n$ fixes $B$ pointwise.

Let $c : \mathbb{R}_+ \to X$ be a geodesic ray pointing towards $\xi$, $c(0) \in B$. For each $n$ there is $r_n > 0$ such that $c$ and $g_n c$ branch at the point $c(r_n)$. In particular, $g_n$ fixes $c(r_n)$ but not $c(r_n + \varepsilon)$ no matter how small $\varepsilon > 0$. We now choose $h_n \in G_\xi$ such that $x_n := h_n c(r_n) \in B$ and notice that the sequence $k_n := h_n g_n h_n^{-1}$ is bounded since $k_n$ fixes $x_n$. We can therefore assume upon extracting that it converges to some $k \in G$; in view of Theorem 5.1, we can further assume that all $k_n$ coincide with $k$ on $B$ and in particular $k$ fixes all $x_n$. Since $\angle_T(k_n \xi, \xi) = \angle_T(g_n \xi, \xi)$, we also have $k \in G_\xi$. Considering any given $n$, it follows now that $k$ fixes the ray from $x_n$ to $\xi$. Thus $k_n$ fixes an initial segment of this ray at $x_n$. This is equivalent to $g_n$ fixing an initial segment at $c(r_n)$ of the ray from $c(r_n)$ to $\xi$, contrary to our construction. \hfill \Box

Here is a first indication that our spaces might resemble symmetric spaces or Euclidean buildings:

**Proposition 6.16.** Let $X$ be a proper CAT(0) space such that the stabiliser of every point at infinity acts cocompactly on $X$. Then any point at infinity is contained in an isometrically embedded standard $n$-sphere in $\partial X$, where $n = \dim \partial X$.

**Proof.** Let $\eta \in \partial X$. There is some standard $n$-sphere $S$ isometrically embedded in $\partial X$ because $X$ is cocompact (Theorem C in [Kle99]). By Lemma 3.1 in [BL05], there is $\xi \in S$ with $\angle_T(\xi, \eta) = \pi$. Let $\vartheta \in S$ be the antipode in $S$ of $\xi$. In view of Lemma 6.14, there is an isometry sending $\vartheta$ to $\eta$. The image of $S$ contains $\eta$. \hfill \Box

We need one more fact for Theorem 1.18. The boundary of a CAT(0) space need not be complete, regardless of the geodesic completeness of the space itself; however, this is the case in our situation in view of Proposition 6.16:

**Corollary 6.17.** Let $X$ be a proper CAT(0) space such that the stabiliser of every point at infinity acts cocompactly on $X$. Then $\partial X$ is geodesically complete.

**Proof.** Suppose for a contradiction that some Tits-geodesic ends at $\xi \in \partial X$ and let $B \subseteq \partial X$ be a small convex Tits-neighbourhood of $\xi$; in particular, $B$ is contractible. Since by Proposition 6.16 there is an $n$-sphere through $\xi$ for $n = \dim \partial X$, the relative homology $H_n(B, B \setminus \{\xi\})$ is non-trivial. Our assumption implies that $B \setminus \{\xi\}$ is contractible by using the geodesic contraction to some point $\eta \in B \setminus \{\xi\}$ on the given geodesic ending at $\xi$. 


This implies $H_n(B, B \setminus \{\xi\}) = 0$, a contradiction. (This argument is adapted from [BH99, II.5.12].)

End of proof of Theorem 1.18. We shall use below that product decompositions preserve geodesic completeness (this follows e.g. from [BH99, I.5.3(3)]). We can reduce to the case where $X$ has no Euclidean factor. By Lemma 2.13, the group $G = \text{Is}(X)$ as well as all stabilisers of points at infinity act minimally. In particular, Proposition 6.3 ensures that $G$ has no fixed point at infinity and we can apply Theorem 1.1 and Addendum 1.4. Therefore, we can from now on assume that $X$ is irreducible. If the identity component $G^0$ is non-trivial, then Theorem 1.11 (see also Theorem 6.4(iii)) ensures that $X$ is a symmetric space, and we are done. We assume henceforth that $G$ is totally disconnected.

For any $\xi \in \partial X$, the collection $\text{Ant}(\xi) = \{\eta : \angle_T(\xi, \eta) = \pi\}$ of antipodes is contained in a $G$-orbit by Lemma 6.14 and hence is Tits-discrete by Proposition 6.15. This discreteness and the geodesic completeness of the boundary (Corollary 6.17) are the assumptions needed for Proposition 4.5 in [Lyt05], which states that $\partial X$ is a building. Since $X$ is irreducible, $\partial X$ is not a (non-trivial) spherical join, see Theorem II.9.24 in [BH99]. Thus, if this building has non-zero dimension, we conclude from the main result of [Lee00] that $X$ is a Euclidean building of higher rank.

If on the other hand $\partial X$ is zero-dimensional, then we claim that it is homogeneous under $G$. Indeed, we know already that for any given $\xi \in \partial X$, the set $\text{Ant}(\xi)$ lies in a single orbit. Since in the present case $\text{Ant}(\xi)$ is simply $\partial X \setminus \{\xi\}$, the claim follows from the fact that $G$ has no fixed point at infinity. Now we argue as in the proof of Theorem 6.4(iii) and deduce that $X$ is an edge-transitive tree. □
Part II. Discrete subgroups of the isometry group

7. An analogue of Borel density

Before discussing our analogue of Borel’s density theorem [Bor60] in Section 7.B below, we present a more elementary phenomenon based on co-amenability.

7.A. Fixed points at infinity. Recall that a subgroup $H$ of a topological group $G$ is co-amenable if any continuous affine $G$-action on a convex compact set (in a Hausdorff locally convex topological vector space) has a fixed point whenever it has an $H$-fixed point. The arguments of Adams–Ballmann [AB98a] imply the following preliminary step towards Theorem 7.4:

**Proposition 7.1.** Let $G$ be a topological group with a continuous isometric action on a proper $\text{CAT}(0)$ space $X$ without Euclidean factor. Assume that the $G$-action is minimal and does not have a global fixed point in $\partial X$.

Then any co-amenable subgroup of $G$ still has no global fixed point in $\partial X$.

**Proof.** Suppose for a contradiction that a co-amenable subgroup $H < G$ fixes $\xi \in \partial X$. Then $G$ preserves a probability measure $\mu$ on $\partial X$ and we obtain a convex function $f : X \to \mathbb{R}$ by integrating Busemann functions against this measure; as in [AB98a], the cocycle equation for Busemann functions (see §1.A) imply that $f$ is $G$-invariant up to constants. The arguments therein show that $f$ is constant and that $\mu$ is supported on flat points. However, in the absence of a Euclidean factor, the set of flat points has a unique circumcentre when non-empty [AB98a, 1.7]; this provides a $G$-fixed point, a contradiction. □

Combining the above with the splitting methods used in Theorem 3.3, we record a consequence showing that the exact conclusions of the Adams–Ballmann theorem [AB98a] hold under much weaker assumptions than the amenability of $G$.

**Corollary 7.2.** Let $G$ be a topological group with a continuous isometric action on a proper $\text{CAT}(0)$ space $X$. Assume that $G$ contains two commuting co-amenable subgroups.

Then either $G$ fixes a point at infinity or it preserves a Euclidean subspace in $X$.

We emphasise that one can easily construct a wealth of examples of highly non-amenable groups satisfying these assumptions. For instance, given any group $Q$, the restricted wreath product $G = \mathbb{Z} \rtimes \bigoplus_{n \in \mathbb{Z}} Q$ contains the pair of commuting co-amenable groups $H_+ = \bigoplus_{n \geq 0} Q$ and $H_- = \bigoplus_{n < 0} Q$, see [MP03]. (In fact, one can even arrange for $H_\pm$ to be conjugated upon replacing $\mathbb{Z}$ by the infinite dihedral group.)

For similar reasons, we deduce the following fixed-point property for R. Thompson’s group

$$F := \langle g_i, i \in \mathbb{N} \mid g_i^{-1}g_jg_i = g_{j+1} \forall j > i \rangle;$$

this fixed-point result explains why the strategy proposed in [Far08b] to disprove amenability of $F$ with the Adams–Ballmann theorem cannot work.

**Corollary 7.3.** Any $F$-action by isometries on any proper $\text{CAT}(0)$ space $X$ has a fixed point in $X$.

**Proof of Corollary 7.2.** We assume that $G$ has no fixed point at infinity. By Proposition 3.1, there is a minimal non-empty closed convex $G$-invariant subspace. Upon considering the Euclidean decomposition [BH99, II.6.15] of the latter, we can assume that $X$ is $G$-minimal and without Euclidean factor and need to show that $G$ fixes a point in $X$.

Let $H_\pm < G$ be the commuting co-amenable groups. In view of Proposition 7.1, both act without fixed point at infinity. In particular, we have an action of $H = H_+ \times H_-$.
without fixed point at infinity and the splitting theorem from [Mon06] provides us with a canonical subspace \( X_+ \times X_- \subseteq X \) with component-wise and minimal \( H \)-action. All of \( \partial X_+ \) is fixed by \( H_- \), which means that this boundary is empty. Since \( X \) is proper, it follows that \( X_+ \) is bounded and hence reduced to a point by minimality. Thus \( H_+ \) fixes a point in \( X \subseteq X \) and co-amenability implies that \( G \) fixes a probability measure \( \mu \) on \( X \). If \( \mu \) were supported on \( \partial X \), the proof of Proposition 7.1 would provide a \( G \)-fixed point at infinity, which is absurd. Therefore \( \mu(X) > 0 \). Now choose a bounded set \( B \subseteq X \) large enough so that \( \mu(B) > \mu(X)/2 \). Then any \( G \)-translate of \( B \) must meet \( B \). It follows that \( G \) has a bounded orbit and hence a fixed point as claimed. \( \square \)

**Proof of Corollary 7.3.** We refer to [CFP96] for a detailed introduction to the group \( F \). In particular, \( F \) can be realised as the group of all orientation-preserving piecewise affine homeomorphisms of the interval [0, 1] that have dyadic breakpoints and slopes \( 2^n \) with \( n \in \mathbb{Z} \). Given a subset \( A \subseteq [0, 1] \) we denote by \( F_A < F \) the subgroup supported on \( A \).

We claim that whenever \( A \) has non-empty interior, \( F_A \) is co-amenable in \( F \). The argument is analogous to [MP03] and to [GM07, §4.F]; indeed, in view of the alternative definition of \( F \) just recalled, one can choose a sequence \( \{g_n\} \) in \( F \) such that \( g_n A \) contains \([1/n, 1-1/n] \) and thus \( F_A \) contains \( F_{1/n,1-1/n} \). Consider the compact space of means on \( F/F_A \), namely finitely additive measures, endowed with the weak* topology from the dual of \( \ell^\infty(F/F_A) \). Any accumulation point \( \mu \) of the sequence of Dirac masses at \( g_n^{-1}F_A \) will be invariant under the union \( F' \) of the groups \( F_{1/n,1-1/n} \). Now \( F' \) is the kernel of the derivative homomorphism \( F \rightarrow 2^\mathbb{Z} \times 2^\mathbb{Z} \) at the pair of points \( \{0, 1\} \). In particular, \( F' \) is co-amenable in \( F \) and thus the \( F' \)-invariance of \( \mu \) implies that there is also a \( F' \)-invariant mean on \( F/F_A \), which is one of the characterisations of co-amenability [Eym72].

Let now \( X \) be any proper CAT(0) space with an \( F \)-action by isometries. We can assume that \( F \) has no fixed point at infinity and therefore we can also assume that \( X \) is minimal by Proposition 3.1. The above claim provides us with many pairs of commuting co-amenable subgroups upon taking disjoint sets of non-empty interior. Therefore, Corollary 7.2 shows that \( X \cong \mathbb{R}^n \) for some \( n \). In particular the isometry group is linear. Since \( F \) is finitely generated (by \( g_0 \) and \( g_1 \) in the above presentation, compare also [CFP96]), Malcev’s theorem [Mal40] implies that the image of \( F \) is residually finite. The derived subgroup of \( F \) (which incidentally coincides with the group \( F' \) introduced above) being simple [CFP96], it follows that it acts trivially. It remains only to observe that two commuting isometries of \( \mathbb{R}^n \) always have a common fixed point in \( \mathbb{R}^n \), which is a matter of linear algebra. \( \square \)

The above reasoning can be adapted to yield similar results for branch groups and related groups; we shall address these questions elsewhere.

### 7.B. Geometric density for subgroups of finite covolume.

The following geometric density theorem generalises Borel’s density (see Proposition 7.8 below) and contains Theorem 1.19 from the Introduction.

**Theorem 7.4.** Let \( G \) be a locally compact group with a continuous isometric action on a proper CAT(0) space \( X \) without Euclidean factor.

If \( G \) acts minimally and without global fixed point in \( \partial X \), then any closed subgroup with finite invariant covolume in \( G \) still has these properties.

**Remark 7.5.** For a related statement without the assumption on the Euclidean factor of \( X \) or on fixed points at infinity, see Theorem 8.14 below.
Proof. Retain the notation of the theorem and let \( \Gamma < G \) be a closed subgroup of finite invariant covolume. In particular, \( \Gamma \) is co-amenable and thus has no fixed points at infinity by Proposition 7.1. By Proposition 3.1, there is a minimal non-empty closed convex \( \Gamma \)-invariant subset \( Y \subseteq X \) and it remains to show \( Y = X \). Choose a point \( x_0 \in X \) and define \( f : X \to \mathbb{R} \) by

\[
f(x) = \int_{G/\Gamma} (d(x,gY) - d(x_0,gY)) \, dg.
\]

This integral converges because the integrand is bounded by \( d(x,x_0) \). The function \( f \) is continuous, convex (by [BH99, II.2.5(1)]) and “quasi-invariant” in the sense that it satisfies

\[
(7.i) \quad f(hx) = f(x) - f(hx_0) \quad \forall h \in G.
\]

Since \( G \) acts minimally and without fixed point at infinity, this implies that \( f \) is constant (see Section 2 in [AB98a]; alternatively, when \( \partial X \) is finite-dimensional, it follows from Theorem 1.6 since (7.i) implies that \( f \) is invariant under the derived subgroup \( G' \)).

In particular, \( d(x,gY) \) is affine for all \( g \). It follows that for all \( x \in X \) the closed set

\[
Y_x = \{ z \in X : d(z,Y) = d(x,Y) \}
\]

is convex. We claim that it is parallel to \( Y \) in the sense that \( d(z,Y) = d(y,Y_x) \) for all \( z \in Y_x \) and all \( y \in Y \). Indeed, on the one hand \( d(z,Y) \) is constant over \( z \in Y_x \) by definition, and on the other hand \( d(y,Y_x) \) is constant by minimality of \( Y \) since \( d(\cdot,Y_x) \) is a convex \( \Gamma \)-invariant function. In particular, \( Y_x \) is \( \Gamma \)-equivariantly isometric to \( Y \) via nearest point projection (compare [BH99, II.2.12]) and each \( Y_x \) is \( \Gamma \)-minimal. At this point, Remarks 39 in [Mon06] show that there is an isometric \( \Gamma \)-invariant splitting

\[
X \cong Y \times T.
\]

It remains to show that the “space of components” \( T \) is reduced to a point. Let thus \( s,t \in T \) and let \( m \) be their midpoint. Applying the above reasoning to the choice of minimal set \( Y_0 \) corresponding to \( Y \times \{m\} \), we deduce again that the distance to \( Y_0 \) is an affine function on \( X \). However, this function is precisely the distance function \( d(\cdot,m) \) in \( T \) composed with the projection \( X \to T \). Being non-negative and affine on \([s,t] \), it vanishes on that segment and hence \( s = t \).

\[ \square \]

Remark 7.6. When \( \Gamma \) is cocompact in \( G \), the proof can be shortened by integrating just \( d(x,gY) \) in the definition of \( f \) above.

Corollary 7.7. Let \( X \) be a proper CAT(0) space without Euclidean factor such that \( G = \text{Is}(X) \) acts minimally without fixed point at infinity, and let \( \Gamma < G \) be a closed subgroup with finite invariant covolume. Then:

(i) \( \Gamma \) has trivial amenable radical.

(ii) The centraliser \( \mathcal{Z}_G(\Gamma) \) is trivial.

(iii) If \( \Gamma \) is finitely generated, then is has finite index in its normaliser \( \mathcal{N}_G(\Gamma) \) and the latter is a finitely generated lattice in \( G \).

Proof. (i) and (ii) follow by the same argument as in the proof of Theorem 1.6. For (iii) we follow [Mar91, Lemma II.6.3]. Since \( \Gamma \) is closed and countable, it is discrete by Baire’s category theorem and thus is a lattice in \( G \). Since it is finitely generated, its automorphism group is countable. By (ii), the normaliser \( \mathcal{N}_G(\Gamma) \) maps injectively to \( \text{Aut}(\Gamma) \) and hence is countable as well. Thus \( \mathcal{N}_G(\Gamma) \), being closed in \( G \), is discrete by applying Baire again. Since it contains the lattice \( \Gamma \), it is itself a lattice and the index of \( \Gamma \) in \( \mathcal{N}_G(\Gamma) \) is finite. Thus \( \mathcal{N}_G(\Gamma) \) is finitely generated. \( \square \)
As pointed out by P. de la Harpe, point (ii) implies in particular that any lattice in \( G \) is ICC (which means by definition that all its non-trivial conjugacy classes are infinite). As is well known, this is the criterion ensuring that the type \( \Pi_1 \) von Neumann algebra associated to the lattice is a factor [Tak02, §V.7].

Finally, we indicate why Theorem 7.4 implies the classical Borel density theorem of [Bor60]. It suffices to justify the following:

**Proposition 7.8.** Let \( k \) be a local field (Archimedean or not), \( G \) a semi-simple \( k \)-group without \( k \)-anisotropic factors, \( X \) the symmetric space or Bruhat-Tits building associated to \( G = G(k) \) and \( L \triangleleft G \) any subgroup. If the \( L \)-action on \( X \) is minimal without fixed point at infinity, then \( L \) is Zariski-dense.

**Proof.** Let \( \bar{L} \) be the (\( k \)-points of the) Zariski closure of \( L \). Then \( \bar{L} \) is semi-simple; this follows e.g. from a very special case of Corollary 4.8, which guarantees that the radical of \( \bar{L} \) is trivial.

In the Archimedean case, we may appeal to Karpelevich–Mostow theorem (see [Kar53] or [Mos55]): any semi-simple subgroup has a totally geodesic orbit in the symmetric space. So the only semi-simple subgroup acting minimally is \( G \) itself.

In the non-Archimedean case, we could appeal to E. Landvogt functoriality theorem [Lan00] which would finish the proof. However, there is an alternative direct and elementary argument which avoids appealing to loc. cit. and goes as follows. First notice that, by the same argument as in the proof of Theorem 6.4(iv), the \( k \)-rank of a semi-simple subgroup acting minimally equals the \( k \)-rank of \( G \) (this holds in all cases, not only in the non-Archimedean one). Therefore, the inclusion of spherical buildings \( \mathcal{B}L \to \mathcal{B}G \) provided by the group inclusion \( L \to G \) has the property that \( \mathcal{B}L \) is a top-dimensional sub-building of \( \mathcal{B}G \). An elementary argument (see [KL06, Lemma 3.3]) shows that the union \( Y \) of all apartments of \( X \) bounded by a sphere in \( \mathcal{B}L \) is a closed convex subset of \( X \). Clearly \( Y \) is \( L \)-invariant, hence \( Y = X \) by minimality. Therefore \( \mathcal{B}L = \mathcal{B}G \), which finally implies that \( \bar{L} = G \). □

7.C. The limit set of subgroups of finite covolume. Let \( X \) be a complete CAT(0) space and \( G \) a group acting by isometries on \( X \). Recall that the limit set \( \Lambda G \) of \( G \) is the intersection of the boundary \( \partial X \) with the closure of the orbit \( G.x_0 \) in \( X = X \cup \partial X \) of any \( x_0 \in X \), this set being independent of \( x_0 \).

**Proposition 7.9.** Let \( G \) be a locally compact group acting continuously by isometries on a complete CAT(0) space \( X \). If \( \Gamma \triangleleft G \) is any closed subgroup with finite invariant covolume, then \( \Lambda \Gamma = \Lambda G \).

Consider the following immediate corollary, which in the special case of Hadamard manifolds follows from the duality condition, see 1.9.16 and 1.9.32 in [Ebe96].

