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AMENABLE ACTIONS, FREE PRODUCTS AND A FIXED POINT PROPERTY

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Abstract

We investigate the class of groups admitting an action on a set with an invariant mean. It turns out that many free products admit interesting actions of this kind. A complete characterization of such free products is given in terms of a fixed point property.

1. *Introduction*

1.A. In the early 20th century, the construction of Lebesgue's measure was followed by the discovery of the Banach–Hausdorff–Tarski paradoxes (see [**21, 19, 4, 32**]; see also [**17**]). This prompted von Neumann [**23**] to study the following general question.

Given a group G acting on a set X, when is there an *invariant mean* on X?

Definition 1.1. An *invariant mean* is a G-invariant map μ from the collection of subsets of X to $[0, 1]$ such that:
(i) $\mu(A \cup B) = \mu(A) + \mu(B)$ when $A \cap B = \varnothing$, and
(ii) $\mu(X) = 1$.
If such a mean exists, the action is called *amenable*.

Remarks 1.2. (1) For the study of the classical paradoxes, one also considers normalisations other than condition (ii) above.

(2) The notion of amenability later introduced by Zimmer [**33**] and its variants [**3**] are different, being in a sense dual to the above.

1.B. The thrust of von Neumann's article was to show that the paradoxes, or the lack thereof, originate in the structure of the group rather than the set X. He therefore proposed the study of *amenable groups* (then 'meßbare Gruppen'), that is, groups whose action on themselves by multiplication is amenable. This direction of research turned out to be most fruitful, with influences on combinatorial group theory, ergodic theory, rigidity and semi-simple groups, harmonic analysis, operator algebras, and so on.

However, the original question remained largely unanswered (compare Greenleaf [**14**, p. 18, Problem], and Pier [**29**, p. 307]). Whilst it is easy to see that any action of an amenable group is amenable, the converse holds *a priori* only for *free* actions, where it is essentially tautological. Besides free actions, von Neumann describes only one other example [**23**, pp. 82–83], which still almost contains a free action of a free group, noting about the general case that "its somewhat complicated character might be found discommodious."

When investigating the general question of the amenability of a G-action on X, a few restrictions are in order (compare Greenleaf [**14**]). First, one should assume the action to be *faithful*, since otherwise one is really investigating a quotient group of G. Next, it is natural

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to consider *transitive* actions, since otherwise X could contain any action (for example, a fixed point, providing an obvious invariant mean) as long as one adds a free orbit for the sake of faithfulness. Thus, we shall focus in this paper on the class of all countable groups that admit a faithful transitive amenable action:

$$\mathcal{A} = \{G \text{ countable} : G \text{ admits a faithful transitive amenable action}\}.$$

(Throughout this paper, *countable* means infinite countable. In the finite case, all our statements either extend trivially or fail for obvious reasons — hence our choice of terminology.)

The only obvious examples of groups in $\mathcal{A}$ are amenable groups (since the action of G on itself is free and transitive). Given that in the classical paradoxes the non-amenability was caused by the presence of a non-Abelian free group, the following posthumous result of E. K. van Douwen is at first sight surprising.

THEOREM (van Douwen [**9**]). *Finitely generated non-Abelian free groups are in* $\mathcal{A}$.

1.C. It is easy to verify that if G has *Kazhdan's property* (T), then any amenable G-action has a finite orbit (Lemma 4.2). Thus for instance $\mathrm{SL}_3(\mathbf{Z})$ is not in $\mathcal{A}$. We propose the following definition.

DEFINITION 1.3. A countable group G has the *fixed point property* (F) if any amenable G-action (on a countable set) has a fixed point.

Such a group is never in $\mathcal{A}$; examples include infinite simple Kazhdan groups. However, G need not be Kazhdan because property (F) is preserved under finite free products (see Section 4 for details). Property (F) can be reformulated without referring to means, as it amounts to the *paradoxicality of any* (*non-trivial*) *G-action* (paragraph 4.H).

We say that G has *virtually* (F) if a finite index subgroup of G has property (F).

REMARK 1.4. The virtual property is much stronger than requiring that G have a finite orbit, since it implies for instance that G has a *minimal* finite index subgroup. Indeed, a group with property (F) cannot have any non-trivial finite quotient F, since F (or $F \times \mathbf{N}$, to be infinite) would be an amenable G-set without fixed point.

1.D. The main result of this paper is that $G * H$ is *always* in $\mathcal{A}$ unless the obvious obstruction occurs, as follows.

THEOREM 1.5. *Let G and H be any countable groups. Then $G * H \in \mathcal{A}$ unless G has property* (F) *and H has virtually* (F) (*possibly upon exchanging G and H*).

Moreover, this provides a necessary and sufficient characterization.

