APPENDIX:
BOUNDEDLY GENERATED GROUPS WITH PSEUDOCHARACTER(S)

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The aim of this appendix is to construct concrete groups which simultaneously:

(1) are boundedly generated;
(2) have Kazhdan's property (T);
(3) have a one-dimensional space of pseudocharacters.

By (3), such groups don't have property (QFA), whilst they have property (FA) by (2); moreover the quasimorphisms in (3) cannot be *bushy* in the sense of [9]. Property (3) has its own interest, as all previous constructions yield infinite-dimensional spaces. (By taking direct products of our examples, one gets any finite dimension.) The examples will be lattices $\widetilde{\Gamma}$ in non-linear simple Lie groups; more precisely, starting with certain higher rank Lie groups H with $\pi_1(H) = \mathbb{Z}$ and suitable lattices $\Gamma < H$, the group $\widetilde{\Gamma}$ will be the preimage of Γ in the universal covering central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{H} \longrightarrow H \longrightarrow 1. \qquad (*)$$

Let us first start with *any* group Γ satisfying the following cohomological properties (we refer to [3] for our use of bounded cohomology):

($3'$) the second bounded cohomology $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$ has dimension one;
($3''$) the natural map $\psi_\Gamma : \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R}) \to \mathrm{H}^2(\Gamma, \mathbb{R})$ is injective;
($3'''$) the image of the natural map $i_\Gamma : \mathrm{H}^2(\Gamma, \mathbb{Z}) \to \mathrm{H}^2(\Gamma, \mathbb{R})$ spans the image of ψ_Γ.

We claim that under these assumptions, there is a central extension $0 \to \mathbb{Z} \to \widetilde{\Gamma} \to \Gamma \to 1$ such that the kernel of $\psi_{\widetilde{\Gamma}} : \mathrm{H}^2_{\mathrm{b}}(\widetilde{\Gamma}, \mathbb{R}) \to \mathrm{H}^2(\widetilde{\Gamma}, \mathbb{R})$ has dimension one.

Proof. By the assumptions, there is $\omega_\mathbb{Z} \in \mathrm{H}^2(\Gamma, \mathbb{Z})$ and $\omega \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$ such that $\psi_\Gamma(\omega) = i_\Gamma(\omega_\mathbb{Z}) \neq 0$. The central extension $0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\Gamma} \xrightarrow{\pi} \Gamma \longrightarrow 1$ associated to $\omega_\mathbb{Z}$ yields a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R}) & \xrightarrow{\psi_\Gamma} & \mathrm{H}^2(\Gamma, \mathbb{R}) & \xleftarrow{i_\Gamma} & \mathrm{H}^2(\Gamma, \mathbb{Z}) \\
\downarrow{\scriptstyle \pi^*_{\mathrm{b},\mathbb{R}}} & & \downarrow{\scriptstyle \pi^*_\mathbb{R}} & & \downarrow{\scriptstyle \pi^*_\mathbb{Z}} \\
\mathrm{H}^2_{\mathrm{b}}(\widetilde{\Gamma}, \mathbb{R}) & \xrightarrow{\psi_{\widetilde{\Gamma}}} & \mathrm{H}^2(\widetilde{\Gamma}, \mathbb{R}) & \xleftarrow{i_{\widetilde{\Gamma}}} & \mathrm{H}^2(\widetilde{\Gamma}, \mathbb{Z})
\end{array}$$

Since $\mathbb{Z}$ is amenable, $\pi^*_{\mathrm{b},\mathbb{R}}$ is an isomorphism [8, 3.8.4] (this is not true in general for $\mathbb{Z}$ coefficients). Setting $\beta := \pi^*_{b,\mathbb{R}}(\omega)$, we are reduced to seeing that $\mathrm{H}^2_{\mathrm{b}}(\widetilde{\Gamma}, \mathbb{R}) = \mathbb{R}\beta$ maps trivially to $\mathrm{H}^2(\widetilde{\Gamma}, \mathbb{R})$. But we have: $\psi_{\widetilde{\Gamma}}(\beta) = \pi^*_\mathbb{R}\big(\psi_\Gamma(\omega)\big) = (\pi^*_\mathbb{R} \circ i_\Gamma)(\omega_\mathbb{Z}) = (i_{\widetilde{\Gamma}} \circ \pi^*_\mathbb{Z})(\omega_\mathbb{Z})$, and $\widetilde{\Gamma}$ was designed as a central extension in order to have $\pi^*_\mathbb{Z}(\omega_\mathbb{Z}) = 0$. □