**Corollary 7.10.** Let \( G \) be a locally compact group with a continuous action by isometries on a proper CAT(0) space. If the \( G \)-action is cocompact, then any lattice in \( G \) has full limit set in \( \partial X \). □

**Proof of Proposition 7.9.** We observe that for any non-empty open set \( U \subseteq G \) there is a compact set \( C \subseteq G \) such that \( U^{-1}\Gamma C = G \). Indeed, (using an idea of Selberg, compare Lemma 1.4 in [Bor60]), it suffices to take \( C \) so large that

\[ \mu(\Gamma C) > \mu(\Gamma \setminus G) - \mu(\Gamma U), \]

where \( \mu \) denotes an invariant measure on \( \Gamma \setminus G \); any right translate of \( \Gamma U \) in \( \Gamma \setminus G \) will then meet \( \Gamma C \).
Now let $\xi \in \Delta G$ and $x_0 \in X$. For any neighbourhood $V$ of $\xi$ in $\partial X$, we shall construct an element in $\Delta \Gamma \cap V$. Let $U \subseteq G$ be a compact neighbourhood of the identity in $G$ such that $U\xi \subseteq V$ and let $\{g_n\}$ be a sequence of elements of $G$ with $g_n x_0$ converging to $\xi$ (one uses nets if $X$ is not separable). In view of the above observation, there are sequences $\{u_n\}$ in $U$ and $\{c_n\}$ in $C$ such that $u_n g_n c_n^{-1} \in \Gamma$. The points $g_n c_n^{-1} x_0$ remain at bounded distance of $g_n x_0$ as $n \to \infty$, and thus converge to $\xi$. Therefore, choosing an accumulation point $u$ of $\{u_n\}$ in $U$, we see that $u\xi$ is an accumulation point of $\{u_n g_n c_n^{-1} x_0\}$, which is a sequence in $\Gamma x_0$. \hfill $\square$

For future use, we observe a variant of the above reasoning yielding a more precise fact in a simpler situation:

**Lemma 7.11.** Let $G$ be a locally compact group with a continuous cocompact action by isometries on a proper CAT(0) space $X$. Let $\Gamma < G$ be a lattice and $c: \mathbb{R}_+ \to X$ a geodesic ray such that $G$ fixes $c(\infty)$. Then there is a sequence $\{\gamma_i\}$ in $\Gamma$ such that $\gamma_i c(i)$ remains bounded over $i \in \mathbb{N}$.

**Proof.** For the same reason as above, there is a compact set $U \subseteq G$ such that $G = U \Gamma U^{-1}$. Choose now $\{g_i\}$ such that $g_i c(i)$ remains bounded and write $g_i = u_i \gamma_i v_i^{-1}$ with $u_i, v_i \in U$. We have

$$d(\gamma_i c(i), c(0)) = d(g_i v_i c(i), u_i c(0)) \leq d(g_i v_i c(i), g_i c(i)) + d(g_i c(i), u_i c(0)) \leq d(v_i c(i), c(i)) + d(g_i c(i), c(0)) + d(u_i c(0), c(0)).$$

This is bounded independently of $i$ because $d(v_i c(i), c(i)) \leq d(v_i c(0), c(0))$ since $c(\infty)$ is $G$-fixed. \hfill $\square$

We shall also need the following:

**Lemma 7.12.** A locally compact group containing a finitely generated subgroup whose closure has finite covolume is compactly generated.

**Proof.** Denoting the closure of the given finitely generated subgroup by $\Gamma$, we can write $G = U \Gamma C$ as in the proof of Proposition 7.9 with both $U$ and $C$ compact. Since $\Gamma$ is a locally compact group containing a finitely generated dense subgroup, it is compactly generated and the conclusion follows. \hfill $\square$

8. CAT(0) lattices

8.A. Preliminaries on lattices. We begin this section with a few well known basic facts about general lattices.

**Proposition 8.1.** Let $G$ be a locally compact second countable group and $N < G$ be a closed normal subgroup.

(i) Given a closed cocompact subgroup $\Gamma < G$, the projection of $\Gamma$ on $G/N$ is closed if and only if $\Gamma \cap N$ is cocompact in $N$.

(ii) Given a lattice $\Gamma < G$, the projection of $\Gamma$ on $G/N$ is discrete if and only if $\Gamma \cap N$ is a lattice in $N$.

**Proof.** See Theorem 1.13 in [Rag72]. \hfill $\square$

The second well known result is straightforward to establish:
Lemma 8.2. Let $G = H \times D$ be a locally compact group. Given a lattice $\Gamma < G$ and a compact open subgroup $Q < D$, the subgroup $\Gamma_Q := \Gamma \cap (H \times Q)$ is a lattice in $H \times Q$, which is commensurated by $\Gamma$.

If moreover $G/\Gamma$ is compact, then so is $(H \times Q)/\Gamma_Q$. \hfill \Box

(As we shall see in Lemma 10.14 below, there is a form of converse.)

Let $X$ be a proper CAT(0) space and $G = \text{Is}(X)$ be its isometry group. Given a discrete group $\Gamma$ acting properly and cocompactly on $X$, then the quotient $G/\Gamma$ is compact and the image of $\Gamma$ in $G$ is a cocompact lattice (note that the kernel of the map $\Gamma \to \text{Is}(X)$ is finite). Conversely, if the quotient $G/\Gamma$ is compact, then any cocompact lattice of $G$ is a discrete group acting properly and cocompactly on $X$.

Lemma 8.3. In the above setting, $G$ is compactly generated and $\Gamma$ is finitely generated.

Proof. For lack of finding a classical reference, we refer to Lemma 22 in [MMS04]. \hfill \Box

8.B. Variations on Auslander’s theorem.

Lemma 8.4. Let $A = \mathbb{R}^n \rtimes O(n)$ and $S$ be a semi-simple Lie group without compact factor. Any lattice $\Gamma$ in $G = A \times S$ has a finite index subgroup $\Gamma^0$ which splits as a direct product $\Gamma^0 \cong \Gamma_A \times \Gamma'$, where $\Gamma_A = \Gamma \cap (A \times 1)$ is a lattice in $(A \times 1)$.

Proof. Let $V = \mathbb{R}^n$ denote the translation subgroup of $A$ and $U$ denote the closure of the projection of $\Gamma$ to $S$. The subgroup $U < S$ is closed of finite covolume; therefore it is either discrete or it contains a semi-simple subgroup of positive dimension by Borel’s density theorem (in fact one could be more precise using the Main Result of [Pra77], but this is not necessary for the present purposes). On the other hand, Auslander’s theorem [Rag72, Theorem 8.24] ensures that the identity component of the projection of $\Gamma$ in $S \times A/U$ is soluble, from which it follows that $U$ has a connected soluble normal subgroup. Thus $U$ is discrete. Therefore, by Proposition 8.1, the group $\Gamma_A = \Gamma \cap (A \times 1)$ is a lattice in $(A \times 1)$. In particular $\Gamma_A$ is virtually Abelian [Flm97, Corollary 4.1.13].

Since the projection of $\Gamma$ to $S$ is a lattice in $S$, it is finitely generated [Rag72, 6.18]. Therefore $\Gamma$ possesses a finitely generated subgroup $\Lambda$ containing $\Gamma_A$ and whose projection to $S$ coincides with the projection of $\Gamma$. Notice that $\Lambda$ is a lattice in $S \times A$ by [Sim96, Theorem 23.9.3]; therefore $\Lambda$ has finite index in $\Gamma$, which shows that $\Gamma$ is finitely generated.

Since $\Gamma$ is normal in $\Gamma_A$, the projection $\Gamma^A$ of $\Gamma$ to $A$ normalises the lattice $\Gamma_A$ and is thus virtually Abelian. Hence $\Gamma^A$ is a finitely generated virtually Abelian group which normalises $\Gamma_A$. Therefore $\Gamma^A$ has a finite index subgroup which splits as a direct product of the form $\Gamma_A \times C$, and the preimage $\Gamma'$ of $C$ in $\Gamma$ is a normal subgroup which intersects $\Gamma_A$ trivially. In particular the group $\Gamma' \cdot \Gamma_A \cong \Gamma' \times \Gamma_A$ is a finite index normal subgroup of $\Gamma$, as desired. \hfill \Box

Lemma 8.5. Let $\Gamma$ be a group containing a subgroup of the form $\Gamma^0 \cong \Gamma_S^0 \times \Gamma_A^0$, where $\Gamma_S^0$ is isomorphic to a lattice in a semi-simple Lie group with trivial centre and no compact factor, and $\Gamma_A^0$ is amenable. If $\Gamma$ commensurates $\Gamma^0$, then $\Gamma$ commensurates both $\Gamma_S^0$ and $\Gamma_A^0$.

Proof. Let $\Gamma^1 \cong \Gamma_S^1 \times \Gamma_A^1$ be a conjugate of $\Gamma^0$ in $\Gamma$. The projection of $\Gamma^0 \cap \Gamma^1$ to $\Gamma_S^0$ is a finite index subgroup of $\Gamma_S^0$. By Borel density theorem, it must therefore have trivial amenable radical. In particular the projection of $\Gamma^0 \cap \Gamma^1$ to $\Gamma_S^0$ is trivial. Therefore the image of the projection of $\Gamma^0 \cap \Gamma^1$ (resp. $\Gamma^0 \cap \Gamma_A^1$) to $\Gamma_S^0$ (resp. $\Gamma_A^0$) is of finite index. The desired assertion follows. \hfill \Box

Proposition 8.6. Let $A = \mathbb{R}^n \rtimes O(n)$, $S$ be a semi-simple Lie group with trivial centre and no compact factor, $D$ be a totally disconnected locally compact group and $G = S \times A \times D$. 


Then any finitely generated lattice $\Gamma < G$ has a finite index subgroup $\Gamma_0$ which splits as a direct product $\Gamma_0 \cong \Gamma_A \times \Gamma'$, where $\Gamma_A \subseteq \Gamma \cap (1 \times A \times D)$ is a finitely generated virtually Abelian subgroup whose projection to $A$ is a lattice.

Proof. Let $Q < D$ be a compact open subgroup. By Lemma 8.2, the intersection $\Gamma^0 = \Gamma \cap (S \times A \times Q)$ is a lattice in $S \times A \times Q$, which is commensurated by $\Gamma$. Since $Q$ is compact, the projection of $\Gamma^0$ to $S \times A$ is a lattice, to which we may apply Lemma 8.4. Upon replacing $\Gamma^0$ by a finite index subgroup (which amounts to replacing $Q$ by an open subgroup), this yields two normal subgroups $\Gamma^0_S, \Gamma^0_A < \Gamma^0$ and a decomposition $\Gamma^0 = \Gamma^0_S \cdot \Gamma^0_A$, where $\Gamma^0_S \cap \Gamma^0_A \subseteq Q$ and $\Gamma^0_A = \Gamma^0 \cap (1 \times A \times Q)$ is a finitely generated virtually Abelian group whose projection to $A$ is a lattice.

By virtue of Lemma 8.5, we deduce that the image of the projection of $\Gamma$ to $A$ commensurates a lattice in $A$. But the commensurator of any lattice in $A$ is virtually Abelian. Therefore, upon replacing $\Gamma$ by a finite index subgroup, it follows that the projection of $\Gamma$ to $A$ normalises the projection of $\Gamma^0_A$. We now define

$$\Gamma_A = \bigcap_{\gamma \in \Gamma} \gamma \Gamma^0_A \gamma^{-1}.$$

Then the projection of $\Gamma_A$ coincides with the projection of $\Gamma^0_A$; in particular it is still a lattice. Furthermore, the subgroup $\Gamma_A$ is normal in $\Gamma$. We now proceed as in the proof of Lemma 8.4. Since the projection of $\Gamma$ to $A$ is finitely generated and virtually Abelian, we may thus find in this group a virtual complement to the image of the projection of $\Gamma_A$.

Let $\Gamma'$ be the preimage of this complement in $\Gamma$. Then, upon replacing $\Gamma$ by a finite index subgroup, the group $\Gamma'$ is normal in $\Gamma$ and $\Gamma = \Gamma_A \cdot \Gamma'$. Since $\Gamma_A$ is normal as well, the commutator $[\Gamma_A, \Gamma']$ is contained in the intersection $\Gamma_A \cap \Gamma'$, which is trivial by construction. This finally shows that $\Gamma \cong \Gamma_A \times \Gamma'$, as desired. \hfill $\square$

Remark 8.7. In the setting of Proposition 8.6, assume that any compact subgroup of $D$ normalised by $\Gamma$ is trivial. Then $\Gamma_A \subseteq 1 \times A \times 1$ and the projection of $\Gamma$ to $S \times D$ is discrete. Indeed, the definition of $\Gamma_A$ given in the proof shows that it is contained in $1 \times A \times \gamma Q \gamma^{-1}$ for all $\gamma \in \Gamma$ and under the current assumptions the intersection $\bigcap_{\gamma \in \Gamma} \gamma Q \gamma^{-1}$ is trivial. The claim about the projection to $S \times D$ follows from Proposition 8.1.

8.C. Lattices, the Euclidean factor and fixed points at infinity. Given a proper CAT(0) space $X$ and a discrete group $\Gamma$ acting properly and cocompactly, it is a well known open question, going back to M. Gromov [Gro93, §6.B.3], to determine whether the presence of an $n$-dimensional flat in $X$ implies the existence of a free Abelian group of rank $n$ in $\Gamma$. (In the manifold case, see problem 65 on Yau’s list [Yau82].) Here we propose the following theorem; the special case where $X/\Gamma$ is a compact Riemannian manifold is the main result of Eberlein’s article [Ebe83] (compare also the earlier Theorem 5.2 in [Ebe80]).

**Theorem 8.8.** Let $X$ be a proper CAT(0) space such that $G = \text{Is}(X)$ acts cocompactly. Suppose that $X \cong \mathbb{R}^n \times X'$.

(i) Any finitely generated lattice $\Gamma < G$ has a finite index subgroup $\Gamma_0$ which splits as a direct product $\Gamma_0 \cong \mathbb{Z}^n \times \Gamma'$.

(ii) If moreover $X$ is $G$-minimal (e.g. if $X$ is geodesically complete), then $\mathbb{Z}^n$ acts trivially on $X'$ and as a lattice on $\mathbb{R}^n$; the projection of $\Gamma$ to $\text{Is}(X')$ is discrete.

We recall that cocompact lattices are automatically finitely generated in the above setting, Lemma 8.3. The following example shows that, without the assumption that $G$ acts minimally, the projection of $\Gamma$ to $\text{Is}(X')$ should not be expected to have discrete image:
Example 8.9. Let $X$ be the closed submanifold of $\mathbb{R}^3$ defined by $X = \{(x, y, z) \in \mathbb{R}^3 \mid 1 \leq z \leq 2\}$ and consider the following Riemannian metric on $X$:

$$ds^2 = dx^2 + z^2 dy^2 + dz^2.$$ 

One readily verifies that it is non-positively curved; thus $X$ is a CAT(0) manifold. Clearly $X$ splits off a one-dimensional Euclidean factor along the $x$-axis. Moreover the group $H \cong \mathbb{R}^2$ of all translations along the $xy$-plane preserves $X$ and acts cocompactly. Let $\Gamma$ be the subgroup of $H$ generated by $a$ and $b$, where

$$a : (x, y, z) \mapsto (x, y, z) + (\sqrt{2}, 1, 0) \quad \text{and} \quad b : (x, y, z) \mapsto (x, y, z) + (1, \sqrt{2}, 0).$$

Then $\Gamma \cong \mathbb{Z}^2$ is a cocompact lattice in $\text{Is}(X)$, but no non-trivial subgroup of $\Gamma$ acts trivially on the $yz$-factor of $X$. The projection of $\Gamma$ to the isometry group of that factor is not discrete (see Proposition 8.1(ii)).

The above result is the converse to the Flat Torus Theorem when it is stated as in [BH99, II.7.1]. In particular we deduce that the dimension of the Euclidean de Rham factor is an invariant of $\Gamma$. In the manifold case, again, this is the main point of [Ebe83].

Corollary 8.10. Let $X$ be a proper CAT(0) space such that $G = \text{Is}(X)$ acts cocompactly and minimally. Let $\Gamma < G$ be a finitely generated lattice.

Then the dimension of the Euclidean factor of $X$ equals the maximal rank of a free Abelian normal subgroup of $\Gamma$.

In order to apply Theorem 1.1 and Addendum 1.4 towards Theorem 8.8, we will need the following.

Theorem 8.11. Let $X$ be a proper CAT(0) space such that $G = \text{Is}(X)$ acts cocompactly and contains a finitely generated lattice. Then $X$ contains a canonical closed convex $G$-invariant $G$-minimal subset $X' \neq \emptyset$ which has no $\text{Is}(X')$-fixed point at infinity.

Consider the immediate corollary.

Corollary 8.12. Let $X$ be a proper CAT(0) space such that $G = \text{Is}(X)$ acts cocompactly and minimally. If $G$ contains a finitely generated lattice, then $G$ has no fixed point at infinity. \hfill $\square$

This shows that the mere existence of a finitely generated lattice imposes restrictions on cocompact CAT(0) spaces; much more detailed results in that spirit will be given in Section 11.

We do not know whether the statement of Corollary 8.12 remains true without the finite generation assumption on the lattice (see Problem 12.3 below).

Example 8.13. We emphasise that the full isometry group of a cocompact proper CAT(0) space may have global fixed points at infinity; in fact, the space might even be homogeneous, as it is the case for E. Heintze’s manifolds [Hei74] mentioned earlier. An even simpler way to construct cocompact proper CAT(0) space with this property is to mimic Example 6.6: Start from a regular tree $T$, assuming for definiteness that the valency is three. Replace every vertex by a congruent copy of an isosceles triangle that is not equilateral, in such a way that its distinguished vertex always points to a fixed point at infinity (of the initial tree). Then the stabiliser $H$ in $\text{Is}(T)$ of that point at infinity still acts faithfully and cocompactly on the modified space $T'$; the construction is so that the isometry group of $T'$ is in fact reduced to $H$. 
We shall also establish a strengthening of Corollary 8.12, which can be viewed as a form of Borel (or geometric) density theorem without assumption about fixed points at infinity.

**Theorem 8.14.** Let $X$ be a proper CAT(0) space such that $G = \text{Is}(X)$ acts cocompactly and minimally. Assume there is a finitely generated lattice $\Gamma < G$. Then $\Gamma$ acts minimally on $X$ and moreover all $\Gamma$-fixed points at infinity are contained in the boundary of the (possibly trivial) Euclidean factor of $X$.

We now turn to the proofs. In the case of a discrete cocompact group $\Gamma = G$, a version of the following was first established by Burger–Schroeder [BS87] (as pointed out in [AB98a, Corollary 2.7]).

**Proposition 8.15.** Let $X$ be a proper CAT(0) space, $G < \text{Is}(X)$ a closed subgroup whose action on $X$ is cocompact and $\Gamma < G$ a finitely generated lattice. Then there exists a $\Gamma$-invariant closed convex subset $Y \subseteq X$ which splits $\Gamma$-equivariantly as $Y = E \times W$, where $E$ is a (possibly 0-dimensional) Euclidean space on which $\Gamma$ acts by translations and such that $\partial E$ contains the fixed point set of $G$ in $\partial X$.

**Proof.** We can assume that there are $G$-fixed points at infinity, since otherwise there is nothing to prove. We claim that for any $G$-fixed point $\xi$ there is a geodesic line $\sigma : \mathbb{R} \to X$ with $\sigma(+\infty) = \xi$ such that any $\gamma \in \Gamma$ moves $\sigma$ to within a bounded distance of itself — and hence to a parallel line by convexity of the metric.

