THE DIXMIER PROBLEM, LAMPLIGHTERS AND BURNSIDE GROUPS

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Abstract. J. Dixmier asked in 1950 whether every non-amenable group admits uniformly bounded representations that cannot be unitarised. We provide such representations upon passing to extensions by abelian groups. This gives a new characterisation of amenability. Furthermore, we deduce that certain Burnside groups are non-unitarisable, answering a question raised by G. Pisier.

1. Introduction

A group G is said to be *unitarisable* if every uniformly bounded representation π of G on a Hilbert space $\mathscr{H}$ is unitarisable, *i.e.* there is an invertible operator S on $\mathscr{H}$ such that $S\pi(\,\cdot\,)S^{-1}$ is a unitary representation. Dixmier [Dix50] proved that all amenable groups are unitarisable and asked whether unitarisability characterises amenability. Since unitarisability passes to subgroups and non-commutative free groups are not unitarisable, every group containing a non-commutative free group is non-unitarisable. For these facts and more background, we refer to Pisier [Pis01, Pis05].

Recently, a criterion was discovered [EMxx] that lead to examples without free subgroups (see [Osixx, EMxx]). We shall improve a strategy proposed in [Mon06] in order to apply ergodic methods to the problem.

Now are our browes bound with Victorious Wreathes[1]

Let G and A be groups. Recall that the associated (restricted) *wreath product*, or *lamplighter group*, is the group

$$A \wr G \;=\; \bigoplus_G A \rtimes G,$$

wherein $\bigoplus_G A$ is the restricted product indexed by G upon which G acts by permutation. We shall be interested in the case where A and hence also $\bigoplus_G A$ is abelian.

Theorem 1. *For any group G, the following assertions are equivalent.*

(i) *The group G is amenable.*
(ii) *The wreath product $A \wr G$ is unitarisable for all abelian groups A.*
(iii) *The wreath product $A \wr G$ is unitarisable for some infinite abelian group A.*

The above theorem leads to a partial answer to a question of G. Pisier, namely whether free Burnside groups are unitarisable (see *e.g.* [Pis05]).

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[1] Shakespeare, *Richard III*, 1:1 (we quote from the 1623 *First Folio*).

Theorem 2. *Let m, n, p be integers with $m, n \geq 2$, $p \geq 665$ and n, p odd. Then the free Burnside group $B(m, np)$ of exponent np with m generators is non-unitarisable.*

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2. Proofs

Let G be a group and $(\pi, \mathscr{H})$ be a unitary representation of G. We write $\mathscr{L}(\mathscr{H})$ for the algebra of bounded operators of $\mathscr{H}$. A map $D\colon G \to \mathscr{L}(\mathscr{H})$ is called a *derivation* if it satisfies the Leibniz rule $D(gh) = D(g)\pi(h) + \pi(g)D(h)$, or equivalently if the map π_D defined by

$$\pi_D(g) = \begin{pmatrix} \pi(g) & D(g) \\ 0 & \pi(g) \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$$

is a group homomorphism. In that case, π_D is a uniformly bounded representation if and only if D is a bounded derivation. Moreover, π_D is unitarisable if and only if D is inner, *i.e.* there is $T \in \mathscr{L}(\mathscr{H})$ such that $D(g) = \pi(g)T - T\pi(g)$. (See Lemma 4.5 in [Pis01] for a proof of this fact.) To set up a cohomological framework for studying this problem, we will view $\mathscr{L}(\mathscr{H})$ as a coefficient G-module whose G-action is given by the conjugation $g \cdot T = \pi(g)T\pi(g)^*$. Then, the space of bounded derivations modulo inner derivations is canonically isomorphic to the first bounded cohomology group $\mathrm{H}^1_{\mathrm{b}}(G, \mathscr{L}(\mathscr{H}))$. Hence, to prove non-unitarisability of G, it suffices to produce a unitary G-representation $(\pi, \mathscr{H})$ for which $\mathrm{H}^1_{\mathrm{b}}(G, \mathscr{L}(\mathscr{H})) \neq 0$.

We now undertake the proof of Theorem 1. It suffices to show that if A is infinite abelian and G is non-amenable, then the wreath product $H = A \wr G$ is non-unitarisable.

We can and shall assume that A and G are countable. Indeed, since amenability is preserved under direct limits, G contains some countable non-amenable group G_0. Further, if A_0 is a countable subgroup of A, then $A_0 \wr G_0$ is a subgroup of $A \wr G$. Thus our claim follows since unitarisability passes to subgroups.