Remarks. 1. $\widetilde{\Gamma}$ has property (T) whenever Γ does. Indeed, since $\psi_\Gamma(\omega) \neq 0$, we have $\omega_\mathbb{Z} \neq 0$ and the corresponding central extension does not split. The claim is now a result due to Serre [4, p. 41].

2. The space of pseudocharacters of $\widetilde{\Gamma}$ is isomorphic to $\mathrm{Ker}(\psi_{\widetilde{\Gamma}})$ modulo the characters of $\widetilde{\Gamma}$; in particular, since property (T) groups have no non-zero characters, $\widetilde{\Gamma}$ satisfies (3) if Γ was chosen with property (T).

3. The group $\widetilde{\Gamma}$ is boundedly generated whenever Γ is so.

In conclusion, it remains to check the existence of groups Γ satisfying (1), (2) and (3′)-(3‴). We obtain two families of examples from the following discussion (see also Remark 4 below).

Let X an irreducible Hermitian symmetric space of non-compact type. Let $H := \mathrm{Isom}(X)^\circ$ be the identity component of its isometry group. We assume that $\pi_1(H) = \mathbb{Z}$, i.e. that $\pi_1(H)$ is torsion-free. We have then a central extension as in $(*)$ above, yielding a class $\omega_{H,\mathbb{Z}}$ in the "continuous" cohomology $\mathrm{H}^2_\mathrm{c}(H,\mathbb{Z})$ (represented by a Borel cocycle); the image ω_H of $\omega_{H,\mathbb{Z}}$ under the natural map $\mathrm{H}^2_\mathrm{c}(H,\mathbb{Z}) \to \mathrm{H}^2_\mathrm{c}(H,\mathbb{R})$ generates $\mathrm{H}^2_\mathrm{c}(H,\mathbb{R})$. For all this, see [5].

Let now $\Gamma < H$ be any lattice and let $\omega_\mathbb{Z}$ be the image of $\omega_{H,\mathbb{Z}}$ under the restriction map $r_\mathbb{Z}$: $\mathrm{H}^2_\mathrm{c}(H,\mathbb{Z}) \to \mathrm{H}^2(\Gamma,\mathbb{Z})$; thus, the corresponding central extension $\widetilde{\Gamma}$ is (isomorphic to) the preimage of Γ in $\widetilde{H}$. Note that, so far, $\omega_\mathbb{Z}$ can be zero. From now on we assume that the rank of X is at least two. This implies on one hand that H and Γ have property (T) [4, 2b.8 and 3a.4]; on the other hand, (3″) is established in [3, Thm. 21]. Furthermore, there are isomorphisms $\mathrm{H}^2_\mathrm{c}(H,\mathbb{R}) \xleftarrow{\psi} \mathrm{H}^2_\mathrm{cb}(H,\mathbb{R}) \xrightarrow{r_\mathbb{R}} \mathrm{H}^2_\mathrm{b}(\Gamma,\mathbb{R})$ (see [3] for the first and the vanishing theorem in [10] for the second). Thus (3′) and (3‴) follow aswell given the above discussion of the cohomology of H.

Finally, we investigate when Γ (and thus $\widetilde{\Gamma}$) can be chosen to be boundedly generated using a result of Tavgen' [14, Theorem B]. We define Γ as integral points of a $\mathbb{Q}$-algebraic group $\underline{H}$ such that the identity component $\underline{H}(\mathbb{R})^\circ$ is $H = \mathrm{Isom}(X)^\circ$. Using Tavgen's theorem requires that $\underline{H}$ be quasi-split over $\mathbb{Q}$. According to Cartan's classification [6, X.6, Table V and §3], the exceptional cases $E\,III$ and $E\,VII$, and the classical series $D\,III$, are excluded because the isometry groups are not quasi-split, and *a fortiori* neither are their $\mathbb{Q}$-forms. Let us check that the remaining types admit quasi-split $\mathbb{Q}$-forms.