Indeed, let $c : \mathbb{R}_+ \to X$ be a geodesic ray with $c(\infty) = \xi$ and let $\{\gamma_i\}$ be as in Lemma 7.11. Then, by Arzelà–Ascoli, there is a subsequence $I \subseteq \mathbb{N}$ and a geodesic line $\sigma : \mathbb{R} \to X$ such that $\sigma(t) = \lim_{i \in I} \gamma_i c(t + i)$ for all $t$. Since each $g \in G$ has bounded displacement along $c$, the sequence $\{\gamma_i g \gamma_i^{-1}\}_{i \in I}$ is bounded and thus we can assume that it converges for all $g$ (recalling that $G$ is second countable, but we shall only consider $g \in \Gamma$ anyway). Since $\Gamma$ is discrete and finitely generated, we can further restrict $I$ so that there is $\gamma_\infty \in \Gamma$ such that

$$\gamma_i \gamma_i^{-1} = \gamma_\infty \gamma_\infty^{-1} \quad \forall \gamma \in \Gamma, i \in I.$$  

Since

$$d(\gamma \gamma_\infty^{-1} \sigma(t), \gamma_\infty^{-1} \sigma(t)) = \lim_{i \in I} d(\gamma_i^{-1} \gamma_\infty \gamma_\infty^{-1} \gamma_i c(t + i), c(t + i))$$  

$$= \lim_{i \in I} d(\gamma c(t + i), c(t + i)) \leq d(\gamma c(0), c(0)),$$

It now follows that every $\gamma \in \Gamma$ has bounded displacement length along the geodesic $\gamma_\infty^{-1} \sigma$. Thus the same holds for the geodesic $\sigma$ which is therefore (by convexity) translated to a parallel line by each element of $\Gamma$ as claimed.

Consider a flat $E \subseteq X$ that is maximal for the property that each element of $\Gamma$ has constant displacement length on $E$. Let $Y$ be the union of all flats that are at finite distance from $E$. One shows that $Y$ splits as $Y \cong E \times W$ for some closed convex $W \subseteq X$ using the Sandwich Lemma [BH99, II.2.12] and Lemma II.2.15 of [BH99] just like in Section 2.B above. The definition of $Y$ shows that $\Gamma$ preserves $Y$ as well as its splitting and acts on the $E$ coordinate by translations.

It remains to show that any $G$-fixed point $\xi \in \partial X$ belongs to $\partial E$. First, $\xi \in \partial Y$ since $\partial Y = \partial X$ by Corollary 7.10; we thus represent $\xi$ by a ray $c : \mathbb{R}_+ \to Y$. Let now $\sigma$ be a geodesic line as provided by the claim. We can assume that $\sigma$ lies in $Y$ because it was constructed from $\Gamma$-translates of $c$ and $Y$ is $\Gamma$-invariant. One can write $\sigma = (\sigma_E, \sigma_W)$ where $\sigma_E, \sigma_W$ are linearly re-parametrised geodesics in $E$ and $W$, see [BH99, I.5.3]. We need to prove that $\sigma_W$ has zero speed. Since any given $\gamma \in \Gamma$ has constant displacement along $\sigma$ and on each of the parallel copies of $E$ individually, its displacement is constant on the union of
all parallel copies of $E$ visited by $\sigma$, which is $E \times \sigma_W(R)$. The latter being again a flat, the maximality of $E$ shows that $\sigma_W$ is constant.

Proof of Theorem 8.11. Let $Y = E \times W \subseteq X$ be as in Proposition 8.15. Recall that $\partial Y = \partial X$ by Corollary 7.10. We claim that $\partial X$ has circumradius $> \pi/2$. Indeed, it would otherwise have a $G$-fixed circumcentre by Proposition 2.1, but this circumcentre cannot belong to $\partial E$ since $E$ is Euclidean; this contradicts Proposition 8.15. We now apply Corollary 2.10. This yields a canonical $G$-invariant closed convex subset $X'$, which is minimal with respect to the property that $\partial X' = \partial X$. It follows in particular by Corollary 7.10 that $\Gamma$ acts minimally on $X'$. Let now $X' = E' \times X_0'$ be the canonical splitting, where $E'$ is the maximal Euclidean factor [BH99, II.6.15]. On the one hand, since $X'$ is $\Gamma$-minimal, Proposition 8.15 applied to $X'$ shows that $G$ has no fixed points in $\partial X_0'$ since $E'$ is maximal as a Euclidean factor. On the other hand, $\text{Is}(E')$ fixes no point at infinity on $E'$. We deduce that $\text{Is}(X') \cong \text{Is}(E') \times \text{Is}(X_0')$ has indeed no fixed point at infinity.

End of proof of Theorem 8.14. Arguing as in the proof of Theorem 8.11, we establish that $X$ is $\Gamma$-minimal. Let $X = X' \times E$ be the canonical splitting, where $E$ is the maximal Euclidean factor. Since any isometry of $X$ decomposes uniquely as isometries of $E$ and $X'$ (II.6.15 in [BH99]), it suffices to show that $\Gamma$ has no fixed point in $\partial X'$. This follows from Proposition 7.1 applied to the $G$-action on $X'$.

End of proof of Theorem 8.8. Assume first that $X$ is $G$-minimal, recalling that this is the case if $X$ is geodesically complete by Lemma 2.13. In view of Corollary 8.12, we can apply Theorem 1.1 and we are therefore in the setting of Proposition 8.6. Since the group $\Gamma_A$ provided by that proposition contains a finite index subgroup isomorphic to $\mathbb{Z}^n$, we have already established (i) under the additional minimality assumption.

In order to show (ii), it suffices by Remark 8.7 to prove that any compact subgroup of $G$ normalised by $\Gamma$ is trivial. This follows from the fact that $X$ is $\Gamma$-minimal, as established in Theorem 8.14.

It remains to prove (i) without the assumption that $X$ is $G$-minimal. Let $Y \subseteq X$ be the $G$-minimal set provided by Theorem 8.11 and let $Y = \mathbb{R}^m \times Y'$ be its Euclidean decomposition. Then $m \geq n$ because of the characterisation of the Euclidean factor in terms of Clifford isometries [BH99, II.6.15]; indeed, any (non-trivial) Clifford isometry of $X$ restricts non-trivially to $Y$ because $Y$ has finite co-diameter. The kernel $F < \Gamma$ of the $\Gamma$-action on $Y$ is finite and thus we can assume that it is central upon replacing $\Gamma$ with a finite index subgroup. Passing to a further finite index subgroup, we know from the minimal case that $\Gamma/F$ splits as $\Gamma/F = \mathbb{Z}^m \times \Lambda'$. Let $\Gamma_{\mathbb{Z}^m}, \Gamma' < \Gamma$ be the pre-images in $\Gamma$ of those factors. Thus we can write $\Gamma = \Gamma_{\mathbb{Z}^m} \cdot \Gamma'$ with $\Gamma_{\mathbb{Z}^m} \cap \Gamma' \subseteq F$. It is straightforward that a finite central extension of $\mathbb{Z}^m$ is virtually $\mathbb{Z}^n$ (see e.g. [BH99, II.7.9]). Therefore $\Gamma$ contains a finite index subgroup isomorphic to $\mathbb{Z}^m \times \Lambda'$ and the result follows since $m \geq n$.

Proof of Corollary 8.10. Notice that a splitting $\Gamma_0 \cong \mathbb{Z}^n \times \Gamma'$ with $\Gamma_0$ normal and $n$ maximal provides a normal subgroup $\mathbb{Z}^n \vartriangleleft \Gamma$ since $\mathbb{Z}^n$ is characteristic in $\Gamma_0$. Therefore, given Theorem 8.8, it only remains to see that a normal $\mathbb{Z}^n \vartriangleleft \Gamma$ of maximal rank forces $X$ to have a Euclidean factor of dimension at least $n$. Otherwise, the projection of $\Gamma$ to the non-Euclidean factor $X'$ would be a lattice by Theorem 8.8(ii) and contain an infinite normal amenable subgroup, contradicting Corollary 7.7(i).

Finally, we record that Theorem 1.21 is contained in Theorem 8.8 and Corollary 8.10 for (i), and Corollary 8.12 and Theorem 8.14 for (ii).
8.D. **Irreducible lattices in CAT(0) spaces.** Recall from the Introduction that a (topological) group is called **irreducible** if no (open) finite index subgroup splits non-trivially as a direct product of (closed) subgroups. For example, any locally compact group acting continuously, properly, minimally, without fixed point at infinity on an irreducible proper CAT(0) space is irreducible by Theorem 1.6.

In particular, an abstract group $\Gamma$ is irreducible if it does not virtually split. This terminology is inspired by the concept of irreducibility for closed manifolds, which means that no finite cover of the manifold splits non-trivially. Of course, the universal cover of such a manifold can still split. Indeed, one gets many classical CAT(0) groups by considering irreducible lattices in products of simple Lie groups or more generally of semi-simple algebraic groups over various local fields.

The latter concept of irreducibility for lattices is defined as follows: A lattice $\Gamma < G = G_1 \times \cdots \times G_n$ in a product of locally compact groups is called an **irreducible lattice** if its projections to any subproduct of the $G_i$'s are dense and each $G_i$ is non-discrete.

The point of this notion (and of the nearly confusing terminology) is that it prevents $\Gamma$ and its finite index subgroups from splitting as a product of lattices in $G_i$. Moreover, if all $G_i$'s are centre-free simple Lie (or algebraic) groups without compact factors, the irreducibility of $\Gamma$ as a lattice is equivalent to its irreducibility as a group in and for itself; this is a result of Margulis [Mar91, II.6.7]. As we shall see in Theorem 8.17 below, a version of this equivalence holds for lattices in the isometry group of a CAT(0) space.

**Remark 8.16.**

(i) The non-discreteness of $G_i$ is often omitted from this definition; the difference is inessential since the notion of a lattice is trivial for discrete groups. Notice however that our definition ensures that all $G_i$ are non-compact and that $n \geq 2$.

(ii) One verifies that any lattice $\Gamma < G = G_1 \times G_2$ is an irreducible lattice in the product $G^* < G$ of the closures $G_i^* < G_i$ of its projections to $G_i$ (provided these projections are non-discrete).

The following geometric version of Margulis' criterion contains Theorem 1.22 from the Introduction.

**Theorem 8.17.** Let $X$ be a proper CAT(0) space, $G < \text{Is}(X)$ a closed subgroup acting cocompactly on $X$, and $\Gamma < G$ a finitely generated lattice.

(i) If $\Gamma$ is irreducible as an abstract group, then for any finite index subgroup $\Gamma_0 < \Gamma$ and any $\Gamma_0$-equivariant splitting $X = X_1 \times X_2$ with $X_1$ and $X_2$ non-compact, the projection of $\Gamma_0$ to both $\text{Is}(X_i)$ is non-discrete.

(ii) If in addition the $G$-action is minimal, then the converse statement holds as well.

**Remark 8.18.** Recall that the $G$-minimality is automatic if $X$ is geodesically complete (Lemma 2.13). Statement (ii) fails completely without minimality (as witnessed for instance by the uncosmopolitanien of an equivariant mane).

**Proof of Theorem 8.17.** Suppose $\Gamma$ irreducible. Let $X' \subseteq X$ be the canonical subspace provided by Theorem 8.11. By Theorem 8.8, the space $X'$ has no Euclidean factor unless $X = \mathbb{R}$ and $\Gamma$ is virtually cyclic, in which case the desired statement is empty.

We first deal with the case when $G$ acts minimally on $X$; by Theorem 7.4 this amounts to assume $X = X'$. Suppose for a contradiction that for $\Gamma_0$ and $X' = X'_1 \times X'_2$ as in the statement, the projection $G_1$ of $\Gamma_0$ to $\text{Is}(X'_1)$ is discrete. Let $G_2$ be the closure of the projection of $\Gamma_0$ to $\text{Is}(X'_2)$ and notice that both $G_i$ are compactly generated since $\Gamma$ and
hence also \( \Gamma_0 \) is finitely generated. The projection \( \Gamma_0 \cap (1 \times G_2) \) to \( G_2 \) is a lattice (by Lemma 8.2 or by Proposition 8.1); being normal, it is cocompact and hence finitely generated. By Theorem 8.14, the group \( \Gamma_0 \), and hence also \( G_2 \), acts minimally and without fixed point at infinity on \( X_2 \). Therefore Corollary 7.7(ii) implies that the centraliser \( \mathcal{Z}_{G_2}(\Gamma_2) \) is trivial. But \( \Gamma_2 \) is discrete, normal in \( G_2 \), and finitely generated. Hence \( \mathcal{Z}_{G_2}(\Gamma_2) \) is open and thus \( G_2 \) is discrete. Therefore, the product \( G_1 \times G_2 \), which contains \( \Gamma_0 \), is a lattice in \( \text{Is}(X'_1) \times \text{Is}(X'_2) \) and thus in \( G \). Now the index of \( \Gamma_0 \) in \( G_1 \times G_2 \) is finite and thus \( \Gamma_0 \) splits virtually, a contradiction.

We now come back to the general case \( X' \subseteq X \) and suppose that \( X \) possesses a \( \Gamma_0 \)-equivariant splitting \( X = X_1 \times X_2 \). The group \( H = \text{Is}(X_1) \times \text{Is}(X_2) < \text{Is}(X) \) contains \( \Gamma_0 \); hence its action on \( X' \) is minimal without fixed point at infinity by Corollary 8.12. Therefore, the splitting theorem [Mon06, Theorem 9] implies that \( X' \) possesses a \( \Gamma_0 \)-equivariant splitting \( X' = X'_1 \times X'_2 \) induced by \( X = X_1 \times X_2 \) via \( H \). Upon replacing \( \Gamma_0 \) by a finite index subgroup, the preceding paragraph thus yields a splitting \( \Gamma_0/F \cong G_1 \times G_2 \) of the image of \( \Gamma_0 \) in \( \text{Is}(X') \), where \( F \) denotes the kernel of the \( \Gamma_0 \)-action on \( X' \). Since \( F \) is finite, so is the projection to \( \text{Is}(X_{3-i}) \) of the preimage \( \widehat{G_i} \) of \( G_i \) in \( \Gamma \), for \( i = 1, 2 \). Therefore upon passing to a finite index subgroup we may and shall assume that \( \widehat{G_i} \) acts trivially on \( \text{Is}(X_{3-i}) \). Now the subgroup of \( \text{Is}(X'_1) \times \text{Is}(X'_2) \) generated by \( \widehat{G_1} \) and \( \widehat{G_2} \) splits \( \widehat{G_1} \times \widehat{G_2} \) and is commensurable to \( \Gamma_0 \), a contradiction.

Conversely, suppose now that the \( G \)-action is minimal and that \( \Gamma = \Gamma' \times \Gamma'' \) splits non-trivially (after possibly having replaced it by a finite index subgroup). If \( X = \mathbb{R}^n \), then reducibility of \( \Gamma \) forces \( n \geq 2 \) and we are done in view of the structure of Bieberbach groups. If \( X \) is not Euclidean but has a Euclidean factor, then Theorem 8.8(ii) provides a discrete projection of \( \Gamma \) to the non-Euclidean factor \( \text{Is}(X') \); furthermore, \( X' \) is indeed non-compact as desired since otherwise by minimality it is reduced to a point, contrary to our assumption.

If on the other hand \( X \) has no Euclidean factor, then \( \Gamma \) acts minimally and without fixed point at infinity by Theorem 8.11. Then the desired splitting is provided by the splitting theorem [Mon06, Theorem 9]. Both projections of \( \Gamma \) are discrete, indeed isomorphic to \( \Gamma' \) respectively \( \Gamma'' \) because the cited splitting theorem ensures componentwise action of \( \Gamma \).

We now briefly turn to uniquely geodesic spaces and to the analogues in this setting of some of P. Eberlein’s results for Hadamard manifolds.

**Theorem 8.19.** Let \( X \) be a proper CAT(0) space with uniquely extensible geodesics such that \( \text{Is}(X) \) acts cocompactly on \( X \).

If \( \text{Is}(X) \) admits a finitely generated non-uniform irreducible lattice, then \( X \) is a symmetric space (without Euclidean factor).

**Proof.** The action of \( \text{Is}(X) \) is minimal by Lemma 2.13 and without fixed point at infinity by Corollary 8.12. Thus we can apply Theorem 1.1 and Addendum 1.4. Notice that \( \text{Is}(X) \) itself is non-discrete since it contains a non-uniform lattice; moreover, if it admits more than one factor in the decomposition of Theorem 1.1, then the latter are all non-discrete by Theorem 8.17. Therefore, we can apply Theorem 6.10 to all factors of \( X \). It remains only to justify that \( X \) has no Euclidean factor; otherwise, Auslander’s theorem (compare also Theorem 8.8) implies \( X = \mathbb{R} \), which is incompatible with the fact that \( \Gamma \) is non-uniform.

The following related result is due to P. Eberlein in the manifold case (Proposition 4.5 in [Ebe82]). We shall establish another result of the same vein later without assuming that geodesics are uniquely extensible (see Theorem 11.6).
**Theorem 8.20.** Let \( X \) be a proper \( \text{CAT}(0) \) space with uniquely extensible geodesics and \( \Gamma < \text{Is}(X) \) be a discrete cocompact group of isometries. If \( \Gamma \) is irreducible (as an abstract group) and \( X \) is reducible, then \( X \) is a symmetric space (without Euclidean factor).

**Proof.** One follows line-by-line the proof of Theorem 8.19. The only difference is that, in the present context, the non-discreteness of the isometry group of each irreducible factor of \( X \) follows from Theorem 8.17 since \( X \) is assumed reducible. \( \square \)

We can now conclude the proof of Theorem 1.27 from the Introduction. The first statement was established in Theorem 6.10. The second follows from Theorem 8.19 and the third from Theorem 8.20 in the uniform case, and from Theorem 8.19 in the non-uniform one. \( \square \)

8.E. **The hull of a lattice.** Let \( X \) be a proper \( \text{CAT}(0) \) space \( X \) such that \( \text{Is}(X) \) acts cocompactly on \( X \). Let \( \Gamma < \text{Is}(X) \) be a finitely generated lattice; note that the condition of finite generation is redundant if \( \Gamma \) is cocompact by Lemma 8.3. Theorem 8.11 provides a canonical \( \text{Is}(X) \)-invariant subspace \( X' \subseteq X \) such that \( G = \text{Is}(X') \) has no fixed point at infinity.

In this section we shall define the hull \( H_\Gamma \) of the lattice \( \Gamma \); this is a locally compact group \( H_\Gamma < \text{Is}(X') \) canonically attached to the situation and containing the image of \( \Gamma \) in \( \text{Is}(X') \).

For simplicity, we first treat the special case where \( \text{Is}(X) \) acts minimally; thus \( X' = X \) and \( G = \text{Is}(X) \). Applying Theorem 1.1 and Addendum 1.4, we see in particular that \( \Gamma \) possesses a canonical finite index normal subgroup \( \Gamma^* = \Gamma \cap G^* \) which is the kernel of the \( \Gamma \)-action by permutation on the set of factors in the decomposition given by Addendum 1.4.

In the classical case when \( X \) is a symmetric space, the closure of the projection of \( \Gamma \) to the isometry \( \text{Is}(X_i) \) of each factor in (1.i) is an open subgroup of finite index, as soon as \( X \) is reducible. This is no longer true in general, even in the case of Euclidean buildings. In fact, the same \( \Gamma \) may (and generally does) occur as lattice in increasingly large ambient groups \( \Gamma < G < G' < G'' < \cdots \). In order to address this issue, we define the hull as follows. Consider the closed subgroup \( H_{\Gamma^*} < G \) which is the direct product of the closure of the images of \( \Gamma^* \) in each of the factors in the decomposition (1.i) of Theorem 1.1. Then set \( H_\Gamma = \Gamma \cdot H_{\Gamma^*} \). In other words, we have inclusions

\[
\Gamma < H_\Gamma < G.
\]

The closed subgroup \( H_{\Gamma^*} \) is nothing but the hull of the lattice \( \Gamma^* \). It coincides with \( H_\Gamma^* = H_\Gamma \cap G^* \). In particular \( H_\Gamma^* = H_{\Gamma^*} \) is a direct product of irreducible groups satisfying all the restrictions of Theorem 1.6 (except for the possible Euclidean motion factor), and the image of \( \Gamma^* \) in each of these factors is dense.

**Remark 8.21.** Notice that \( \Gamma \) is always a lattice in \( H_\Gamma \) (by [Rag72, Lemma 1.6]). We emphasise that \( H_\Gamma \) is non-discrete and that \( \Gamma^* \) is an irreducible lattice in \( H_{\Gamma^*} \) (in the sense of §8.D) as soon as \( \Gamma \) is reducible as a group and \( X \) is reducible; this follows from Theorem 8.17.

We now define the hull \( H_\Gamma < G \) in the general situation \( G = \text{Is}(X') \) with \( X' \subseteq X \) given by Theorem 8.11. Since \( \text{Is}(X) \backslash X \) is cocompact, it follows that \( X' \) is \( \tau \)-dense in \( X \) for some \( r > 0 \) and the canonical map \( \text{Is}(X) \to G \) is proper. Let \( F_\Gamma < \Gamma \) be the finite kernel of the induced map \( \Gamma \to G \) and write \( \Gamma' := \Gamma / F_\Gamma \). Then the hull of \( \Gamma \) is defined by \( H_\Gamma := H_{\Gamma'} \) (reducing to the above case).