For example, we see that for any countable G, H, the free product $G * H$ is in $\mathcal{A}$ as soon as one of the groups is residually finite or non-finitely-generated or amenable. Thus, for instance, $G * \mathrm{SL}_3(\mathbf{Z})$ is in $\mathcal{A}$ for any countable group G. Furthermore, Theorem 1.5 leads to the following dichotomy.

COROLLARY 1.6. *Let $G = *_{i=1}^{n} G_i$ be any free product of $2 \leqslant n \leqslant \infty$ countable groups. Then either $G \in \mathcal{A}$ or G has virtually* (F).

Moreover, the latter occurs if and only if $n \neq \infty$, all G_i with $i > 1$ have property (F) *and G_1 virtually has* (F) (*possibly upon reordering the factors*).

Incidentally, this shows that a group in $\mathcal{A}$ can be the increasing union of groups with property (F) (see paragraph 4.E).

1.E. We summarize below some structural properties of the class $\mathcal{A}$. Most of these properties either are elementary or follow easily from known results and from Theorem 1.5. Statement (iv) provides an interesting contrast to Theorem 1.5. See Section 4 for definitions and details.

PROPOSITION 1.7. *For any countable groups G, H, the following properties hold.*

(i) *G amenable $\Rightarrow G \in \mathcal{A}$.*
(ii) *$G, H \in \mathcal{A} \Leftrightarrow G \times H \in \mathcal{A}$.*
(iii) *$G, H \in \mathcal{A} \Rightarrow G * H \in \mathcal{A}$.*
However, $G * H \in \mathcal{A} \not\Rightarrow G, H \in \mathcal{A}$.
(iv) *$G, H \in \mathcal{A} \not\Rightarrow G \rtimes H \in \mathcal{A}$, even if $G = \mathbf{Z}^2$.*
(v) *Any countable group embeds into a group in $\mathcal{A}$.*
(vi) *Assume that $H < G$ is co-amenable. Then $H \in \mathcal{A} \Rightarrow G \in \mathcal{A}$.*
However, $G \in \mathcal{A} \not\Rightarrow H \in \mathcal{A}$, even if $G = H \rtimes \mathbf{Z}$.
(vii) *G has Kazhdan's property (T) $\Rightarrow G \notin \mathcal{A}$.*
(viii) *Let $H \triangleleft G$ be a normal subgroup that is not of finite exponent. If the pair (H, G) has the relative property (T), then $G \notin \mathcal{A}$.*
(ix) *R. Thompson's group F is in $\mathcal{A}$; non-amenable Tarski monsters are not.*

REMARK 1.8. Concerning property (vi): a subgroup $H < G$ is called *co-amenable* if the G-action on $X = G/H$ is amenable. Thus we could rephrase all questions about $\mathcal{A}$ by studying groups G that admit some co-amenable subgroup H such that the intersection of all conjugates of H is trivial. (See [**11, 22, 28**] for more on co-amenability.)

REMARK 1.9. The starting point of this paper was our observation that one can give a very short proof of van Douwen's result that $\mathbf{Z} * \mathbf{Z} \in \mathcal{A}$, as follows. If σ is a transitive permutation on some set X, then any *generic* choice (in Baire's sense) of another permutation τ defines a faithful transitive *amenable* action of a free group with σ, τ as free generators. The idea that generic transformations generate a free group has been used in, for example, [**10, 8, 13, 2, 1**]. Another, as yet unpublished, simple proof of van Douwen's result was communicated to us by R. I. Grigorchuk [**15**]. Using generic permutations, one can further establish that $\mathcal{A}$ is closed under free product.

1.F. The organisation of this paper is as follows. Section 2 gathers basic facts about amenable actions. Section 3 is concerned with the proof of Theorem 1.5, which proceeds in two steps. We first establish that $G * H \in \mathcal{A}$ unless *both* G and H have virtually (F) (Theorem 3.3). A different argument (Theorem 3.4) implies as a corollary that the remaining case of Theorem 1.5 also holds true. Section 4 supplies the proofs of the remaining statements and examples of this introduction.

2. *Generalities*

2.A. A *G-set* is a countable set endowed with an action of the countable group G; a *G-map* is a G-equivariant map between G-sets. Unless otherwise stated, G itself is considered as a G-set under left multiplication. The group of bijections of X is denoted by $X!$. By functoriality one has the following lemma.

LEMMA 2.1. *Let $X \to Y$ be a G-map of G-sets. If the G-action on X is amenable, then so is its action on Y.* □

This shows notably that any action of an amenable group is amenable. In the anti-functorial direction, one can check that the next lemma holds.