Let $\mathbf{F}$ be a countable non-commutative free group. The proof relies on the following two facts. (1) $\mathrm{H}^1_{\mathrm{b}}(\mathbf{F}, \mathscr{L}(\ell_2\mathbf{F})) \neq 0$, see the proof of Theorem 2.7* in [Pis01]. (2) Every non-amenable countable group admits a free type II_1 action whose orbits contain the orbits of a free $\mathbf{F}$-action ([GLxx]), as described below. The strategy of the proof is to induce $\mathrm{H}^1_{\mathrm{b}}(\mathbf{F}, \mathscr{L}(\ell_2\mathbf{F}))$ through this "randembedding" in the sense of [Mon06].

We henceforth consider a non-amenable countable group G and the corresponding Bernoulli shift action on the compact metrisable product space $X = [0,1]^G$ endowed with the product of the Lebesgue measures. Gaboriau and Lyons prove in [GLxx] that the resulting equivalence relation $\mathscr{R} \subseteq X \times X$ contains the equivalence relation of some free measure-preserving $\mathbf{F}$-action upon X. In particular, we have commuting G- and $\mathbf{F}$-actions on $\mathscr{R}$ given by the action on the first, respectively the second coordinate. These actions preserve the σ-finite measure on $\mathscr{R}$ provided by integrating over X the counting measure on orbits. Each of these actions admits a fundamental domain; let $Y \subseteq \mathscr{R}$ be a fundamental domain for $\mathbf{F}$. We may now forget the orbit equivalence relation and view $\mathscr{R}$ just as a standard measure space with a measure-preserving $G \times \mathbf{F}$-action such that G admits a fundamental domain X of finite measure and $\mathbf{F}$ admits a fundamental domain Y. We identify $\mathscr{R}$ with $Y \times \mathbf{F}$ in such a way that $t^{-1}y \in \mathscr{R}$ corresponds to $(y,t) \in Y \times \mathbf{F}$. Then, $s \in \mathbf{F}$ acts on $Y \times \mathbf{F}$ by $s(y,t) = (y, ts^{-1})$ and $g \in G$ acts by $g(y,t) = (g \cdot y, \alpha(g,y)t)$, where $g \cdot y \in Y$ is the (essentially) unique element in $\mathbf{F}gy \cap Y \subset \mathscr{R}$ and $\alpha(g,y) \in \mathbf{F}$ is the (essentially) unique element such that $\alpha(g,y)gy = g \cdot y$. It follows that α satisfies the cocycle relation $\alpha(gh,y) = \alpha(g, h\cdot y)\alpha(h,y)$.

We now consider any countable infinite abelian group A. We claim that A has a representation into the unitaries of the von Neumann algebra $L^\infty(Y)$ whose image generates $L^\infty(Y)$ as a von Neumann algebra. By construction, Y is a standard Borel space with a σ-finite non-atomic measure. Furthermore, as far as the present claim is concerned, we may temporarily assume this measure finite since only its measure class is of relevance. Since A is countably infinite, its Pontryagin dual $\widehat{A}$ (for A endowed with the discrete topology) is a non-discrete compact metrisable group. In other words, we have reduced to the case where we may assume that Y is $\widehat{A}$ endowed with a Haar measure. Fourier transform establishes an isomorphism between $L^\infty(\widehat{A})$ and the group von Neumann algebra $L(A) \subseteq \mathscr{L}(\ell_2 A)$, which is by definition generated by the unitary regular representation of A; this proves the claim.

Returning to the main argument, we view A in the unitary group of $L^\infty(Y) \cong L^\infty(Y) \otimes \mathbf{C}1_{\mathbf{F}} \subset L^\infty(\mathscr{R})$. Since A and $gAg^{-1} \subset L^\infty(Y)$ commute, this gives rise to a unitary representation of $H = A \wr G$ on $L^2(\mathscr{R})$. We will prove that $\mathrm{H}^1_{\mathrm{b}}(H, \mathscr{L}(L^2(\mathscr{R}))) \neq 0$.