In other words, \( \Gamma \) sits in \( H_\Gamma \) only modulo the canonical finite kernel \( F_\Gamma \). In fact, \( F_\Gamma \) is even canonically attached to \( \Gamma \) viewed as an abstract group.
\textbf{Lemma 8.22.} \(F_{\Gamma}\) is a (necessarily unique) maximal finite normal subgroup of \(\Gamma\). Moreover, \(X\) is \(\Gamma\)-minimal.

\textit{Proof.} The \(\Gamma\)'-action on \(X\) is minimal by an application of Theorem 8.14 and therefore every finite normal subgroup of \(\Gamma\) is trivial. Since moreover the \(\Gamma\)-action on \(X\) is proper, it follows that a normal subgroup of \(\Gamma\) is finite if and only if it lies in \(F_{\Gamma}\). \(\Box\)

For later references, we record the following expected fact.

\textbf{Lemma 8.23.} Assume that \(\Gamma\) is irreducible. If \(X\) is reducible, then \(H_{\Gamma}\) contains the identity component of \(G := \text{Is}(X')\). In fact \((H_{\Gamma})^0 = G^0\) is a semi-simple Lie group with trivial centre and no compact factor.

\textit{Proof.} By Theorem 8.8, the hypotheses on \(\Gamma\) imply that \(X\) has no Euclidean factor. Thus each almost connected factor of \(G^*\) is a simple Lie group with trivial centre and no compact factor. The projection of \(\Gamma^*\) to each of these factors is non-discrete by Theorem 8.17 and the assumption made on \(X\'). Its closure is semi-simple and Zariski dense by Theorem 7.4 and Proposition 7.8. The result follows. \(\Box\)

\textbf{8.F. On the canonical discrete kernel.} Let \(G = G_1 \times G_2\) be a locally compact group and \(\Gamma < G\) be an irreducible lattice. It follows from irreducibility that the projection to \(G_i\) of the kernel of the projection \(\Gamma \rightarrow G_{j\neq i}\) is a normal subgroup of \(G_i\). In other words, we have a canonical discrete normal subgroup \(\Gamma_i \triangleleft G_i\) defined by

\[\Gamma_i = \text{Proj}_{G_i}(\Gamma \cap (G_1 \times 1))\]

(and likewise for \(\Gamma_2\)) which we call the \textit{canonical discrete kernel} of \(G_i\) (depending on \(\Gamma\)).

We observe that the image

\[\Gamma = \Gamma/(\Gamma_1 \cdot \Gamma_2)\]

of \(\Gamma\) in the \textit{canonical quotient} \(G_1/\Gamma_1 \times G_2/\Gamma_2\) is still an irreducible lattice (see Proposition 8.1(ii)) and has the additional property that it projects injectively into both factors.

In this subsection, we collect some basic facts on lattices in (products of) totally disconnected locally compact groups, adapting ideas of M. Burger and Sh. Mozes (see Propositions 2.1 and 2.2 in [BM00b]).

\textbf{Proposition 8.24.} Let \(\Gamma < G = G_1 \times G_2\) be an irreducible lattice. Assume that \(G_2\) is totally disconnected, compactly generated and without non-trivial compact normal subgroup. If \(\Gamma\) is residually finite, then canonical the discrete kernel \(\Gamma_2 = \Gamma \cap (1 \times G_2)\) commutes with the discrete residual \(G_2^{(\infty)}\).

Recall that the \textit{discrete residual} \(G^{(\infty)}\) of a topological group \(G\) is by definition the intersection of all open normal subgroups. It is important to remark that, by Corollary 5.12 the discrete residual of a non-discrete compactly generated locally compact group without non-trivial compact normal subgroup is necessarily non-trivial.

\textit{Proof of Proposition 8.24.} By a slight abuse of notation, we shall identify \(G_2\) with the subgroup \(1 \times G_2\) of \(G\). Given a finite index normal subgroup \(\Gamma_0 \triangleleft \Gamma\), the intersection \(\Gamma_0 \cap \Gamma_2\) is a discrete normal subgroup of \(G_2\) (by irreducibility), contained as a finite index subgroup in \(\Gamma_2\). Thus \(G_2\) acts by conjugation on the finite quotient \(\Gamma_2/\Gamma_0 \cap \Gamma_2\). In particular the kernel of this action is a finite index closed normal subgroup, which is thus open. Therefore, the discrete residual \(G_2^{(\infty)}\) acts trivially on \(\Gamma_2/\Gamma_0 \cap \Gamma_2\). In other words, this means that \([\Gamma_2, G_2^{(\infty)}] \subseteq \Gamma_0 \cap \Gamma_2\).
Assume now that $\Gamma$ is residually finite. The preceding argument then shows that the commutator $[\Gamma_2, G_2^{(\infty)}]$ is trivial, as desired. □

**Proposition 8.25.** Let $\Gamma < G = G_1 \times G_2$ be a cocompact lattice in a product of compactly generated locally compact groups. Assume that $G_2$ is totally disconnected and that the centraliser in $G_1$ of any cocompact lattice of $G_1$ is trivial. If the discrete kernel $\Gamma_2 = \Gamma \cap (1 \times G_2)$ is trivial, then the quasi-centre $\mathcal{Z}(G_2)$ is topologically locally finite.

**Proof.** Let $S \subseteq \mathcal{Z}(G_2)$ be a finite subset of the quasi-centre. Then $G_2$ possesses a compact open subgroup $U$ which centralises $S$. By Lemma 8.2 the group $\Gamma_U = \Gamma \cap (G_1 \times U)$ is a cocompact lattice in $G_1 \times U$. In particular, there is a finite generating set $\mathcal{T} \subseteq \Gamma_U$. By a lemma of Selberg [Sel60], the group $\mathcal{Z}_T(\mathcal{T})$ is a cocompact lattice in $\mathcal{Z}(G_2)(\mathcal{T})$. But $\mathcal{Z}(G_2)(\mathcal{T}) = \mathcal{Z}_G(\Gamma_U) \subseteq 1 \times G_2$ since the projection of $\Gamma_U$ to $G_1$ is a cocompact lattice. Since the discrete kernel $\Gamma \cap (1 \times G_2)$ is trivial by hypothesis, the centraliser $\mathcal{Z}_T(\mathcal{T})$ is trivial and, hence, $\mathcal{Z}(G_2)(\mathcal{T})$ is compact. By construction $S$ is contained in $\mathcal{Z}(G_2)(\mathcal{T})$, which yields the desired result. □

8.G. Residually finite lattices.

**Theorem 8.26.** Let $X$ be a proper CAT(0) space such that $\Is(X)$ acts cocompactly and minimally. Let $\Gamma < \Is(X)$ be a finitely generated lattice. Assume that $\Gamma$ is irreducible and residually finite. Then we have the following (see Section 8.E for the notation):

(i) $\Gamma^*$ acts faithfully on each irreducible factor of $X$.

(ii) If $\Gamma$ is cocompact and $X$ is reducible, then for any closed subgroup $G < \Is(X)$ containing $H_{\Gamma^*}$, we have $\mathcal{Z}(G) = \mathcal{Z}(G^*) = 1$. Furthermore $\soc(G^*)$ is a direct product of $r$ non-discrete closed subgroups, each of which is characteristically simple, where $r$ is the number of irreducible factors of $X$.

(We emphasise that the irreducibility assumption concerns $\Gamma$ as an abstract group; compare however Remark 8.21.)

**Proof.** If $X$ is irreducible, there is nothing to prove. We assume henceforth that $X$ is reducible. In view of Theorem 8.8, $X$ has no Euclidean factor. Moreover, Corollary 8.12 implies that $\Is(X)$ fixes no point at infinity. In particular, $\Gamma$ and $H_{\Gamma^*}$ act minimally without fixed point at infinity by Theorem 7.4.

Let $H_1, \ldots, H_r$ be the irreducible factors of $H_{\Gamma^*}$; thus $r$ coincides with the number of irreducible factors of $X$. In view of Theorem 8.17, the group $\Gamma^*$ is an irreducible lattice in this product. By Corollary 1.7 and Theorem 7.4, each $H_i$ is either a centre-free simple Lie group or totally disconnected with trivial amenable radical. If $H_1$ is a simple Lie group, then it has no non-trivial discrete normal subgroup and hence

$$(\Gamma^*)_1 := \Gamma^* \cap (H_1 \times 1 \times \cdots \times 1) = 1.$$ 

If $H_1$ is totally disconnected, then by Proposition 8.24 the canonical discrete kernel $(\Gamma^*)_1$ commutes with the discrete residual $H_1^{(\infty)}$, which is non-trivial by Corollary 5.12. Thus $\mathcal{Z}(H_1^{(\infty)}) = 1$ by Theorem 1.6 and hence $(\Gamma^*)_1 = 1$.

Assertion (i) now follows from a straightforward induction on $r$.

Assume next that $\Gamma$ is cocompact. Let $G_1, \ldots, G_r$ be the irreducible factors of $G^*$. By Lemma 8.23 and Proposition 8.25, and in view of Part (i), for each totally disconnected factor $G_i$, the quasi-centre $\mathcal{Z}(G_i)$ is topologically locally finite. Its closure is thus amenable, hence trivial by Theorem 1.6. Moreover, the quasi-centre of each almost connected factor is trivial as well by Lemma 8.23.
Clearly the projection of the quasi-centre of \(G^*\) to the irreducible factor \(G_i\) is contained in \(\mathcal{L}(G_i)\). This shows that \(\mathcal{L}(G^*)\) is trivial. Hence so is \(\mathcal{L}(G)\), since it contains \(\mathcal{L}(G^*)\) as a finite index subgroup and since \(G\) has no non-trivial finite normal subgroup by Corollary 4.8. Now the desired conclusion follows from Proposition 5.16. □

9. CAT(0) superrigidity

9.A. CAT(0) superrigidity for some classical non-uniform lattices. Let \(\Gamma\) be a non-uniform lattice in a simple (real) Lie group \(G\) of rank at least 2. By [LMR00, Theorem 2.15], unipotent elements of \(\Gamma\) are exponentially distorted. This means that, with respect to any finitely generating set of \(\Gamma\), the word length of \(|u^n|\) is an \(O(\log n)\) when \(u\) is a unipotent. More generally an element \(u\) is called *distorted* if \(|u^n|\) is sublinear. If \(\Gamma\) is virtually boundedly generated by unipotent elements, one can therefore apply the following fixed point principle:

**Lemma 9.1.** Let \(\Lambda\) be a group which is virtually boundedly generated by distorted elements. Then any isometric \(\Gamma\)-action on a complete CAT(0) space such that elements of zero translation length are elliptic has a global fixed point.

**Proof.** For any \(\Gamma\)-action on a CAT(0) space, the translation length of a distorted element is zero. Thus every such element has a fixed point; the assumption on \(\Gamma\) now implies that all orbits are bounded, thus providing a fixed point [BH99, II.2.8(1)]. □

Bounded generation is a strong property, which conjecturally holds for all (non-uniform) lattices of a higher rank semi-simple Lie group. It is known to hold for arithmetic groups in split or quasi-split algebraic groups of a number field \(K\) of \(K\)-rank \(\geq 2\) by [Tav90], as well as in a few cases of isotropic but non-quasi-split groups [ER06].

As noticed in a conversation with Sh. Mozes, Lemma 9.1 yields the following elementary superrigidity statement.

**Proposition 9.2.** Let \(\Lambda = \text{SL}_n(\mathbb{Z}[\frac{1}{p_1\cdots p_k}])\) with \(n \geq 3\) and \(p_i\) distinct primes and set \(H = \text{SL}_n(\mathbb{Q}_{p_1}) \times \cdots \times \text{SL}_n(\mathbb{Q}_{p_k})\).

Given any isometric \(\Lambda\)-action on any complete CAT(0) space such that every element of zero translation length is elliptic, there exists a \(\Lambda\)-invariant closed convex subspace on which the given action extends uniquely to a continuous \(H\)-action by isometries.

**Proof.** Let \(X\) be a complete CAT(0) space endowed with a \(\Lambda\)-action as in the statement. The subgroup \(\Gamma = \text{SL}_n(\mathbb{Z}) < \Lambda\) fixes a point by Lemma 9.1. The statement now follows because \(\Gamma\) is the intersection of \(\Lambda\) with the *open* subgroup \(\text{SL}_n(\mathbb{Z}_{p_1}) \times \cdots \times \text{SL}_n(\mathbb{Z}_{p_k})\) of \(H\); for later use, we isolate this elementary fact as Lemma 9.3 below. □

**Lemma 9.3.** Let \(H\) be a topological group, \(U < H\) an open subgroup, \(\Lambda < H\) a dense subgroup and \(\Gamma = \Lambda \cap U\). Any \(\Lambda\)-action by isometries on a complete CAT(0) space with a \(\Gamma\)-fixed point admits a \(\Lambda\)-invariant closed convex subspace on which the action extends continuously to \(H\).

**Proof.** Let \(X\) be the CAT(0) space and \(x_0 \in X\) a \(\Gamma\)-fixed point. For any finite subset \(F \subseteq \Lambda\), let \(Y_F \subseteq X\) be the closed convex hull of \(F x_0\). The closed convex hull \(Y\) of \(\Lambda x_0\) is the closure of the union \(Y_\infty\) of the directed family \(\{Y_F\}\). Therefore, since the action is isometric and \(Y\) is complete, it suffices to show that the \(\Lambda\)-action on \(Y_\infty\) is continuous for the topology induced on \(\Lambda\) by \(H\). Equivalently, it suffices to prove that all orbital maps \(\Lambda \to Y_\infty\) are continuous at \(1 \in \Lambda\). This is the case even for the discrete topology on \(Y_\infty\) because the pointwise fixator of each \(Y_F\) is an intersection of finitely many conjugates of \(\Gamma\), the latter being open by definition. □
The same arguments as below show that Theorem 1.16 holds for any lattice of a higher-rank semi-simple Lie group which is boundedly generated by distorted elements (and accordingly Theorem 1.17 generalises to suitable (S-)arithmetic groups).

**Proof of Theorems 1.16 and 1.17.** We start with the case $\Gamma = \text{SL}_n(\mathbb{Z})$. By Theorem 1.11, we obtain a closed convex subspace $X'$ which splits as a direct product

$$X' \cong X_1 \times \cdots \times X_p \times Y_0 \times Y_1 \times \cdots \times Y_q$$

in an $\text{Is}(X')$-equivariant way, where $Y_0 \cong \mathbb{R}^a$ is the Euclidean factor. Each totally disconnected factor $D_i$ of $\text{Is}(X')^*$ acts by semi-simple isometries on the corresponding factor $Y_i$ of $X'$ by Corollary 5.3. Therefore, by Lemma 9.1 for each $i = 0, \ldots, q$, the induced $\Gamma$-action on $Y_i$ has a global fixed point, say $y_i$. In other words $\Gamma$ stabilises the closed convex subset

$$Z := X_1 \times \cdots \times X_p \times \{y_0\} \times \cdots \times \{y_q\} \subseteq X.$$

Note that the isometry group of $Z$ is an almost connected semi-simple real Lie group $L$. Combining Lemma VII.5.1 and Theorems VII.5.15 and VII.6.16 from [Mar91], it follows that the Zariski closure of the image of $\Gamma$ in $L$ is a commuting product $L_1 \times L_2$, where $L_1$ is compact, such that the corresponding homomorphism $\Gamma \to L_2$ extends to a continuous homomorphism $G \to L_2$. We define $Y \subseteq Z$ as the fixed point set of $L_1$. Now $L_2$, and hence $\Gamma$, stabilises $Y$. Therefore the continuous homomorphism $G \to L_2$ yields a $G$-action on $Y$ which extends the given $\Gamma$-action, as desired.

Applying Theorem 6.4(iv) to the pair $L_2 < L$ acting on $Z$, we find in particular that $L_2$ has no fixed point at infinity in $Y$. Thus, upon replacing $Y$ by a subspace, it is $L_2$-minimal. Now Theorem 7.4 implies that the $\Gamma$- and $G$-actions on $Y$ are minimal and without fixed point in $\partial Y$ (although there might be fixed points in $\partial X$).

Turning to Theorem 1.17, the only change is that one replaces Lemma 9.1 by Proposition 9.2. \qed

**9.B. CAT(0) superrigidity for irreducible lattices in products.** The aim of this section is to state a version of the superrigidity theorem [Mon06, Theorem 6] with CAT(0) targets. The original statement from loc. cit. concerns actions of lattices on arbitrary CAT(0) spaces, with reduced unbounded image. The following statement shows that, when the underlying CAT(0) space is nice enough, the assumption on the action can be considerably weakened.

**Theorem 9.4.** Let $\Gamma$ be an irreducible uniform (or square-integrable weakly cocompact) lattice in a product $G = G_1 \times \cdots \times G_n$ of $n \geq 2$ locally compact $\sigma$-compact groups. Let $X$ be a proper CAT(0) space with finite-dimensional boundary. Given any $\Gamma$-action on $X$ without fixed point at infinity, if the canonical closed convex $\Gamma$-invariant $\Gamma$-minimal subset $Y \subseteq X$ provided by Theorem 3.3(B.iii) has no Euclidean factor, then the $\Gamma$-action on $Y$ extends to a continuous $G$-action by isometries.

**Remark 9.5.** Although the above condition on the Euclidean factor in the $\Gamma$-minimal subspace $Y$ might seem awkward, it cannot be avoided, as illustrated by Example 64 in [Mon06]. Notice however that if $\Gamma$ has the property that any isometric action on a finite-dimensional Euclidean space has a global fixed (for example if $\Gamma$ has Kazhdan’s property (T)), then any minimal $\Gamma$-invariant subspace has no Euclidean factor.

**Proof of Theorem 9.4.** Let $Y \subseteq X$ be the canonical subspace provided by Theorem 3.3(B.iii). Then $\text{Is}(Y)$ acts minimally on $Y$, without fixed point at infinity. In particular we may apply Theorem 1.1 and Addendum 1.4. In order to show that the $\Gamma$-action on $Y$ extends to a
continuous $G$-action, it is sufficient to show that the induced $\Gamma$-action on each irreducible factor of $Y$ extends to a continuous $G$-action, factoring through some $G_i$. But the induced $\Gamma$-action on each irreducible factor of $Y$ is reduced by Corollary 2.8. Thus the result follows from [Mon06, Theorem 6].

9.C. **Strong rigidity for product spaces.** Superrigidity should contain, in particular, strong rigidity à la Mostow. This is indeed the content of Theorem 9.6 below, where an isomorphism of lattices is shown to extend to an ambient group. However, in contrast to the classical case of symmetric spaces, which are homogeneous, the full isometry group does not in general determine the space since CAT(0) spaces are in general not homogeneous. Another difference is that the hull of a lattice, as described in Section 8.E, is generally smaller than the full isometry group of the ambient CAT(0) space.

In view of the definition of the hull, the following statement is non-trivial only when $X$ (or an invariant subspace) is reducible; this is expected since we want to use superrigidity for irreducible lattices in products.

**Theorem 9.6.** Let $X,Y$ be proper CAT(0) spaces and $\Gamma, \Lambda$ discrete cocompact groups of isometries of $X$, respectively $Y$, not splitting (virtually) a $\mathbb{Z}^n$ factor.

Then any isomorphism $\Gamma \cong \Lambda$ determines an isomorphism $H_\Gamma \cong H_\Lambda$ such that the following commutes:

$$
\begin{array}{ccc}
\Gamma & \longrightarrow & \Lambda \\
\downarrow & & \downarrow \\
H_\Gamma & \cong & H_\Lambda
\end{array}
$$

Theorem 9.6 provides a partial answer to Question 21 in [FHT08].

**Remark 9.7.** The assumption on $\mathbb{Z}^n$ factors is equivalent to excluding Euclidean factors from $X$ (or its canonical invariant subspace) by Theorem 8.8. On the one hand, this assumption is really necessary for the theorem to hold, even for symmetric spaces, since one can twist the product using a $\Gamma$-action on the Euclidean factor when $H^1(\Gamma) \neq 0$ (compare [LY72, §4]). On the other hand, since Bieberbach groups are obviously Mostow-rigid, Theorem 9.6 together with Theorem 8.8 give us as complete as possible a description of the situation with $\mathbb{Z}^n$ factors.

