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BOUNDS FOR COHOMOLOGY CLASSES

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Let G be a simple Lie group (connected and with finite centre). Consider the continuous cohomology $\mathrm{H}^*(G, \mathbf{R})$ of G, which can be defined for instance with the familiar bar-resolutions of the Eilenberg–MacLane cohomology, except that the cochains are required to be *continuous* maps on G (or equivalently smooth or just measurable).

CONJECTURE 18.1. *Every cohomology class of* $\mathrm{H}^*(G, \mathbf{R})$ *is* bounded, *i.e. is represented by a bounded cocycle.*

Recall that $\mathrm{H}^*(G, \mathbf{R})$ is isomorphic to the algebra of invariant differential forms on the symmetric space associated to G, hence to a relative cohomology of Lie algebras and thus moreover to the cohomology of the *compact dual space* associated to G. It is however not understood how these isomorphisms interact with boundedness of cochains (compare Dupont [6]).

We emphasise also that, unlike for discrete groups, $\mathrm{H}^*(G, \mathbf{R})$ does not coincide with the cohomology of the classifying space BG. There is however a natural transformation $\mathrm{H}^*(BG, \mathbf{R}) \to \mathrm{H}^*(G, \mathbf{R})$ and we refer to its image as the *primary characteristic classes*. By a difficult result of M. Gromov [7], the latter are indeed bounded; M. Bucher-Karlsson gave a simpler proof of this fact in her thesis [1].

In order to prove the above conjecture, it would suffice to establish the boundedness of the *secondary invariants* of Cheeger–Simons; indeed, Dupont–Kamber proved that the latter together with the primary classes generate $\mathrm{H}^*(G, \mathbf{R})$ as an algebra.

An important example where boundedness was established very recently is the class of the *volume form* of the associated symmetric space. Using estimates by Connell–Farb [5], Lafont–Schmidt [8] provided bounded cocycles in all cases except $SL_3(\mathbf{R})$, the latter case being settled by M. Bucher-Karlsson [2] (a previous proof of R. Savage [11] is incorrect). It follows that the fundamental class of closed locally symmetric spaces is bounded; as explained by M. Gromov, this provides a non-zero lower bound for the *minimal volume* of such a manifold, i.e. a non-trivial lower bound for its volume with respect to *any* (suitably normalised) Riemannian metric.

Many more questions are related to the above conjecture via the following steps listed in increasing order of refinement: (i) find a bounded cocycle representing a given class; (ii) establish a sharp numerical bound for that class; (iii) determine the equivalence class of the cocycle up to boundaries of bounded cochains only.

The latter point leads one to introduce the (continuous) *bounded cohomology* H^*_b of groups or spaces, where all cochains are required to be bounded. There is then an obvious natural transformation

$$\mathrm{H}^*_\mathrm{b}(-,\mathbf{R}) \longrightarrow \mathrm{H}^*(-,\mathbf{R}) \tag{$*$}$$

and the above conjecture amounts to the surjectivity of that map for a connected simple Lie group with finite centre. As of now, there is not a single simple Lie group for which $\mathrm{H}^*_\mathrm{b}(G,\mathbf{R})$ is known; all the partial results are however consistent with a positive answer to the following:

QUESTION 18.2. *Is the map* $(*)$ *an isomorphism?*

For instance, the answer is yes in degree two [3] (and trivially yes in degrees $0,1$); for $G = SL_n(\mathbf{R})$, it is also yes in degree three (see [4] for $n=2$ and [10] for $n \geq 3$).

The functor H^*_b is quite interesting for discrete groups as well and has found applications notably to representation theory, dynamics, geometry and ergodic theory. This notwithstanding, *there is not a single countable group for which* $\mathrm{H}^*_\mathrm{b}(-,\mathbf{R})$ *is known*, unless it is known to vanish in all degrees (e.g. for amenable groups). In any case, the map $(*)$ fails dramatically either to be injective or surjective in many examples. Most known results regard the degree two, with for instance a large supply of groups having an infinite-dimensional $\mathrm{H}^2_\mathrm{b}(-,\mathbf{R})$, including the non-Abelian free group F_2. Interestingly, the surjectivity of the map $(*)$ (with more general coefficients) in degree two *characterises non-elementary Gromov-hyperbolic groups* (Mineyev [9]).

It appears that new techniques are required in higher degrees. Here is a test on which to try them:

QUESTION 18.3. *For which degrees n is* $\mathrm{H}^n_{\mathrm{b}}(F_2, \mathbf{R})$ *non-trivial ?*

It is known to be non-trivial for $n = 2, 3$. (Triviality for $n = 1$ and non-triviality for $n = 0$ are elementary to check for any group.)

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