LEMMA 2.2. *Let X be a G-set with an invariant mean μ, and let $Y \subseteq X$ be a G-invariant subset. If $\mu(Y) \neq 0$, then $\mu/\mu(Y)$ yields an invariant mean on Y.* □

2.B. Recall that a subgroup $H < G$ is *co-amenable* if the G-action on G/H is amenable (for example, if H has finite index in G). This is equivalent to the following relative fixed point property (see [**11**]).

> *Every continuous affine G-action on a convex compact subset of a locally convex space with an H-fixed point has a G-fixed point.*

Applying this to the space of means on a G-set X, one deduces the next lemma.

LEMMA 2.3. *Let X be a G-set and H a co-amenable subgroup. If the H-action on X is amenable, then so is the G-action.* □

2.C. Let H be a countable group, $L < H$ a subgroup and Z an L-set. If Z were a coset space $Z = L/M$ for some subgroup $M < L$, one would obtain a related H-set X by setting $X = H/M$. This construction can be generalized to the arbitrary L-set Z as follows.

DEFINITION 2.4. The *induced* H-set is the quotient $X = L \setminus (Z \times H)$ of $Z \times H$ by the diagonal L-action; the H-action on itself by *right* multiplication (by the inverse elements) turns X into an H-set.

It is straightforward to verify the following statement.

LEMMA 2.5. *If the L-action on Z is faithful, transitive or free, then the H-action on X has the corresponding property. The converse holds for the latter two properties, but not for faithfulness.* □

LEMMA 2.6. *The H-action on X has a finite orbit if and only if L is of finite index in H and has a finite orbit in Z.*

Proof. Suppose that H has a finite orbit; that is, some finite index subgroup $H_0 < H$ fixes a point in X. Let (z, h) be a representative in $Z \times H$ for this point. Thus, for every $h_0 \in H_0$ there is $\ell = \ell(h_0) \in L$ such that $(z, hh_0) = (\ell z, \ell h)$. In particular, $\ell(h_0) = hh_0h^{-1}$, and hence $hH_0h^{-1} \subseteq L$; moreover, hH_0h^{-1} fixes z. Since hH_0h^{-1} has finite index in H, this shows at once that L has finite index in H, and that the L-orbit of z is finite.

The converse is straightforward, and will not be used. □

As for amenability, we have the following lemma.

LEMMA 2.7. *Suppose that the L-action on Z is amenable. Then the H-action on X is amenable if and only if L is co-amenable in H.*

REMARK 2.8. However, even if Z is of the form L/M for M normal in L, it can happen that the H-action on X is amenable and $L < H$ is co-amenable whilst the L-action on Z is not amenable; see [**22, 28**].

Remark 2.9. Combining the two last lemmata, we obtain the first claim of Proposition 1.7(vi): a group containing a co-amenable subgroup in $\mathcal{A}$ is itself in $\mathcal{A}$.

Proof of Lemma 2.7. The map $Z \to Z \times \{e\} \subseteq Z \times H$ descends to an L-map $Z \to X$. Therefore, by Lemma 2.1, the L-action on X is amenable. Thus, if L is co-amenable in H, the H-action on X is amenable, by Lemma 2.3. The converse follows from Lemma 2.1, since there is a canonical H-map $X \to H/L$. □

Lemma 2.10. *Suppose that $L \lhd H$ is normal and co-amenable. Then L has property* (F) *if and only if every amenable H-set has an L-fixed point.*

Proof. Necessity is obvious. Conversely, let Z be an amenable L-set; L fixes a point in the induced H-set X, by Lemma 2.7. If this point is represented by (z, h) in $L \setminus (Z \times H)$, then z is fixed by $hLh^{-1} = L$. □

2.D. The following characterization, originating in Følner's work [**12**], is well known; see [**30**] for a proof.

Theorem 2.11. *A G-action on a set X is amenable if and only if for any finite subset $S \subseteq G$ and any $\varepsilon > 0$ there exists a finite subset $A \subseteq X$ such that*

$$|A \triangle sA| < \varepsilon |A| \qquad \forall s \in S.$$

We call such a set an *(S, ε)-Følner set.*

Remark 2.12. After summation over $s \in S$ (and adjusting ε accordingly), the above inequality is additive with respect to decomposing A along the partition of X into G-orbits. Therefore, given S and ε, we can find an (S, ε)-Følner set contained in a single G-orbit.

Since we consider the case where G is countable, it follows from Theorem 2.11 that the action is amenable if and only if there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of finite non-empty sets $A_n \subseteq X$ such that for every $s \in G$ one has

$$\lim_{n\to\infty} \frac{|A_n \triangle sA_n|}{|A_n|} = 0.$$

Definition 2.13. A sequence as above is called a *Følner sequence* for the G-action on X.

Remark 2.14. It suffices to check that $\lim_{n\to\infty} |A_n \triangle sA_n|/|A_n| = 0$ for all s in some set generating G.