**Proof.** Let $X' \subseteq X$ be the subset provided by Theorem 8.11. We retain the notation $F_\Gamma < \Gamma$ and $\Gamma' = \Gamma / F_\Gamma < \text{Is}(X')$ from Section 8.E and recall from Lemma 8.22 that $F_\Gamma$ depends only on $\Gamma$ as abstract group and that $X'$ is $\Gamma'$-minimal. We define $Y'$, $F_\Lambda$ and $\Lambda'$ in the same way and have the corresponding lemma. In particular, it follow that the isomorphism $\Gamma \cong \Lambda$ descends to $\Gamma' \cong \Lambda'$. Therefore, we can and shall assume from now on that $X$ and $Y$ are minimal and $\Gamma < H_\Gamma < \text{Is}(X)$, $\Lambda < H_\Lambda < \text{Is}(Y)$. By Theorem 8.8, we know that $X,Y$ have no Euclidean factor. Thus $\Gamma, \Lambda$ have no fixed point at infinity by Theorem 8.14. We claim that $\Gamma$ has a finite index subgroup $\Gamma^\dagger$ which decomposes as a product $\Gamma^\dagger = \Gamma_1 \times \cdots \times \Gamma_s$ of irreducible factors, with $s$ maximal for this property. Indeed, otherwise we could apply the splitting theorem of [Mon06] to a chain a finite index subgroups and contradict the properness of $X$. We write $\Lambda^\dagger = \Lambda_1 \times \cdots \times \Lambda_s$ for the corresponding groups in $\Lambda$. Combining the splitting theorem with Addendum 1.4, it follows from the definition of the hull that it is sufficient to prove the statement for $s = 1$. We assume henceforth that $\Gamma$, and hence also $\Lambda$, is irreducible. Furthermore, if $X$ and $Y$ are both irreducible, then $H_\Gamma = \Gamma$ and $H_\Lambda = \Lambda$ and the desired statement is empty. We now assume that $X$ is reducible.
By Theorem 7.4, the lattice $\Gamma$ (resp. $\Lambda$) acts minimally without fixed point at infinity on $X$ (resp. $Y$). Theorem 9.4 yields a continuous morphism $f : H^+_\Gamma \to H^+_\Lambda$, which shows in particular (by the splitting theorem [Mon06]) that $Y$ is reducible as well. A second application of Theorem 9.4 yields a second continuous morphism $f' : H^+_\Lambda \to H^+_\Gamma$. Notice that the respective restrictions to $\Gamma^*$ and $\Lambda^*$ coincide with the given isomorphism and its inverse. In particular $f' \circ f$ (resp. $f' \circ f$) is the identity on $\Gamma$ (resp. $\Lambda$). By definition of the hull, it follows that $f' \circ f$ (resp. $f' \circ f$) is the identity on $H^+_\Gamma$ (resp. $H^+_\Lambda$). The desired result finally follows, since there is a canonical isomorphism $\Gamma/\Gamma^* \cong H^+_\Gamma/H^+_\Gamma$ and since the action of $H^+_\Gamma/H^+_\Gamma$ on $H^+_\Gamma$ is canonically determined by the action of $\Gamma/\Gamma^*$ on $\Gamma^*$. □

The above proof shows in particular that amongst spaces that are $\Gamma$-minimal without Euclidean factor, the number of irreducible factors depends only upon the group $\Gamma$. If we combine this with Theorem 8.14, Corollary 8.10 and Theorem 8.8(ii), we obtain that the number of factors in the “de Rham” decomposition (1.ii) is an invariant of the group:

**Corollary 9.8.** Let $X$ be a proper CAT(0) space and $\Gamma < \text{Is}(X)$ be a group acting properly discontinuously and cocompactly.

Then any other such space admitting a proper cocompact $\Gamma$-action has the same number of factors in (1.ii) and the Euclidean factor has same dimension. □

(We recall that minimality is automatic when $X$ is geodesically complete; Lemma 2.13.)

## 10. Arithmeticity of abstract lattices

The main goal of this section is to prove Theorem 1.29, which we now state in a slightly more general form. Following G. Margulis [Mar91, IX.1.8], we shall say that a simple algebraic group $G$ defined over a field $k$ is **admissible** if none of the following holds:

- char($k$) = 2 and $G$ is of type $A_1$, $B_n$, $C_n$ or $F_4$.
- char($k$) = 3 and $G$ is of type $G_2$.

A semi-simple group will be said admissible if all its factors are.

**Theorem 10.1.** Let $\Gamma < G = G_1 \times \cdots \times G_n$ be an irreducible finitely generated lattice, where each $G_i$ is any locally compact group.

If $\Gamma$ admits a faithful Zariski-dense representation in an admissible semi-simple group (over any field), then the amenable radical $R$ of $G$ is compact and the quasi-centre $\mathcal{Z}(G)$ is virtually contained in $\Gamma \cdot R$. Furthermore, upon replacing $G$ by a finite index subgroup, the quotient $G/R$ splits as $G^+ \times \mathcal{Z}(G/R)$ where $G^+$ is a semi-simple algebraic group and the image of $\Gamma$ in $G^+$ is an arithmetic lattice.

Since the projection map $G \to G/R$ is proper, the statement of Theorem 10.1 implies in particular that $\mathcal{Z}(G/R)$ is discrete.

**Corollary 10.2.** Let $G = G_1 \times \cdots \times G_n$ be a product of locally compact groups. Assume that $G$ admits a finitely generated irreducible lattice with a faithful Zariski-dense representation in a semi-simple group over some field of characteristic $\neq 2, 3$.

Then $G$ is a compact extension of a direct product of a semi-simple algebraic group by a discrete group. □

To be more precise, the arithmeticity conclusion of Theorem 10.1 means the following.

There exists a global field $K$, a connected semi-simple $K$-anisotropic $K$-group $H$ and a finite set $\Sigma$ of valuations of $K$ such that:

(i) The quotient $\Gamma := \Gamma/\Gamma \cap (R \cdot \mathcal{Z}(G))$ is commensurable with the arithmetic group $H(K(\Sigma))$, where $K(\Sigma)$ is the ring of $\Sigma$-integers of $K$. Moreover, $\Sigma$ contains all Archimedean
valuations \( v \) for which \( H \) is \( K_v \)-isotropic, where \( K_v \) denotes the \( v \)-completion of \( K \). In particular, by Borel–Harish-Chandra and Behr–Harder reduction theory, the diagonal embedding realises \( H(K(\Sigma)) \) as a lattice in the product \( \prod_{v \in \Sigma} H(K_v) \).

(ii) The group \( G^+ \) is isomorphic to \( \prod_{v \in \Sigma} H(K_v)^+ \) and this isomorphism implements the commensurability of \( \Gamma \) with \( H(K(\Sigma)) \). For background references, including on \( H(K_v)^+ \), see [Mar91, I.3].

In contrast to statements in [Mon05], there is no assumption on the subgroup structure of the factors \( G_i \) in Theorem 10.1, which may not even be irreducible. The nature of the linear representation is however more restricted.

Another improvement is that no (weak) cocompactness assumption is made on \( \Gamma \). In particular, under the same algebraic restrictions on the factors \( G_i \) as in loc. cit., we obtain the following arithmeticity vs. non-linearity alternative for all finitely generated lattices.

**Corollary 10.3.** Let \( \Gamma < G = G_1 \times \cdots \times G_n \) be an irreducible finitely generated lattice, where each \( G_i \) is a locally compact group such that every non-trivial closed normal subgroup is cocompact. Then one of the following holds:

(i) Every finite-dimensional linear representation of \( \Gamma \) in characteristic \( \neq 2, 3 \) has virtually soluble image.

(ii) \( G \) is a semi-simple algebraic group and \( \Gamma \) is an arithmetic lattice.

The hypothesis made on each factor \( G_i \) may be used to describe to some extent its structure independently of the existence of a lattice in \( G \), see Proposition 5.13; in particular each \( G_i \) is monolithic. However, we will not appeal to this preliminary description of the \( G_i \) when proving Corollary 10.3: the structural information will instead be obtained \textit{a posteriori}.

**Remark 10.4.** In [Mon05], the conclusion (i) was replaced by finiteness of the image. This follows from the current conclusion in the more restricted setting of loc. cit. thanks to Y. Shalom’s superrigidity for characters [Sha00], unless of course \( G_i \) admits (virtually) a non-zero continuous homomorphism to \( \mathbb{R} \) (after all in the current setting we can have \( G_i = \mathbb{R} \)). It is part of the assumptions in [Mon05] that no such homomorphism exists, so that Corollary 10.3 indeed generalises loc. cit.

10.A. **Superrigid pairs.** For convenience, we shall use the following terminology. Let \( J \) be a topological group and \( \Lambda < J \) any subgroup. We call the pair \((\Lambda, J)\) \textit{superrigid} if for any local field \( k \) and any connected absolutely almost simple adjoint \( k \)-group \( H \), every abstract homomorphism \( \Lambda \rightarrow H(k) \) with unbounded Zariski-dense image extends to a continuous homomorphism of \( J \).

**Proposition 10.5.** Let \((\Lambda, J)\) be a superrigid pair with \( J \) locally compact and \( \Lambda \) finitely generated with closure of finite covolume in \( J \).

If \( \Lambda \) admits a faithful representation in an admissible semi-simple group (over any field) with Zariski-dense image, then the amenable radical \( R \) of \( J \) is compact and the quasi-centre \( \mathcal{Z}(J) \) is virtually contained in \( \Lambda \cdot R \). Furthermore, upon replacing \( J \) by a finite index subgroup, the quotient \( J/R \) splits as \( J^+ \times \mathcal{Z}(J/R) \) where \( J^+ \) is a semi-simple algebraic group and the image of \( \Lambda \) in \( J^+ \) is an arithmetic lattice.

(We point out again that in particular the direct factor \( \mathcal{Z}(J/R) \) is discrete.)

One might expect that Theorem 10.1 could now be proved by establishing in complete generality that finitely generated irreducible lattices in products of locally compact groups
form a superrigid pair. For uniform lattices, or more generally weakly cocompact square-summable lattices, this is indeed true and was proved in [Mon06]. We do not have a proof in general and shall eschew this difficulty by giving first an independent proof of the compactness of the amenable radical (Corollary 10.13 below) and using the residual finiteness of finitely generated linear groups before proceeding with Proposition 10.5.

Nevertheless, we do have a general proof as soon as the groups are totally disconnected.

**Theorem 10.6.** Let $\Gamma < G = G_1 \times \cdots \times G_n$ be an irreducible finitely generated lattice, where each $G_i$ is any locally compact group.

If $G$ is totally disconnected, then $(\Gamma, G)$ is a superrigid pair.

(As we shall see in Proposition 10.10, one can drop the finite generation assumption in the simpler case where $\Gamma$ projects faithfully to some factor $G_i$.)

**Proof of Proposition 10.5.** We will largely follow the ideas of Margulis, deducing arithmeticity from superrigidity [Mar91, Chapter IX]. It is assumed that the reader has a copy of [Mon05] at hand, since it contains a similar reasoning under different hypotheses. The characteristic assumption in loc. cit. will be replaced by the current admissibility assumption.

The group $J$ (and hence also all finite index subgroups and factors) is compactly generated by Lemma 7.12. Let $\tau : \Lambda \to H$ be the given faithful representation. Upon replacing $\Lambda$ and $J$ by finite index subgroups and post-composing $\tau$ with the projection map $H \to H/\mathcal{Z}(H)$, we shall assume henceforth that $H$ is adjoint and Zariski-connected. The representation $\tau : \Lambda \to H$ need no longer be faithful, but it still has finite kernel. As in [Mon05, (3.3)], in view of the assumption that $\Lambda$ is finitely generated, we may assume that $H$ is defined over a finitely generated field $K$. This is the first of two places where the admissibility assumption is used in loc. cit. (following VIII.3.22 and IX.1.8 in [Mar91]).

By Tits’ alternative [Tit72], the amenable radical of $\Lambda$ is soluble-by-locally-finite and thus locally finite since $\tau(\Lambda)$ is Zariski-dense and $H$ is semi-simple. The finite generation of $K$ implies that this radical is in fact finite (see e.g. Corollary 4.8 in [Web73]), thus trivial by Zariski-density since $H$ is adjoint. (The finite generation of $K$ is essential in positive characteristic since for algebraically closed fields there is always a locally finite Zariski-dense subgroup [BGM04].)

It now follows that if $J$ is a compact extension of a discrete group, then the latter has trivial amenable radical and thus all the conclusions of Proposition 10.5 hold trivially. Therefore, we assume henceforward that $J$ is not compact-by-discrete.

Let $H = H_1 \times \cdots \times H_k$ be the decomposition of $H$ into its simple factors. We shall work with the factors $H_i$ one at a time. Let $\tau_i : \Lambda \to H_i$ be the induced representation of $\Lambda$. Notice that $\tau_i$ need not be faithful; however, it has Zariski-dense (and in particular infinite) image.

We let $\Sigma_i$ denote the set of all (inequivalent representatives of) valuations $v$ of $K$ such that the image of $\tau_i(\Lambda)$ is not relatively compact in $H_i(K_v)$ (for the Hausdorff topology); observe that this image is still Zariski-dense. Then $\Sigma_i$ is non-empty since $\tau_i(\Lambda)$ is infinite, see [BG07, Lemma 2.1].

By hypothesis, there exists a continuous representation $J \to H_i(K_v)$ for each $v \in \Sigma_i$, extending the given $\Lambda$-representation. We denote by $N_v < J$ the kernel of this representation. Let $I \subseteq \{1, \ldots, k\}$ be the set of all those indices $i$ such that $J/N_v$ is non-discrete for each $v \in \Sigma_i$.

We claim that the set $I$ is non-empty.
Indeed, for each index \( j \notin I \), there exists \( v_j \in \Sigma_j \) such that \( N_{v_j} \) is open in \( J \). Thus the kernel

\[
J^+ = \bigcap_{j \notin I} N_{v_j}
\]

of the continuous representation \( J \to \prod_{j \notin I} H_j(K_{v_j}) \) is open.

By assumption the closure of \( \Lambda \) in \( J \) has finite covolume. Therefore, for each open subgroup \( F < J \), the closure of \( \Lambda \cap F \) has finite covolume in \( F \). It follows in particular that \( \Lambda \cap F \) is infinite unless \( F \) is compact.

These considerations apply to the open subgroup \( J^+ < J \). Since \( J \) is not compact-by-discrete, we deduce that \( \Lambda \cap J^+ \) is infinite. Therefore the restriction to \( \Lambda \) of the representation \( J \to \prod_{j \in I} H_j(K_{v_j}) \) has infinite kernel and, hence, it does not factor through \( \tau : \Lambda \to H(K) \).

In particular it cannot coincide with the given representation \( \tau : \Lambda \to H \). Thus \( I \) is non-empty.

We claim that for each \( i \in I \), the set \( \Sigma_i \) is finite.

Let \( i \in I \) and \( v \in \Sigma_i \). The arguments of [Mon05, (3.7)] show that the isomorphic image of \( J/N_v \) in \( H_i(K_v) \) contains \( H_i(K_v)^+ \). These arguments use again the admissibility assumption because the appeal to a result of R. Pink [Pin98]; the fact that the latter hold in the admissible case is explicit in the table provided in Proposition 1.6 of [Pin98]. Furthermore, it follows from Tits’ simplicity theorem [Tit64] combined with [BT73, 6.14] that each \( J/N_v \) is quasi-simple in the sense of the definition made on p. 28. Moreover, an application of [BT73, 8.13] shows that the various quotients \( (J/N_v)_{v \in \Sigma_i} \) are pairwise non-isomorphic. In particular the normal subgroups \( (N_v)_{v \in \Sigma_i} \) are pairwise distinct.

Let \( D_i = \bigcap_{v \in \Sigma_i} N_v \) and recall that \( J/D_i \) is compactly generated. Projecting each \( N_v \) to \( J/D_i \), we obtain a family of pairwise distinct normal subgroups of \( J/D_i \) indexed by \( \Sigma_i \) such that each corresponding quotient is quasi-simple, non-discrete and non-compact. Therefore, the desired claim follows from Corollary 5.15.

In particular, appealing again to [BT73, Corollaire 8.13], we obtain a continuous map

\[
J \to \prod_{v \in \Sigma_i} H_i(K_v)
\]

which we denote again by \( \tau_i^J \). The kernel of \( \tau_i^J \) is \( D_i \). Upon replacing \( J \) and \( \Lambda \) by finite index subgroups we may assume that the image of \( \tau_i^J \) coincides in fact with \( \prod_{v \in \Sigma_i} H_i(K_v)^+ \), compare [Mon05, (3.9)].

We claim that \( R := J^+ \cap D \) is compact and that \( J = J^+ \cdot D \), where \( D \) is defined by \( D = \bigcap_{i \in I} D_i \).

We first show that \( R = J^+ \cap D \) is compact. Assume for a contradiction that this is not the case. Then, given a compact open subgroup \( U \) of \( J \), the intersection \( \Lambda_0 = \Lambda \cap (U \cdot R) \) is infinite: this follows from the same argument as above, using the assumption that the closure of \( \Lambda \) has finite covolume.

For each index \( j \notin I \), we have \( J^+ \subseteq N_{v_j} \) and we deduce that the image of \( \Lambda_0 \) in \( H_j(K_{v_j}) \) is finite, since it is contained in the image of \( U \). Equivalently, the subgroup \( \tau_j(\Lambda_0) < H_j(K) \) is finite. It follows in particular that \( \tau_i(\Lambda_0) \) is infinite for some \( i \in I \). By [BG07, Lemma 2.1], there exists \( v \in \Sigma_i \) such that the image of \( \Lambda_0 \) in \( H_i(K_v) \) is unbounded. This is absurd since \( D \subseteq N_v \) and hence the image of \( \Lambda_0 \) in \( H_i(K_v) \) is contained in the image of the compact subgroup \( U \). This shows that the intersection \( R \) is indeed compact.

At this point we know that the quotient \( J/D \) is isomorphic to a subgroup of the product

\[
\prod_{i \in I} \prod_{v \in \Sigma_i} H_i(K_v)^+
\]
which projects surjectively onto each factor of the form $\prod_{v \in \Sigma} H_v(K_v)^+$. Using again the Goursat-type argument as in Corollary 5.15, we find that $J/D$ is indeed isomorphic to a finite product of non-compact non-discrete simple groups $H_v(K_v)^+$. In particular the quotient $J/D$ has no non-trivial open normal subgroup. Since $J^+$ is open and normal in $J$, we deduce that $J = J^+ \cdot D$, thereby establishing the claim.

By the very nature of the statement, we may replace $J$ by the quotient $J/R$ without any loss of generality, since $R$ is compact. In view of this further simplification, the preceding claim implies that $J \cong J^+ \times D$. In particular $D$ is discrete.

It now follows as in [Mon05, (3.11)] that $K$ is a global field, and that the image of $\Lambda$ in the semi-simple group $J/D$ is an arithmetic lattice (compare [Mon05, (3.13)]). Therefore, by Proposition 8.1, the intersection $\Lambda \cap D$ is a lattice in $D$ and, hence, the discrete normal subgroup $D$ is virtually contained in $\Lambda$. As $J \cong J^+ \times D$ and $J^+$ has trivial quasi-centre, it follows that the quasi-centre of $J$ coincides with $D$. This finishes the proof. 

For later use, we single out a (simpler) version of an argument referred to above.

**Lemma 10.7.** Let $H$ be an admissible connected absolutely almost simple adjoint $k$-group $H$, where $k$ is a local field. Let $J$ be a locally compact group with a continuous unbounded Zariski-dense homomorphism $\tau : J \to H(k)$. Then any compact normal subgroup of $J$ is contained in the kernel of $\tau$.

**Proof.** Let $K \triangleleft J$ be a compact normal subgroup. The Zariski closure of $\tau(K)$ is normalised by the Zariski-dense group $\tau(J)$ and therefore it is either $H(k)$ or trivial. We assume the former since otherwise we are done.

We claim that we can assume $k$ non-Archimedean. Otherwise, either $k = \mathbb{R}$ or $k = \mathbb{C}$. In the first case, $\tau(K)$ coincides with its Zariski closure by Weyl’s algebraicity theorem [Vin94, 4.2.1] so that $H(k)$ is compact in which case the lemma is void by the unboundedness assumption. In the second case, one can reduce to the case $\tau(J) \subseteq H(\mathbb{R})$ as in [Mon05, (3.5)] and thus $\tau(K) = 1$ as before since $H(\mathbb{R})$ is also simple; the claim is proved.