Case $C\,I$: this corresponds to Siegel's upper half-spaces $\mathrm{Sp}_{2n}(\mathbb{R})/\mathrm{U}(n)$. The standard symplectic forms with all coefficients equal to one define $\mathbb{Q}$-split algebraic subgroups of SL_{2n} [2, V.23.3]. For each $n \geq 2$, the lattice $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z}) := \mathrm{Sp}_{2n}(\mathbb{Q}) \cap \mathrm{SL}_{2n}(\mathbb{Z})$ satisfies all the required properties. The corresponding symmetric space X has rank n and dimension $n(n+1)$.

Case $A\,III$: this corresponds to $\mathrm{SU}(p,q)/\mathrm{S}\big(\mathrm{U}(p)\times\mathrm{U}(q)\big)$ with $p \geq q$. In view of the Satake-Tits diagrams [12, II §3], the corresponding isometry groups which are quasi-split over $\mathbb{R}$ are those for which $p = q$ or $p = q+1$. The Hermitian form $h := \bar{x}_1 x_{2n} - \bar{x}_2 x_{2n-1} + \ldots - x_1 \bar{x}_{2n}$ (resp. $\bar{x}_1 x_{2n+1} - \bar{x}_2 x_{2n} + \ldots - x_1 \bar{x}_{2n+1}$), where $^-$ denotes the conjugation of $\mathbb{Q}(i)$, defines a $\mathbb{Q}$-form of the isometry group $\mathrm{SU}(n,n)$ (resp. $\mathrm{SU}(n+1,n)$). The matrices of $\mathrm{SL}_{2n}(\mathbb{Z}[i])$ (resp. $\mathrm{SL}_{2n+1}(\mathbb{Z}[i])$) preserving h provide suitable groups Γ.

Remarks. 1. What we call *bounded generation*, following *e.g.* [11, §A.2 p.575] and [13], is what Tavgen' calls *finite width*, while bounded generation in [14] is defined with respect to a generating system.

2. To have bounded generation, we restricted ourselves to arithmetic subgroups of quasi-split groups, which prevents from constructing the groups Γ as uniform lattices (the Godement compactness criteron requires $\mathbb{Q}$-anisotropic groups [11, Theorem 4.12], which are so to speak opposite to split and quasi-split groups). The underlying deeper problem is to know whether boundedly generated *uniform* lattices exist [13, Introduction].

3. Given the cohomological vanishing results of [3], [10], the only possibilities for Γ to be a lattice in (the k-points of) a simple group over a local field k is the case we considered: $k = \mathbb{R}$, rank at least two and Hermitian structure. In particular, the non-Archimedean case is excluded. As far as bounded generation only is concerned, there is an even stronger obstruction in positive characteristic: any boundedly generated group that is linear in positive characteristic is virtually Abelian [1].

4. A case in Cartan's classification was not alluded to above. This is the type $BD\,I$, corresponding to $\mathrm{SO}(p,q)^\circ/\big(\mathrm{SO}(p)\times\mathrm{SO}(q)\big)$ with $p \geq q = 2$. First, $\mathrm{SO}(2,2)^\circ$ is not simple and the associated

symmetric space is not irreducible (it is the product of two hyperbolic disks). For $p \geq 3$, the fundamental group $\pi_1(\mathrm{SO}(p,2)^\circ)$ has torsion since it is $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ [7, I.7.12.3], but lattices in $H = \mathrm{SO}(p,2)^\circ$ still enjoy properties (2) and (3′)-(3‴). For bounded generation, since a symmetric non-degenerate bilinear form defines a split (resp. quasi-split) orthogonal group if and only if $p - q \leq 1$ (resp. $p - q \leq 2$) [2, V.23.4], suitable groups Γ are provided by lattices $\mathrm{SO}(Q) \cap \mathrm{SL}_n(\mathbb{Z})$, with Q a non-degenerate quadratic form on $\mathbb{Q}^n$ of signature $(3,2)$ or $(4,2)$ over $\mathbb{Q}$.

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