We write $N = \bigoplus_G A$. Since N is amenable and $\mathscr{L}(L^2(\mathscr{R}))$ is a dual module, a weak-$*$ averaging argument shows that there is a canonical isomorphism

$$\mathrm{H}^*_{\mathrm{b}}(H, \mathscr{L}(L^2(\mathscr{R}))) \;\cong\; \mathrm{H}^*_{\mathrm{b}}(G, \mathscr{L}(L^2(\mathscr{R}))^N)$$

(see Corollary 7.5.10 in [Mon01]). With the identification $\mathscr{R} = Y \times \mathbf{F}$, one has

$$\mathscr{L}(L^2(\mathscr{R}))^N \;=\; N' \cap \mathscr{L}(L^2(\mathscr{R})) \;=\; L^\infty(Y) \,\bar{\otimes}\, \mathscr{L}(\ell_2\mathbf{F}) \;\cong\; L^\infty(Y, \mathscr{L}(\ell_2\mathbf{F}))$$

(see Theorem IV.5.9 in [Tak02]). Keeping track of the G-representation, one sees that $g \in G$ acts on $L^\infty(Y, \mathscr{L}(\ell_2\mathbf{F}))$ by $(g\cdot f)(y) = \tau_{\alpha(g, g^{-1}\cdot y)}(f(g^{-1}\cdot y))$, where τ denotes the $\mathbf{F}$-action on $\mathscr{L}(\ell_2\mathbf{F})$. For ease of notation, we denote the coefficient $\mathbf{F}$-module $\mathscr{L}(\ell_2\mathbf{F})$ by V. Then, one further has a G-isomorphism

$$L^\infty(Y, V) \;\cong\; L^\infty(\mathscr{R}, V)^{\mathbf{F}},$$

where $f \in L^\infty(Y,V)$ corresponds to $\tilde{f} \in L^\infty(\mathscr{R}, V)^{\mathbf{F}}$ defined by $\tilde{f}(y,t) = \tau_t^{-1}(f(y))$. Now, $\mathbf{F}$ acts on $L^\infty(\mathscr{R}, V)$ by $(s\cdot F)(z) = \tau_s(F(s^{-1}z))$ and G acts by $(g\cdot F)(z) = F(g^{-1}z)$. Since both the $\mathbf{F}$-action and the G-action on $\mathscr{R}$ admit a fundamental domain, Proposition 4.6 in [MS06] implies that

$$\mathrm{H}^*_{\mathrm{b}}(G, L^\infty(\mathscr{R}, V)^{\mathbf{F}}) \;\cong\; \mathrm{H}^*_{\mathrm{b}}(\mathbf{F}, L^\infty(\mathscr{R}, V)^G) \;\cong\; \mathrm{H}^*_{\mathrm{b}}(\mathbf{F}, L^\infty(X, V)).$$

(See also Proposition 5.8 in [Mon06].) Since $X = \mathscr{R}/G$ has a finite $\mathbf{F}$-invariant measure, the inclusion $V \hookrightarrow L^\infty(X,V)$ has a G-equivariant left inverse. It follows that the corresponding morphism

$$\mathrm{H}^*_{\mathrm{b}}(\mathbf{F}, V) \longrightarrow \mathrm{H}^*_{\mathrm{b}}(\mathbf{F}, L^\infty(X, V))$$

is an injection. Therefore, putting all identifications together, we conclude that there are injections

$$\mathrm{H}^*_{\mathrm{b}}(\mathbf{F}, \mathscr{L}(\ell_2\mathbf{F})) \longrightarrow \mathrm{H}^*_{\mathrm{b}}(H, \mathscr{L}(L^2(\mathscr{R})))$$

in all degrees. Since $\mathrm{H}^1_{\mathrm{b}}(\mathbf{F}, \mathscr{L}(\ell_2\mathbf{F})) \neq 0$, this completes the proof. □

Analysing the proof at the level of derivations, one observes that the above injection maps $D\colon \mathbf{F} \to \mathscr{L}(\ell_2\mathbf{F})$ to $\tilde{D}\colon H \to \mathscr{L}(L^2(Y, \ell_2\mathbf{F}))$ defined by

$$(\tilde{D}(ag)\xi)(y) = a(y)D(\alpha(g, g^{-1}\cdot y))\xi(g^{-1}\cdot y),$$

where $a \in N$ is viewed as an element of $L^\infty(Y)$, $g \in G$ and $\xi \in L^2(Y, \ell_2\mathbf{F})$.

Proof of Theorem 2. By a theorem of Adyan [Ady82], the free Burnside group $G = B(2, p)$ is non-amenable. Therefore, Theorem 1 implies that $(\bigoplus_{\mathbf{N}} \mathbf{Z}/n\mathbf{Z}) \wr G$ is non-unitarisable. Notice that this wreath product is a countably generated group of exponent np. Therefore, by the universal property of free Burnside groups, it is a quotient of $B(\aleph_0, np)$. In particular, the latter is non-unitarisable. It was shown by Širvanjan [Šir76] that $B(\aleph_0, np)$ embeds into $B(2, np)$ which is therefore also non-unitarisable. Finally, each $B(m, np)$ surjects onto $B(2, np)$ as long as $m \geq 2$, concluding the proof. □

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