Following now an idea from [Sha00, p. 41] (see also the explanations in Section (3.7) of [Mon05]), one uses [Fin98] to deduce that $\tau(K)$ is open upon possibly replacing $k$ by a closed subfield (the admissibility assumption enters as in the proof of Proposition 10.5). We can still denote this subfield by $k$ because it accommodates the whole image $\tau(J)$, see again [Mon05, (3.7)]. Now $\tau(J)$ is an unbounded open subgroup and hence contains $H(k)^+$ by a result of J. Tits (see [Pra82]; this also follows from the Howe–Moore theorem [HM79] which however is posterior to Tits’ result). This implies that the compact group $\tau(K)$ is trivial since $H(k)^+$ is simple by [Tit64]. \hfill $\Box$

10.B. Boundary maps. We record two statements extracted from Margulis’ work in the form most convenient for us.

**Proposition 10.8.** Let $J$ be a second countable locally compact group with a measure class preserving action on a standard probability space $B$. Let $\Lambda < J$ be a dense subgroup with a Zariski-dense unbounded representation $\tau : \Gamma \to H(k)$ to a connected absolutely almost $k$-simple adjoint group $H$ over an arbitrary local field $k$.

If there is a proper $k$-subgroup $L < H$ and a $\Lambda$-equivariant non-essentially-constant measurable map $B \to H(k)/L(k)$, then $\tau$ extends to a continuous homomorphism $J \to H(k)$.

**Proof.** The argument is given by A’Campo–Burger in the characteristic zero case at the end of Section 7 in [AB94] (pp. 18–19). This reference considers homogeneous spaces for $B$ but
this restriction is never used. The general statement is referred to in [Bur95] and details are given in [Bon04].

**Proposition 10.9.** Let $\Gamma$ be a countable group with a Zariski-dense unbounded representation $\Gamma \to H(k)$ to a connected absolutely almost $k$-simple adjoint group $H$ over an arbitrary local field $k$. Let $B$ be a standard probability space with a measure class preserving $\Gamma$-action that is amenable in Zimmer’s sense [Zim84] and such that the diagonal action on $B^2$ is ergodic.

Then there is a proper $k$-subgroup $L < H$ and a $\Gamma$-equivariant non-essentially-constant measurable map $B \to H(k)/L(k)$.

**Proof.** Again, this is proved in [AB94] for the characteristic zero case (and $B$ homogeneous) and the necessary adaptations to the general case are explained in [Bon04].

We shall need these specific statements below. They first appeared within the proof of Margulis’ commensurator superrigidity, which can adapted as follows using [Bur95] and Lemma 9.3, providing a first step towards Theorem 10.6.

**Proposition 10.10.** Let $G = G_1 \times G_2$ be a product of locally compact $\sigma$-compact groups and $\Lambda < G$ be an irreducible lattice. Assume that the projection of $\Lambda$ to $G_1$ is injective and that $G_2$ admits a compact open subgroup. Then the pair $(\Lambda, G)$ is superrigid.

**Proof.** We claim that one can assume $G$ second countable. As explained in [Mon06, Proposition 61], $\sigma$-compactness implies the existence of a compact normal subgroup $K < G$ meeting $\Lambda$ trivially and such that $G/K$ is second countable. Applying the statement to $G/K$ together with the image of $\Lambda$ therein yields the general statement since the projection of $\Lambda$ to $G/K$ is an isomorphism; this proves the claim.

Let $\tau: \Lambda \to H(k)$ be as in the definition of superrigid pairs and let $U < G_2$ be a compact open subgroup. Set $\Lambda_U = \Lambda \cap (G_1 \times U)$. By the injectivity assumption and Lemma 8.2, we can consider $\Lambda_U$ as a lattice in $G_1$ which is commensurated by (the image of the projection of $\Lambda$). We distinguish two cases.

Assume first that $\tau(\Lambda_U)$ is unbounded in the locally compact group $H(k)$. We may then apply Margulis’ commensurator superrigidity in its general form proposed by M. Burger [Bur95, Theorem 2.A], see [Bon04] for details. This yields a continuous map $J \to H(k)$ factoring through $G_1$ and extending the given $\Lambda$-representation, as desired.

Assume now that $\tau(\Lambda_U)$ is bounded, which is equivalent to $\Lambda_U$ fixing a point in the symmetric space or Bruhat–Tits building associated to $H(k)$. Then Lemma 9.3 yields a continuous map $J \to H(k)$ factoring through $G_2$. \qed

10.C. Radical superrigidity.

**Theorem 10.11.** Let $G$ be a locally compact group, $R < G$ its amenable radical, $\Gamma < G$ a finitely generated lattice and $F$ the closure of the image of $\Gamma$ in $G/R$.

Then any Zariski-dense unbounded representation of $\Gamma$ in any connected absolutely almost simple adjoint $k$-group $H$ over any local field $k$ arises from a continuous representation of $F$ via the map $\Gamma \to F$.

(In particular, the pair $(\Gamma/(\Gamma \cap R), F)$ is superrigid.)

**Proof.** Notice that $G$ is $\sigma$-compact since it contains a finitely generated, hence countable, lattice. (In fact $G$ is even compactly generated by Lemma 7.12.) Set $J = G/R$. There exists a standard probability $J$-space $B$ on which the $\Gamma$-action is amenable and such that the diagonal $\Gamma$-action on $B^2$ is ergodic; it suffices to choose $B$ to be the Poisson boundary
of a symmetric random walk with full support on $J$. Indeed: (i) The $J$-action is amenable as was shown by Zimmer [Zim78]; this implies that the $G$-action is amenable since $R$ is an amenable group and thus that the $\Gamma$-action is amenable since $\Gamma$ is closed in $G$ (see [Zim84, 5.3.5]). (ii) The diagonal action of any closed finite covolume subgroup $F < J$ on $B^2$ is ergodic in view of the ergodicity with coefficients of $J$, and hence the same holds for dense subgroups of $F$. For detailed background on this strengthening of ergodicity introduced in [BM02] and on the Poisson boundary in general, we refer the reader to [Kai03].

Let now $k$ be a local field, $H$ a connected absolutely almost simple $k$-group and $\Gamma \to H(k)$ a Zariski-dense unbounded representation. We can apply Proposition 10.9 and obtain a proper subgroup $L < H$ and a $\Gamma$-equivariant map $B \to H(k)/L(k)$. Writing $\Lambda$ for the image of $\Gamma$ in $J$, we can therefore apply Proposition 10.8 with $F$ instead of $J$ and the conclusion follows.

**Remark 10.12.** An examination of this proof shows that one has also the following related result. Let $J$ be a second countable locally compact group and $\Lambda \subseteq J$ a dense countable subgroup whose action on $J$ by left multiplication is amenable. Then the pair $(\Lambda, J)$ is superrigid. Indeed, one can again argue with Propositions 10.8 and 10.9 because it is easy to check that in the present situation any amenable $J$-space is also amenable for $\Lambda$ viewed as a discrete group. Related ideas were used by R. Zimmer in [Zim87].

**Corollary 10.13.** Let $G$ be a locally compact group and $\Gamma < G$ a finitely generated lattice.

If $\Gamma$ admits a faithful Zariski-dense representation in an admissible semi-simple group (over any field), then the amenable radical of $G$ is compact.

**Proof.** Let $R$ be the amenable radical of $G$, $F$ be the closure of the image of $\Gamma$ in $G/R$ and $J < G$ the preimage of $F$ in $G$. The content of Theorem 10.11 is that the pair $(\Gamma, J)$ is superrigid. Since in addition $\Gamma$ is closed and of finite covolume in $J$ (see [Rag72, Lemma 1.6]), we may apply Proposition 10.5 and deduce that the amenable radical of $J$ is compact. The conclusion follows since $R < J$. □

**10.D. Lattices with non-discrete commensurators.** The following useful trick allows to realize the commensurator of any lattice in a locally compact group $G$ as a lattice in a product $G \times D$. A similar reasoning in the special case of automorphism groups of trees may be found in [BG02, Theorem 6.6].

**Lemma 10.14.** Let $\Lambda$ be a group and $\Gamma < \Lambda$ a subgroup commensurated by $\Lambda$. Let $D$ be the completion of $\Lambda$ with respect to the left or right uniform structure generated by the $\Lambda$-conjugates of $\Gamma$. Then $D$ is a totally disconnected locally compact group.

If furthermore $G$ is a locally compact group containing $\Lambda$ as a dense subgroup such that $\Gamma$ is discrete (resp. is a lattice) in $G$, then the diagonal embedding of $\Lambda$ in $G \times D$ is discrete (resp. is an irreducible lattice).

The above lemma is in some sense a converse to Lemma 8.2. In the special case where one starts with a lattice satisfying a faithfulness condition, this relation becomes even stronger.

**Lemma 10.15.** Let $G, H$ be locally compact groups and $\Lambda < G \times H$ a lattice. Assume that the projection of $\Lambda$ to $G$ is faithful and that both projections are dense. Let $U < H$ be a compact open subgroup, set $\Gamma = \Lambda \cap (G \times U)$ as in Lemma 8.2 and consider the group $D$ as in Lemma 10.14 (upon viewing $\Lambda$ as a subgroup of $G$). Define the compact normal subgroup $K < H$ as the core $K = \bigcap_{h \in H} hUh^{-1}$ of $U$ in $G$.

Then the map $\Lambda \to D$ induces an isomorphism of topological groups $H/K \cong D$. 
Proof of Lemma 10.14. One verifies readily the condition given in [Bon60] (TG III, §3, No 4, Théorème 1) ensuring that the completion satisfies the axioms of a group topology. We emphasise that it is part of the definition of the completion that $D$ is Hausdorff; in other words $D$ is obtained by first completing $\Lambda$ with respect to the group topology as defined above, and then dividing out the normal subgroup consisting of those elements which are not separated from the identity.

Let $U$ denote the closure of the projection of $\Gamma$ to $D$. By definition $U$ is open. Notice that it is compact since it is a quotient of the profinite completion of $\Gamma$ by construction. In particular $D$ is locally compact.

By a slight abuse of notation, let us identify $\Gamma$ and $\Lambda$ with their images in $D$. We claim that $U \cap \Lambda = \Gamma$. Indeed, let $\{\lambda_n\}_{n \geq 0}$ be a sequence of elements of $\Lambda$ such that $\lim_n \lambda_n = \lambda \in \Lambda$. Since $\lambda \Gamma \lambda^{-1}$ is a neighbourhood of the identity in $\Lambda$ (with respect to the topology induced from $D$), it follows that $\gamma_n \lambda^{-1} \in \lambda \Gamma \lambda^{-1}$ for $n$ large enough. Thus $\lambda \in \gamma_n \Gamma = \Gamma$.

Assume now that $\Gamma$ is discrete and choose a neighbourhood $V$ of the identity in $G$ such that $\Gamma \cap V = 1$. In view of the preceding claim the product $V \times U$ is a neighbourhood of the identity in $G \times D$ which meets $\Lambda$ trivially, thereby showing that $\Lambda$ is discrete.

Assume finally that $\Gamma$ is a lattice in $G$ and let $F$ be a fundamental domain. Then $F \times U$ is a fundamental domain for $\Lambda$ in $G \times D$, which has finite volume since a Haar measure for $G \times D$ may be obtained by taking the product of respective Haar measures for $G$ and $D$. Thus $\Lambda$ has finite covolume in $G \times D$. □

Proof of Lemma 10.15. In order to construct a continuous homomorphism $\pi: H \to D$, it suffices to check that any net in $\Lambda$ whose image in $H$ converges to the identity also converges to the identity in $D$; this follows from the definitions of $\Gamma$ and $D$ since the net is eventually in any $\Lambda$-conjugate of $U$. Notice that $\pi$ has dense image.

We claim that the kernel of $\pi$ is $\bigcap_{\lambda \in \Lambda} \lambda U \lambda^{-1}$. Indeed, if on the one hand $k \in \ker(\pi)$ is the limit of the images in $H$ of a net $\{\lambda_i\}$ in $\Lambda$, then for any $\lambda$ we have eventually $\lambda_i \in \lambda^{-1} \Gamma \lambda \subseteq \lambda^{-1} (G \times U) \lambda$ so that indeed $k \in \lambda^{-1} U \lambda$ since $U$ is closed. Conversely, if $k \in \bigcap_{\lambda \in \Lambda} \lambda U \lambda^{-1}$ is limit of images of $\{\lambda_i\}$, then, since $U$ is open, for any $\lambda$ the image of $\lambda_i$ is eventually in $\lambda U \lambda^{-1}$, hence in $\lambda \Gamma \lambda^{-1}$ so that $\pi(\lambda_i) \to 1$. This proves the claim.

Now it follows that $\ker(\pi)$ is indeed the core $K$ of the statement since $U$ is compact. The fact that $\pi$ is onto and open follows from the existence of a compact open subgroup in $H$. □

Theorem 10.16. Let $G$ be a locally compact group and $\Gamma < G$ be a lattice. Assume that $G$ possesses a finitely generated dense subgroup $\Lambda$ such that $\Gamma < \Lambda < \mathrm{Comm}_G(\Gamma)$.

If $\Lambda$ admits a faithful Zariski-dense representation in an admissible semi-simple group (over any field), then the amenable radical $R$ of $G$ is compact and the quasi-centre $\mathcal{Z}(G)$ is virtually contained in $\Gamma \cdot R$. Furthermore, upon replacing $G$ by a finite index subgroup, the quotient $G/R$ splits as $G^+ \times \mathcal{Z}(G/R)$ where $G^+$ is a semi-simple algebraic group and the image of $\Gamma$ in $G^+$ is an arithmetic lattice.

Proof. Let $J = G \times D$, where $D$ is the totally disconnected locally compact group provided by Lemma 10.14. As a totally disconnected group, it has numerous compact open subgroups (for instance the closure of $\Gamma$). We shall view $\Lambda$ as an irreducible lattice in $J$. The projection of $\Lambda$ to $G$ is faithful by construction. By Proposition 10.10, the pair $(\Lambda, J)$ is superrigid. This allows us to apply Proposition 10.5. Since the amenable radical $R_G$ of $G$ is contained in the amenable radical $R_J$ of $J$, it is compact. Furthermore, the quasi-centre of $G$ is contained in the quasi-centre of $J$ and the centre-free group $G/R_G$ is a direct factor of $J^+ \times \mathcal{Z}(J/R_J)$; the desired conclusions follow. □
10.E. Lattices in products of Lie and totally disconnected groups.

**Theorem 10.17.** Let $\Gamma < G = S \times D$ be a finitely generated irreducible lattice, where $S$ is a connected semi-simple Lie group with trivial centre and $D$ is a totally disconnected locally compact group. Let $\Gamma_D < D$ be the canonical discrete kernel of $D$.

Then $D/\Gamma_D$ is a profinite extension of a semi-simple algebraic group $Q$ and the image of $\Gamma$ in $S \times Q$, which is isomorphic to $\Gamma/\Gamma_D$, is an arithmetic lattice.

**Corollary 10.18.** In particular, $D$ is locally profinite by analytic.

A family of examples will be constructed in Section 11.C below, showing that the statement cannot be simplified even in a geometric setting (see Remark 11.7).

**Proof of Theorem 10.17.** By the very nature of the statement, we can factor out the canonical discrete kernel. Therefore, we shall assume henceforth that the projection map $\Gamma \to S$ is injective. We can also assume that $S$ has no compact factors. Since $S$ is connected with trivial centre, there is a Zariski connected semi-simple adjoint $\mathbb{R}$-group $H$ without $\mathbb{R}$-anisotropic factors such that $S = H(\mathbb{R})$. Notice that the injectivity of $\Gamma \to S$ is preserved when passing to finite index subgroups.

By Proposition 10.10, the pair $(\Gamma, G)$ is superrigid. We can therefore apply Proposition 10.5. In particular, $D$ has compact amenable radical and therefore, in view of the statement of Theorem 10.17, we can assume that this radical is trivial. Given the conclusion of Proposition 10.5, it only remains to show that the quasi-centre $Z^\Gamma(G)$ of $G$ is trivial. We now know that $Z^\Gamma(G)$ is virtually contained in $\Gamma$; since on the other hand $S$ has trivial quasi-centre, $Z^\Gamma(G) \subseteq 1 \times D$. In other words, $Z^\Gamma(G)$ is contained in the discrete kernel $\Gamma_D$, which has been rendered trivial. This completes the proof.

We have treated Theorem 10.17 as a port of call on the way to Theorem 10.1. In fact, one can also describe lattices in products of groups with a simple algebraic factor over an arbitrary local field and in most cases without assuming finite generation a priori. We record the following statement, which will not be used below.

**Theorem 10.19.** Let $k$ be any local field and $G$ an admissible connected absolutely almost simple adjoint $k$-group. Let $H$ be any compactly generated locally compact group admitting a compact open subgroup. Let $\Gamma < G(k) \times H$ be an irreducible lattice. In case $k$ has positive characteristic and the $k$-rank of $G$ is one, we assume $\Gamma$ cocompact.

Then $H/\Gamma_H$ is a compact extension of a semi-simple algebraic group $Q$ and the image of $\Gamma$ in $G(k) \times Q$ is an arithmetic lattice.

There is no assumption whatsoever on the compactly generated locally compact group $H$ beyond admitting a compact open subgroup; recall that the latter is automatic if $H$ is totally disconnected [Bou71, III §4 No 6]. Notice that a posteriori it follows from arithmeticity that $\Gamma$ is finitely generated; in the proof below, finite generation will be established in two steps.

**Proof of Theorem 10.19.** We factor out the canonical discrete kernel $\Gamma_H$ and assume henceforth that it is trivial. This does not affect the other assumptions and thus we choose some compact open subgroup $U < H$. We write $G = G(k)$ and consider $\Gamma_U = \Gamma \cap (G \times U)$ as in Lemma 8.2. Since we factored out the canonical discrete kernel, we can consider $\Gamma_U$ as a lattice in $G$ commensurated by the dense subgroup $\Gamma < G$. Moreover, $\Gamma_U$ is finitely generated; indeed, either we have simultaneously rank$_k(G) = 1$ and char($k) > 0$, in which case we assumed $\Gamma$ cocompact, so that $\Gamma_U$ is cocompact in the compactly generated group $G(k)$ (again Lemma 8.2) and hence finitely generated [Mar91, I.0.40]; or else, $\Gamma_U$ is known
to be finitely generated by applying, as the case may be, either Kazhdan’s property, or
the theory of fundamental domains, or the cocompactness of $p$-adic lattices — we refer to
Margulis, Sections (3.1) and (3.2) of Chapter IX in [Mar91].

We can now apply Margulis’ arithmeticity [Mar91, 1.(1)] and deduce that $G$ is defined
over a global field $K$ and that $\Gamma_U$ is commensurable to $G(K(S))$ for some finite set of places
$S$; in short $\Gamma_U$ is $S$-arithmetic. (The idea to obtain first this preliminary arithmeticity of
$\Gamma_U$ was suggested by M. Burger.) It follows that $\Gamma$ is rational over the global field $K$
see Theorem 3.b in [Bor66] (loc. cit. is formulated for the Lie group case; see [Wor07,
Lemma 7.3] in general).

Since the pair $(\Gamma, G \times H)$ is superrigid (for instance by Proposition 10.10), only the $a
priori$ lack of finite generation for $\Gamma$ prevents us from applying Proposition 10.5. However, a
good part of the proof of that proposition is already secured here since $\Gamma$ has been shown to
be rational over a global field. We now proceed to explain how to adapt the remaining part
of that proof to the current setting. We use those elements of notation introduced in the
proof of Proposition 10.5 that do not conflict with present notation and review all uses of
finite generation that are either explicit in the proof of Proposition 10.5 or implicit through
references to [Mon05].

The compact generation of $G \times H$ is an assumption rather than a consequence of Lemma 7.12.

We also used finite generation in order to pass to a finite index subgroup of $\Gamma$ contained in
$G(K_v)^+$ for all valuations $v \in \Sigma$. We shall postpone this step, so that the whole argument
provides us with maps from $G \times H$ to a product $Q$ of factors that lie in-between $G(K_v)^+$ and $G(K_v)$. In particular all these factors are quasi-simple and we can still appeal
to Corollary 5.15 as before. Notice however that at the very end of the proof, once finite
generation is granted, we can invoke the argument that $G(K_v)/G(K_v)^+$ is virtually torsion
Abelian [BT73, 6.14] and thus reduce again to the case where $\Gamma$ is contained in $G(K_v)^+$. We now justify that the image of $\Gamma$ in $G \times Q$ is discrete because previously this followed from [Mon05, (3.13)] which relies on finite generation. If $\Gamma$ were not discrete, an application of [BG07, Lemma 2.1] would provide a valuation $v \notin \Sigma$ with $\Gamma$ unbounded in $G(K_v)$, which is absurd.

We are now in a situation where $G \times H$ maps to $G \times Q$ with cocompact finite covolume
image and injectively on $\Gamma$; therefore the discreteness of the image of $\Gamma$ implies that this
map is proper and hence $H$ is a compact extension of $Q$. Pushing forward the measure on
$(G \times H)/\Gamma$, we see that the image of $\Gamma$ in $G \times Q$ is a lattice. Now $\Gamma$ is finitely generated (see
above references to [Mar91, IX]) and thus the proof is completed as in Proposition 10.5.
The discrete factor occurring in the conclusion of the latter proposition is trivial for the
same reason as in the proof of Theorem 10.17. $\square$

10.F. Lattices in general products. We begin with the special case of totally disconnected groups.

Proof of Theorem 10.6. An issue that we need to deal with is that the projection of $\Gamma$ to $G_1$
is $a priori$ not faithful. In order to circumvent this difficulty, we proceed to a preliminary
construction.

Let $\iota : \Gamma \to \hat{\Gamma}$ be the canonical map to the profinite completion of $\Gamma$ and denote its kernel
by $\Gamma^{(1)}$; in other words, $\Gamma^{(1)}$ is the finite residual of $\Gamma$. Let $\hat{G}_1$ denote the locally compact
group which is defined as the closure of the image of $\Gamma$ in $G_1 \times \hat{\Gamma}$ under the product map
$\text{proj}_1 \times \iota$, where $\text{proj}_1 : G \to G_1$ is the canonical projection. Since $\text{proj}_1(\Gamma)$ is dense in $G_1$
and $\hat{\Gamma}$ is compact, the canonical map $\hat{G}_1 \to G_1$ is surjective. In other words, the group $\hat{G}_1$
is a compact extension of $G_1$. 
We now define \( G'_1 = G_2 \times \cdots \times G_n \) and \( \hat{G} = \hat{G}_1 \times G'_1 \). Then \( \Gamma \) admits a diagonal embedding into \( \hat{G} \) through which the injection of \( \Gamma \) in \( G \) factors. We will henceforth identify \( \Gamma \) with its image in \( \hat{G} \) and consider \( \Gamma \) as an irreducible lattice of \( \hat{G} \).

We claim that the pair \((\Gamma, \hat{G})\) is superrigid.

The argument is a variation on the proof of Proposition 10.10. Let \( \tau : \Gamma \to \text{H}(k) \) be as in the definition of superrigid pairs. Since \( \tau(\Gamma) \) is finitely generated and linear, it is residually finite [Mal40]. This means that \( \tau \) factors through \( \Gamma := \Gamma / \Gamma^{(f)} \). Let \( U < G'_1 \) be a compact open subgroup, \( \Gamma_U = \Gamma \cap (\hat{G}_1 \times U) \) and \( \Gamma_{U'} = \Gamma_U / (\Gamma_U \cap \Gamma^{(f)}) \). By construction and Lemma 8.2, we can consider \( \Gamma_{U'} \) as a lattice in \( \hat{G}_1 \) commensurated by \( \hat{\Gamma} \). Arguing as in Proposition 10.10, when \( \tau(\Gamma_U) \) is unbounded one applies commensurator superrigidity yielding a continuous map \( J : \hat{\Gamma} \to \text{H}(k) \) and extending the map \( \Gamma \to \text{H}(k) \) and hence also \( \tau \). When \( \tau(\Gamma_U) \) is bounded, one applies Lemma 9.3 instead and the resulting extension factors through \( G'_1 \). This proves the claim.

In order to conclude that the pair \((\Gamma, \hat{G})\) is also superrigid, it now suffices to apply Lemma 10.7.

\[ \square \]

**Corollary 10.20.** Theorem 10.1 holds in the particular case of totally disconnected groups.

**Proof.** Theorem 10.6 provides the hypothesis needed for Proposition 10.5. \[ \square \]

We now turn to the general case \( \Gamma < G = G_1 \times \cdots \times G_n \) of Theorem 10.1. The main part of the remaining proof will consist of a careful analysis of how the lattice \( \Gamma \) might sit in various subproducts hidden in the factors \( G_i \) or their finite index subgroups once the amenable radical has been trivialised. It will turn out that \( \Gamma \) is virtually a direct product \( \Gamma' \times \Gamma'' \), where \( \Gamma' \) is an irreducible lattice in a product \( S' \times D' \) with \( S' \) a semi-simple Lie (virtual) subproduct of \( G \) and \( D' \) a totally disconnected subgroup of \( G \) whose position will be clarified; as for \( \Gamma'' \), it is an irreducible lattice in a semi-simple Lie group \( S'' \) that turns out to satisfy the assumptions of Margulis’ arithmeticity. Of course, any of the above factors might well be trivial.

The amenable radical is compact by Corollary 10.13 and hence we can assume that it is trivial. The group \( G \) (and hence also all finite index subgroups and factors) is compactly generated by Lemma 7.12. Upon regrouping the last \( n-1 \) factors and in view of the definition of an irreducible lattice (see p. 52), we can assume \( G = G_1 \times G_2 \). We apply the solution to Hilbert’s fifth problem (compare Theorem 4.6) and write \( G_i = S_i \times D_i \) after replacing \( G \) and \( \Gamma \) with finite index subgroups. Here \( S_i \) are connected semi-simple centre-free Lie groups without compact factors and \( D_i \) totally disconnected compactly generated with trivial amenable radical. Set \( S = S_1 \times S_2 \) and \( D = D_1 \times D_2 \). Thus \( \Gamma \) is a lattice in \( G = S \times D \). Notice that if \( S \) is trivial, then \( G \) is totally disconnected and we are done by Theorem 10.6. We assume henceforth that \( S \) is non-trivial. The main remaining obstacle is that the lattice \( \Gamma \) need not be irreducible with respect to the product decomposition \( G = S \times D \).

Observe that the closure \( \text{proj}_{D_i}(\Gamma) \) of the projection of \( \Gamma \) to \( D_i \) has trivial amenable radical.

Indeed \( \text{proj}_{D_i}(\Gamma) \) is dense in \( \Gamma_i \) for \( i = 1, 2 \), hence the projection \( \text{proj}_{D_i}(\Gamma) \to D_i \) has dense image. The desired claim follows since \( G \), and hence \( D_i \), has trivial amenable radical.

Let \( U < D \) be a compact open subgroup and set \( \Gamma_U = \Gamma \cap (S \times U) \). By Lemma 8.2, the projection \( \text{proj}_S(\Gamma_U) \) of \( \Gamma_U \) to \( S \) is a lattice which is commensurated by \( \text{proj}_S(\Gamma) \). The lattice \( \text{proj}_S(\Gamma_U) \) possesses a finite index subgroup which admits a canonical splitting into finitely many irreducible groups \( \Gamma^1 \times \cdots \times \Gamma^r \), compare Theorem 8.17. Furthermore each \( \Gamma^i \)
is an irreducible lattice in a semi-simple subgroup $S^i < S$ which is obtained by regrouping some of the simple factors of $S$.

Since the projection of $\Gamma$ to each $G_1$ and $G_2$, and hence to $S_1$ and $S_2$, is dense, it follows that the projection of $\Gamma$ to each simple factor of $S$ is dense. We now consider the projection of $\Gamma$ to the various factors $S^i$. In view of the preceding remark and the fact that $\Gamma^i$ is an irreducible lattice in $S^i$, it follows that $\text{proj}_{S^i}(\Gamma)$ is either dense in $S^i$ or discrete and contains $\Gamma^i$ with finite index, see [Mar91, IX.2.7]. Let now

$$S' = \langle S^i \mid \text{proj}_{S^i}(\Gamma) \text{ is non-discrete} \rangle \text{ and } S'' = \langle S^i \mid \text{proj}_{S^i}(\Gamma) \text{ is discrete} \rangle.$$ 

We claim that the projection of $\Gamma$ to $S'$ is dense.

If this failed, then by [Mar91, IX.2.7] there would be a subproduct of some simple factors of $S'$ on which the projection of $\Gamma$ is a lattice. Since each $\Gamma^i$ is irreducible, this subproduct is a regrouping $S^{i_1} \times \cdots \times S^{i_p}$ of some factors $S^i$. Now the projection of $\Gamma$ is a lattice in this subgroup, hence it contains the product $\Gamma^{i_1} \times \cdots \times \Gamma^{i_p}$ with finite index and thus projects discretely to each $S^{i_j}$. This contradicts the definition of $S'$ and proves the claim.

Our next claim is that $\Gamma$ has a finite index subgroup which splits as $\Gamma' \times \Gamma''$, where $\Gamma'' = \text{proj}_{S''}(\Gamma)$ and $\Gamma'$ is a lattice in $S' \times D$.

In order to establish this, we define

$$\Gamma' = \text{Ker}(\text{proj} : \Gamma \to S'') \quad \text{and} \quad \Gamma'' = \bigcap_{\gamma \in \Gamma} \gamma \Gamma_U \gamma^{-1}.$$ 

Notice that $\Gamma'$ and $\Gamma''$ are both normal subgroups of $\Gamma$. Since $\text{proj}_D(\Gamma'')$ is a compact subgroup of $D$ normalised by $\text{proj}_D(\Gamma)$, which has trivial amenable radical, it follows that $\Gamma'' \subset S' \times S'' \times 1$. Therefore, the intersection $\Gamma' \cap \Gamma''$ is a normal subgroup of $\Gamma$ contained in $S' \times 1 \times 1$. In view of the preceding claim, we deduce that $\Gamma' \cap \Gamma'' = 1$. Thus $\langle \Gamma' \cup \Gamma'' \rangle < \Gamma$ is isomorphic to $\Gamma' \times \Gamma''$. Since $\Gamma'' \cap \Gamma' \cap \Gamma_U$ projects to a lattice in $S'$ which commutes with the projection of $\Gamma''$, we deduce moreover that $\text{proj}_{S''}(\Gamma'') = 1$, or equivalently that $\Gamma'' < 1 \times S'' \times 1$.

Since the projection of $\Gamma$ to $S''$ has discrete image by definition, it follows from Proposition 8.1 that $\Gamma' < S' \times 1 \times D$ projects onto a lattice in $S' \times D$. On the other hand, the very definition of $S''$ implies $\text{proj}_{S''}(\Gamma)$ contains $\text{proj}_{S''}(\Gamma_U)$, and hence also $\text{proj}_{S''}(\Gamma'')$, as a finite index subgroup. In particular, this shows that $\Gamma' \times \Gamma''$ is a lattice in $S' \times S'' \times D$.

Since it is contained in the lattice $\Gamma$, we finally deduce that the index of $\Gamma' \times \Gamma''$ in $\Gamma$ is finite.

We observe that we have in particular obtained a lattice $\Gamma'' < S''$ with $S''$ non-simple and $\Gamma''$ irreducible (unless both $\Gamma''$ and $S''$ are trivial), because the projection of $\Gamma$ to any simple Lie group factor is dense: indeed, any simple factor must be a factor of some $G_i$ and $\Gamma$ projects densely on $G_i$. It follows from Margulis’ arithmeticity theorem [Mar91, Theorem 1.(1')] that $\Gamma''$ is an arithmetic lattice in $S''$.

Turning to the other lattice, we remark that $\Gamma'$ admits a faithful Zariski-dense representation in a semi-simple group, obtained by reducing the given representation of $\Gamma$. Furthermore, notice that the projection of $\Gamma$ to $S'$ coincides (virtually) with the projection of $\Gamma$. In particular it has dense image. Therefore, setting $D' = \text{proj}_D(\Gamma')$, we may now view $\Gamma'$ as an irreducible lattice in $S' \times D'$. We may thus apply Theorem 10.17. Notice that the same argument as before shows that $D'$ has trivial amenable radical.
We claim that the canonical discrete kernel \( \Gamma'_{D'} \) is in fact a direct factor of \( D' \).

Indeed, since \( \Gamma \) is residually finite by Malcev’s theorem [Mal40], Proposition 8.24 ensures that \( \Gamma'_{D'} \) centralises the discrete residual \( D'^{(\infty)} \). In particular \( D'^{(\infty)} \cap \Gamma'_{D'} = 1 \) since \( D' \) has trivial amenable radical. Furthermore, since \( D'/\Gamma'_{D'} \) is a semi-simple group, its discrete residual has finite index. In particular, upon replacing \( D' \) by a finite index subgroup we have \( D' \cong D'^{(\infty)} \times \Gamma'_{D'} \) as desired. It also follows that \( \Gamma'_{D'} \) itself admits a Zariski-dense representation in a semi-simple group.

It remains to consider again the projection maps \( \text{proj}_{D_i} : D \to D_i \). Restricting these maps to \( D' \) and using the fact that \( \text{proj}_{D_i}(D') \) is dense, we obtain that \( D_i \cong \text{proj}_{D_i}(D'^{(\infty)}) \times D'_i \), where \( D'_i = \text{proj}_{D_i}(\Gamma'_{D'}) \). The final conclusion follows by applying Corollary 10.20 to the irreducible lattice \( \Gamma'_{D'} < D'_1 \times D'_2 \).

**Proof of Corollary 10.3.** Since \( \Gamma \) is finitely generated and irreducible, all \( G_i \) are compactly generated (alternatively, apply Lemma 7.12).

We claim that all projections \( \Gamma \to G_i \) are injective. Indeed, if not, then \( \text{chain} \) there is \( j \) such that the canonical discrete kernel \( \Gamma_{G_j} \) is non-trivial. It is then cocompact, which implies that the projection

\[
G_j/\Gamma_{G_j} \times \prod_{i \neq j} G_i \to \prod_{i \neq j} G_i
\]

is proper. This is a contradiction the fact that the image of \( \Gamma \) in the left hand side above is discrete whilst it is dense in the right hand side, proving the claim.

Suppose given a linear representation of \( \Gamma \) in characteristic \( \neq 2,3 \) whose image is not virtually soluble. Arguing as in [Mon05], we can reduce to the case where we have a Zariski-dense representation \( \tau : \Gamma \to H(K) \) in a non-trivial connected adjoint absolutely simple group \( H \) over a finitely generated field \( K \). Since \( \tau(\Gamma) \) is infinite, we can choose a completion \( k \) of \( K \) for which \( \tau(\Gamma) \) is unbounded [BG07, 2.1].

Part of the argument in [Mon05] is devoted to proving that the representation is a posteriori faithful. One can adapt the entire proof to the present setting, but we propose an alternative line of reasoning using an amenability theorem from [BS06]. Suppose towards a contradiction that the kernel \( \Gamma_0 \triangleleft \Gamma \) of \( \tau \) is non-trivial. Since the projections are injective, the closure \( N_i \) of the image of \( \Gamma_0 \) in \( G_i \) is a non-trivial closed subgroup, which is normal by irreducibility and hence is cocompact. Then Theorem 1.3 in [BS06] implies that \( \Gamma/\Gamma_0 \) is amenable, contradicting the fact that \( \tau(\Gamma) \) is not virtually soluble in view of Tits’ alternative [Tit72].

At this point we can conclude by Theorem 10.1. \( \square \)

### 11. Geometric arithmeticity

11.A. **CAT(0) lattices and parabolic isometries.** We now specialise the various arithmeticity results of Section 10 to the case of lattices in CAT(0) spaces and combine them with some of our geometric results.

Recall that a parabolic isometry is called neutral if it has zero translation length; the following contains Theorem 1.25 from the Introduction.

**Theorem 11.1.** Let \( X \) be a proper CAT(0) space with cocompact isometry group and \( \Gamma \triangleleft G := \text{Is}(X) \) be a finitely generated lattice. Assume that \( \Gamma \) is irreducible and that \( G \) contains a neutral parabolic isometry. Then one of the following assertions holds:

(i) \( G \) is a non-compact simple Lie group of rank one with trivial centre.
(ii) There is a subgroup $\Gamma_D \subseteq \Gamma$ normalised by $G$, which is either finite or infinitely generated and such that the quotient $\Gamma/\Gamma_D$ is an arithmetic lattice in a product of semi-simple Lie and algebraic groups.

**Proof.** Let $X' \subseteq X$ be the canonical subspace provided by Theorem 8.11; notice that $X'$ still admits a neutral parabolic isometry. Theorem 1.1 and its addendum now apply to $X'$. The space $X'$ has no Euclidean factor: indeed, otherwise Theorem 8.8 would imply $X' = \mathbb{R}$, which has no parabolic isometries. The kernel of the $\Gamma$-action on $X'$ is finite and we will include it in the subgroup $\Gamma_D$ below.

We distinguish two cases according as $X'$ has one or more factors.

In the first case, $\text{Is}(X')$ cannot be totally disconnected since otherwise Corollary 5.3(i) rules out neutral parabolic isometries. Thus $\text{Is}(X')$ is a non-compact simple Lie group with trivial centre. If its real rank is one, we are in case (i); otherwise, $\Gamma$ is arithmetic by Margulis’ arithmeticity theorem [Mar91, Theorem 1.1(1’)] and we are in case (ii).

For the rest of the proof we treat the case of several factors for $X'$; let $\Gamma^*$ and let $H_{\Gamma^*}$ be as in Section 8.E. Note that $H_{\Gamma^*}$ acts cocompactly on each irreducible factor of $X'$. Furthermore, each irreducible factor of $H_{\Gamma^*}$ is non-discrete by Theorem 8.17. Therefore $H_{\Gamma^*}$ is a product of the form $S \times D$ (possibly with one trivial factor), where $S$ is a semi-simple Lie group with trivial centre and $D$ is a compactly generated totally disconnected group without discrete factor.

By Corollary 5.3(i), the existence of a neutral parabolic isometry in $G$ implies that $\text{Is}(X')$ is not totally disconnected. Lemma 8.23 ensures that the identity component of $\text{Is}(X')$ is in fact contained in $H_{\Gamma^*}$. Therefore, upon passing to a finite index subgroup, the identity component of $\text{Is}(X')$ coincides with $S$.

If $D$ is trivial, then $H_{\Gamma^*} = S$ is a connected semi-simple Lie group containing $\Gamma$ as an irreducible lattice. Since $S$ is non-simple, it has higher rank and we may appeal again to Margulis’ arithmeticity theorem; thus we are done in this case. Otherwise, $D$ is non-trivial and we may then apply Theorem 10.17. It remains to check that the normal subgroup $\Gamma_D < \Gamma$, if non-trivial, is not finitely generated. But we know that $\Gamma_D$ is a discrete normal subgroup of $D$. By Theorem 7.4, the lattice $\Gamma$, and hence also $H_{\Gamma^*}$, acts minimally without fixed point at infinity on each irreducible factor of $X'$. Therefore, Corollary 4.8 ensures that $D$ has no finitely generated discrete normal subgroup, as desired. $\square$

Here is another variation, of a more geometric flavour; this time, it is not required that there be a neutral parabolic isometry:

**Theorem 11.2.** Let $X$ be a proper geodesically complete $\text{CAT}(0)$ space with cocompact isometry group and $\Gamma < \text{Is}(X)$ be a finitely generated lattice. Assume that $\Gamma$ is irreducible and residually finite.

If $G := \text{Is}(X)$ contains any parabolic isometry, then $X$ is a product of symmetric spaces and Bruhat-Tits buildings. In particular, $\Gamma$ is an arithmetic lattice unless $X$ is a real or complex hyperbolic space.

**Proof.** We maintain the notation of the previous proof and follow the same arguments. We do not know a priori whether there exists a neutral parabolic isometry. However, under the present assumption that $X$ is geodesically complete, Corollary 5.3(iii) shows that the existence of any parabolic isometry is enough to ensure that $\text{Is}(X')$ is not totally disconnected. Thus the conclusion of Theorem 11.1 holds. In case (i), Theorem 6.4(iii) ensures that $X$ is a rank one symmetric space and we are done. We now assume that (ii) holds and define $D$ as in the proof of Theorem 11.1.
The canonical discrete kernel $\Gamma_D$ is trivial by Theorem 8.26. Since $D$ has no non-trivial compact normal subgroup by Corollary 4.8, it follows from Theorem 10.17 that $D$ is a totally disconnected semi-simple algebraic group. Therefore, the desired result is a consequence of Theorem 6.4(iii).

For the record, we propose a variant of Theorem 11.2:

**Theorem 11.3.** Let $X$ be a proper geodesically complete CAT(0) space with cocompact isometry group and $\Gamma < \text{Is}(X)$ be a finitely generated lattice. Assume that $\Gamma$ is irreducible and that every normal subgroup of $\Gamma$ is finitely generated.

If $G := \text{Is}(X)$ contains any parabolic isometry, then $X$ is a product of symmetric spaces and Bruhat-Tits buildings of total rank $\geq 2$. In particular, $\Gamma$ is an arithmetic lattice.

**Proof.** As for Theorem 11.2, we can apply Theorem 11.1. We claim that case (i) is ruled out under the current assumptions. Indeed, a lattice in a simple Lie group of rank one is relatively hyperbolic (see [Far98] or [Osi06]) and as such has numerous infinitely generated normal subgroups (and is even SQ-universal, see [Gro87] or [Del96] for the hyperbolic case and [AMO07] for the general relative case). In case (ii) the discrete kernel $\Gamma_D$ is trivial and rank one is excluded as in case (i) if the group is Archimedean; if it is non-Archimedean, then there are no non-uniform finitely generated lattices (see [BL01]) and thus $\Gamma$ is again Gromov-hyperbolic which contradicts the assumption on normal, subgroups as before.

We can now complete the proof of some results stated in the Introduction.

**Proof of Theorem 1.23.** If $\Gamma$ is residually finite, then Theorem 11.2 yields the desired conclusion; it therefore remains to consider the case where $\Gamma$ is not residually finite. We follow the beginning of the proof of Theorem 11.2 until the invocation of Theorem 8.26, since the latter no longer applies. However, we still know that there is a non-trivial Lie factor in $\text{Is}(X')$ and therefore we apply Theorem 10.17 in order to obtain the desired conclusion about the lattice $\Gamma$. As for the symmetric space factor of the space, it is provided by Theorem 6.4(iii).

**Proof of Corollary 1.24.** One implication is given by Theorem 1.23. For the converse, it suffices to recall that unipotent elements exist in all semi-simple Lie groups of positive real rank.

11.B. **Arithmeticity of linear CAT(0) lattices.** We start by considering CAT(0) lattices with a linear non-discrete linear commensurator:

**Theorem 11.4.** Let $X$ be a proper geodesically complete CAT(0) space with cocompact isometry group and $\Gamma < \text{Is}(X)$ be a finitely generated lattice. Assume that $\text{Is}(X)$ possesses a finitely generated subgroup $\Lambda$ containing $\Gamma$ as a subgroup of infinite index, and commensurating $\Gamma$.

If $X$ is irreducible and $\Lambda$ possesses a faithful finite-dimensional linear representation (in characteristic $\neq 2, 3$), then $X$ is a symmetric space or a Bruhat-Tits building; in particular $\Gamma$ is an arithmetic lattice.

**Remark 11.5.** Several examples of irreducible CAT(0) spaces $X$ of dimension $> 1$ admitting a discrete cocompact group of isometries with a non-discrete commensurator in $\text{Is}(X)$ have been constructed by F. Haglund [Hag98] and A. Thomas [Tho06] (see also [Hag, Théorème A] and [BT]). In all cases that space $X$ is endowed with walls; in particular $X$ is the union of two proper closed convex subspaces. This implies in particular that $X$ is not a Euclidean building. Therefore, Theorem 11.4 has the following consequence: in the aforementioned examples of Haglund and Thomas, either the commensurator of the lattice
is nonlinear, or it is the union of a tower of lattices. In fact, as communicated to us by F. Haglund, for most of these lattices the commensurator contains elliptic elements of infinite order; this implies right away that the commensurator is not an ascending union of lattices and, hence, it is nonlinear. Note on the other hand that it is already known that $\text{Is}(X)$ is mostly nonlinear in these examples, since it contains closed subgroups isomorphic to the full automorphism group of regular trees.

**Proof of Theorem 11.4.** Since $X$ is irreducible and the case $X = \mathbb{R}$ satisfies the conclusions of the theorem, we assume henceforth that $X$ has no Euclidean factor.

The $\text{Is}(X)$-action on $X$ is minimal by Lemma 2.13 and has no fixed point at infinity by Corollary 8.12. In particular, we can apply Theorem 1.11: either $\text{Is}(X)$ is totally disconnected or it is simple Lie group with trivial centre and $X$ is the associated symmetric space. In the latter case, Margulis' arithmeticity theorem finishes the proof. We assume henceforth that $\text{Is}(X)$ is totally disconnected.

Let $G$ denote the closure of $\Lambda$ in $\text{Is}(X)$. Note that $G$ acts minimally without fixed point at infinity, since it contains a subgroup, namely $\Gamma$, which possesses these properties by Theorem 7.4. In particular $G$ has trivial amenable radical by Theorem 1.6 and thus the same holds for the dense subgroup $\Lambda < G$. In particular any faithful representation of $\Lambda$ to an algebraic group yields a faithful representation of $\Lambda$ to an adjoint semi-simple algebraic group with Zariski-dense image, to which we can apply Theorem 10.16. As we saw, the group $G$ has no non-trivial compact (in fact amenable) normal subgroup and furthermore $G$ is irreducible since $X$ is so, see Theorem 1.6. The fact that the lattice $\Gamma$ has infinite index in $\Lambda$ rules out the discrete case. Therefore $G$ is a simple algebraic group and $\Gamma$ an arithmetic lattice.

It remains to deduce that $X$ has the desired geometric shape. This will follow from Theorem 6.4(iii) provided we show that $\partial X$ is finite-dimensional and that $G$ has full limit set. The first fact holds since $X$ is cocompact; the second is provided by Corollary 7.10. □

Remark 11.5 illustrates that Theorem 11.4 fails dramatically if one assumes only that $\Gamma$ is linear. However, passing now to the case where $X$ is reducible, the linearity of $\Gamma$ is enough to establish arithmeticity, independently of any assumption on commensurators, the result announced in Theorem 1.26 in the Introduction.

**Theorem 11.6.** Let $X$ be a proper geodesically complete CAT(0) space with cocompact isometry group and $\Gamma < \text{Is}(X)$ be a finitely generated lattice. Assume that $\Gamma$ is irreducible and possesses some faithful linear representation (in characteristic $\neq 2, 3$).

If $X$ is reducible, then $\Gamma$ is an arithmetic lattice and $X$ is a product of symmetric spaces and Bruhat-Tits buildings.

**Proof.** In view of Theorem 8.8, we can assume that $X$ has no Euclidean factor. The $\text{Is}(X)$-action on $X$ is minimal by Lemma 2.13 and has no fixed point at infinity by Corollary 8.12. In particular, we can apply Theorem 1.11 to obtain decompositions of $\text{Is}(X)$ and $X$ in which the factors of $X$ corresponding to connected factors of $\text{Is}(X)$ are isometric to symmetric spaces. There is no loss of generality in assuming $\Gamma^* = \Gamma$ in the notation of Section 8.E. Let now $G$ be the hull of $\Gamma$. By Remark 8.21, the group $\Gamma$ is an irreducible lattice in $G$.

Since $\text{Is}(X)$ acts minimally without fixed point at infinity, it follows from Corollary 7.7 that $\Gamma$ has trivial amenable radical. In particular any faithful representation of $\Gamma$ to an algebraic group yields a faithful representation of $\Gamma$ to an adjoint semi-simple algebraic group with Zariski-dense image, to which we can apply Theorem 10.1.
The group $G$ has no non-trivial compact normal subgroup e.g. by minimality. Furthermore the discrete factor is trivial by Theorem 8.17. Therefore $G$ is a simple algebraic group and $\Gamma$ an arithmetic lattice.

It remains to deduce that $X$ has the desired geometric shape and this follows exactly as in the proof of Theorem 11.4.

11.C. A family of examples. We shall now construct a family of lattices $\Gamma < G = S \times D$ as in the statement of Theorem 10.17 (see also Theorem 11.2) with the following additional properties:

(i) There is a proper $\text{CAT}(0)$ space $Y$ with $D < \text{Is}(Y)$ such that the $D$-action is cocompact, minimal and without fixed point at infinity. In particular, setting $X = X_S \times Y$, where $X_S$ denotes the symmetric space associated to $S$, the $\Gamma$-action on $X$ is properly discontinuous (in fact free), cocompact, minimal, without fixed point at infinity.

(ii) The canonical discrete kernel $\Gamma_D \triangleleft D$ is infinite (in fact, it is a free group of countable rank).

(iii) The profinite kernel of $D/\Gamma_D \to Q$ is non-trivial.

Remark 11.7. Since $D$ is minimal, it has no compact normal subgroup and thus we see that the profinite extension appearing in Theorem 10.17 cannot be eliminated.

We begin with a general construction:

Let $g$ be (the geometric realisation of) a locally finite graph (not reduced to a single point) and let $Q < \text{Is}(g)$ a closed subgroup whose action is vertex-transitive. In particular, $Q$ is a compactly generated totally disconnected locally compact group. We point out that any compactly generated totally disconnected locally compact group can be realised as acting on such a graph by considering Schreier graphs $g$, see [Mon01, §11.3]; the kernel of this action is compact and arbitrary small. On the other hand, if $Q$ is a non-Archimedean semi-simple group, one can also take very explicit graphs drawn on the Bruhat–Tits building of $Q$, e.g. the 1-skeleton (this part is inspired by [BM00a, 1.8], see also [BMZ04]).

Let moreover $C$ be an infinite profinite group and choose a locally finite rooted tree $t$ with a level-transitive $C$-action for which every infinite ray has trivial stabiliser. For instance, one can choose the coset tree associated to any nested sequence of open subgroups with trivial intersection, see the proof of Théorème 15 in §6 on p. 82 in [Ser77]. We define a locally finite graph $\mathfrak{h}$ with a $C \times Q$-action as the 1-skeleton of the square complex $t \times g$. Let $a = \mathfrak{h}$ be the universal cover of $\mathfrak{h}$, $\Lambda = \pi_1(\mathfrak{h})$ and define the totally disconnected locally compact group $D$ by the corresponding extension

$$1 \longrightarrow \Lambda \longrightarrow D \longrightarrow C \times Q \longrightarrow 1.$$ 

Proposition 11.8. There exists a proper $\text{CAT}(0)$ space $Y$ such that $D$ sits in $\text{Is}(Y)$ as a closed subgroup whose action is cocompact, minimal and without fixed point at infinity.

Proof. One verifies readily the following:

Lemma 11.9. Let $a$ be (the geometric realisation of) a locally finite simplicial tree and $D < \text{Is}(a)$ any subgroup. Let $x \in a$ be a vertex and let $Y$ be the completion of the metric space obtained by assigning to each edge of $a$ the length $2^{-r}$, where $r$ is the combinatorial distance from this edge to the nearest point of the orbit $D.x$. 

Then \( Y \) is a proper CAT(0) space with a cocompact continuous isometric \( D \)-action. Moreover, if the \( D \)-action on \( a \) was minimal or without fixed point at infinity, then the corresponding statement holds for the \( D \)-action on \( Y \).

Apply the lemma to the tree \( a = h \) considered earlier. We claim that the \( D \)-action on \( a \) is minimal. Clearly it suffices to show that the \( \Lambda \)-action is minimal. Note that \( \Lambda \) acts transitively on each fibre of \( p : h \to h \). Thus it is enough to show that the convex hull of a given fibre meets every other fibre. Consider two distinct vertices \( v, v' \in h \). The product nature of \( h \) makes it clear that \( v \) and \( v' \) are both contained in a common minimal loop based at \( v \). This loop lifts to a geodesic line in \( h \) which meets the respective fibres of \( v \) and \( v' \) alternately and periodically. In particular, this construction yields a geodesic segment joining two points in the fibre above \( v \) and containing a point sitting above \( v' \), whence the claim.

Since \( \Lambda \) acts freely and minimally on the tree \( a \) which is not reduced to a line, it follows that \( \Lambda \) fixes no end of \( a \). Thus the lemma provides a proper CAT(0) space \( Y \) with a cocompact minimal isometric \( D \)-action, without fixed point at infinity. It remains to show that \( D < \text{Is}(Y) \) is closed. This holds because the totally disconnected groups \( \text{Is}(a) \) and \( \text{Is}(Y) \) are isomorphic; indeed, the canonical map \( a \to Y \) induces a continuous surjective homomorphism \( \text{Is}(a) \to \text{Is}(Y) \), which is thus open.

**Remark 11.10.** The above construction gives an example of a proper CAT(0) space with a totally disconnected cocompact and minimal group of isometries such that not all point stabilisers are open. Consider indeed the points added when completing. Their stabilisers map to \( Q \) under \( D \to (C \times Q) \) and hence cannot be open. In other words, the action is not smooth in the terminology of [Cap07]. Notice however that the set of points with open stabiliser is necessarily a dense convex invariant set.

We shall now specialise this general construction to yield our family of examples. Let \( K, H, \Sigma, K(\Sigma) \) be as described after Theorem 10.1 on p. 60. We write \( \Sigma_f, \Sigma_\infty \subseteq \Sigma \) for the subsets of finite/infinite places and assume that both are non-empty. Let \( S = \prod_{v \in \Sigma_f} H(K_v)^+ \) and \( Q = \prod_{v \in \Sigma_\infty} H(K_v)^+ \). The group \( \Delta = H(K(\Sigma)) \cap (S \times Q) \) is an irreducible cocompact lattice in \( S \times Q \). Let \( C \) be any profinite group with a dense inclusion \( \Delta \to C \). We now embed \( \Delta \) diagonally in \( S \times C \times Q \); clearly \( \Delta \) is a cocompact lattice. Let \( \Gamma < G = S \times D \) be its pre-image. Then \( \Gamma \) is a cocompact lattice since it contains the discrete kernel of the canonical map \( G \to S \times C \times Q \). It is clearly irreducible and therefore provides an example that the structure of the description in the conclusion of Theorem 10.17 cannot be simplified. Furthermore, the normal subgroup appearing in Theorem 11.1(ii) is also unavoidable.

We end this section with a few supplementary remarks on the preceding construction:

(i) If the profinite group \( C \) has no discrete normal subgroup, then \( \Gamma_D = \pi_1(h) \) coincides with the quasi-centre of \( D \). This would be the case for example if \( C = H(K_v) \) and \( H \) is almost \( K \)-simple of higher rank, where \( v \) is a non-Archimedean valuation such that \( H \) is \( K_v \)-anisotropic. In particular, in that situation \( \Gamma_D \) is the unique maximal discrete normal subgroup of \( D \) and the quotient \( D/\Gamma_D \) has a unique maximal compact normal subgroup. Thus the group \( G \) admits a unique decomposition as in Theorem 10.17 in this case.

(ii) We emphasise that, even though \( D/\Gamma_D \) decomposes as a direct product \( C \times Q \) in the above construction, the group \( D \) admits no non-trivial direct product decomposition, since it acts minimally without fixed point at infinity on a tree (see Theorem 1.6).
(iii) The fact that $D/\Gamma_D$ decomposes as a direct product $C \times Q$ is not a coincidence. In fact, this happens always provided that every cocompact lattice in $S$ has the Congruence Subgroup Property (CSP). Indeed, given a compact open subgroup $U$ of $D/\Gamma_D$, the intersection $\Gamma_U$ of $\Gamma \cap (S \times U)$ is an irreducible lattice in $S \times U$ with trivial canonical discrete kernels. By (CSP), upon replacing $\Gamma_U$ by a finite index subgroup (which amounts to replace $U$ by an open subgroup), the profinite completion $\widehat{\Gamma_U}$ splits as the product over all primes $p$ of the pro-$p$ completions $(\widehat{\Gamma_U})_p$, which are just-infinite. Thus the canonical surjective map $\widehat{\Gamma_U} \to U$ shows that $U$ is a direct product. This implies that the maximal compact normal subgroup of $D/\Gamma_D$ is a direct factor.

(iv) According to a conjecture of Serre's (footnote on page 489 in [Ser70]), if $S$ has higher rank then every irreducible lattice in $S$ has (CSP). (See [Rag04] for a recent survey on this conjecture.)

12. Some open questions

We conclude by collecting some perhaps noteworthy questions that we have encountered while working on this paper, but that we have not been able to answer.

It is well known that the Tits boundary of a proper CAT(0) space with cocompact isometry group is necessarily finite-dimensional (see [Kle99, Theorem C]). It is quite possible that the same conclusion holds under a much weaker assumption.

**Question 12.1.** Let $X$ be a proper CAT(0) space such that Is($X$) has full limit set. Is the boundary $\partial X$ finite-dimensional?

A positive answer to this question would show in particular that the second set of assumptions — denoted (b) — in Theorem 6.4 is in fact redundant.

Let $G$ be a simple Lie group acting continuously by isometries on a proper CAT(0) space $X$. The Karpelevich–Mostow theorem ensures that there exists a convex orbit when $X$ is a symmetric space of non-compact type. This statement, however, cannot be generalised to arbitrary $X$ in view of Example 6.7.

**Question 12.2.** Let $G$ be a simple Lie group acting continuously by isometries on a proper CAT(0) space $X$. If the action is cocompact, does there exist a convex orbit?

It is shown in Theorem 6.4(iii) that if $X$ is geodesically complete, then the answer is positive. It is good to keep in mind Example 6.6, which shows that the natural analogue of this question for a simple algebraic group over a non-Archimedean local field has a negative answer. More optimistically, one can ask for a convex orbit whenever the simple Lie group acts on a complete (not necessarily proper) CAT(0) space, but assuming the action non-evanescent (in the sense of [Mon06]). A positive answer would imply superrigidity statements upon applying it to spaces of equivariant maps.

Many of our statements on CAT(0) lattices require the assumption of finite generation. One should of course wonder for each of them whether it remains valid without this assumption. One instance where this question is especially striking is the following (see Corollary 8.12).

**Question 12.3.** Let $X$ be a proper CAT(0) space which is minimal and cocompact. Assume that Is($X$) contains a lattice. Is it true that Is($X$) has no fixed point at infinity?
Notice that $\text{Is}(X)$ is compactly generated (see Lemma 8.3). In particular, if $\text{Is}(X)$ is discrete, then it is finitely generated and is itself a lattice. Therefore, in view of Corollary 8.12, one can assume that $\text{Is}(X)$ is non-discrete in Question 12.3.

We have seen in Corollary 6.12 that if the isometry group of a proper CAT(0) space $X$ is non-discrete in a strong sense, then $\text{Is}(X)$ comes close to being a direct product of topologically simple groups.

**Question 12.4.** Retain the assumptions of Corollary 6.12. Is it true that $\text{soc}(G^*)$ is a product of simple groups? Is it cocompact in $G$, or at least does $G$ have compact Abelianisation?

Clearly Corollary 6.12 reduces the question to the case where $\text{Is}(X)$ is totally disconnected. One can also ask if $\text{soc}(G^*)$ is compactly generated (which is the case e.g. if it is cocompact in $G$). If so, we obtain additional information by applying Proposition 5.11.

In the above situation one furthermore expects that the geometry of $X$ is encoded in the structure of $\text{Is}(X)$. In precise terms, we propose the following.

**Question 12.5.** Retain the assumptions of Corollary 6.12. Is it true that any proper cocompact action of $G$ on a proper CAT(0) space $Y$ yields an equivariant isometry $\partial X \to \partial Y$ between the Tits boundaries? Or an equivariant homeomorphism between the boundaries with respect to the cône topology?

The discussion around Corollary 10.3 (cf. Remark 10.4) suggests the following.

**Question 12.6.** Let $\Gamma < G = G_1 \times \cdots \times G_n$ be an irreducible finitely generated lattice, where each $G_i$ is a locally compact group. Does every character $\Gamma \to \mathbb{R}$ extend continuously to $G$?

Y. Shalom [Sha00] proved that this is the case when $\Gamma$ is cocompact and in some other situations.

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IHÉS, Route de Chartres 35, 91440 Bures-sur-Yvette, France
E-mail address: caprace@ihes.fr

EPFL, 1015 Lausanne, Switzerland
E-mail address: nicolas.monod@epfl.ch