2.E. The next proposition provides a technical but useful characterization of the virtual property (F).

Proposition 2.15. *Let H be a countable group, and assume that H does not have virtually* (F). *Then there exist an H-set Y and a Følner sequence $\{A_n\}_{n\in\mathbf{N}}$ such that:*

(i) *each A_n is contained in a single H-orbit $Hy_n \subseteq Y$, and*
(ii) *the cardinality $|Hy_n|$ converges to infinity in $\mathbf{N} \cup \{\aleph_0\}$.*

Moreover, we can assume that each H-orbit in Y contains at most one set A_n.

Statement (ii) means that for all $N \in \mathbf{N}$ there are at most finitely many finite orbits of size less than N amongst the Hy_n.

Proof of Proposition 2.15. Let $H = \{h_n : n \in \mathbf{N}\}$ be an enumeration of H. We claim that for every $n \in \mathbf{N}$ there are an H-set Y_n and a finite subset $A_n \subseteq Y_n$ contained in a single H-orbit $Hy_n \subseteq Y_n$ such that:

(1) $\forall\, i \leqslant n: \ |A_n \vartriangle h_i A_n| < \frac{1}{n}|A_n|$, and

(2) $|Hy_n| > n$.

This implies the statement of the lemma upon considering $Y = \bigsqcup_{n\in\mathbf{N}} Y_n$. Thus we consider for a contradiction the smallest $n \in \mathbf{N}$ for which the claim fails. Then every amenable H-set X has an orbit of size at most n. Indeed, by Theorem 2.11 X contains a finite subset satisfying statement (1), and we can assume that it lies in a single H-orbit by Remark 2.12; therefore statement (2) has to fail. Considering the special case where X is a disjoint union of finite quotients of H, we conclude that H has a *minimal* finite index subgroup L.

In order to obtain a contradiction, we shall now prove that L has property (F). Thus, let Z be any amenable L-set. Since the induced H-set X is amenable by Lemma 2.7, it contains a finite H-orbit, by the previous discussion. By Lemma 2.6, Z has a finite L-orbit and hence a fixed point, since L has no finite index proper subgroup. This is the desired contradiction. □

2.F. An idea of Kazhdan [**20**] yields the following lemma.

Lemma 2.16. *Let G be a countable group. If G is not finitely generated, then G has an amenable action without finite orbits.*

Proof. Let X be the disjoint union of all coset spaces G/H, where H ranges over the family of finitely generated subgroups of G. Then X is a countable set with a natural G-action; there are no finite orbits since G is not finitely generated. For any finite subset $S \subseteq G$, the trivial coset $\langle S\rangle$ in $G/\langle S\rangle$ is fixed by every $s \in S$; thus, the set $A = \{\langle S\rangle\}$ is an (S,ε)-Følner set for any $\varepsilon > 0$ and the action is amenable. □

3. *Amenable actions of free products*

3.A. We first explain why the restriction in Theorem 1.5 is an obvious obstruction; the rest of Section 3 will be devoted to proving that this is the only obstruction.

Lemma 3.1. *Let G be a group with property* (F) *and H a group with virtually* (F). *Then $G * H$ has virtually* (F).

In particular, such a group $G * H$ cannot belong to the class $\mathcal{A}$.

Proof of Lemma 3.1. Let $H_0 < H$ be a finite index subgroup with property (F); since H_0 is a minimal finite index subgroup, it is normal in H. We claim that the kernel L of the canonical morphism $G * H \to H/H_0$ has property (F). By Lemma 2.10, it suffices to find a L-fixed point in any $G * H$-set X with an invariant mean μ. Fix coset representatives $h_1, \ldots, h_n \in H$ for H/H_0. Let X^G be the (non-empty!) set of G-fixed points. By property (F), the G-action on $X \setminus X^G$ is not amenable. Therefore, $\mu(X \setminus X^G) = 0$; indeed, otherwise, Lemma 2.2 would yield a G-invariant mean on $X \setminus X^G$. It follows that $\mu(X^G) = 1$, and likewise $\mu(X^{H_0}) = 1$. Since μ is invariant, we deduce further that $\mu(h^{-1}X^G) = 1$ for all $h \in H$. Therefore, the set

$$X^{H_0} \cap \bigcap_{i=1}^{n} h_i^{-1} X^G$$

has mean one, and hence is non-empty. For any x in this set, the H-orbit $Hx = \{h_1 x, \ldots, h_n x\}$ is G-fixed, and therefore consists of L-fixed points. □

The above proof also shows that the next lemma holds.

LEMMA 3.2. *If G and H have property* (F), *then so does $G * H$.* □

3.B. We now establish a result slightly weaker than Theorem 1.5.

THEOREM 3.3. *Let G, H be any countable groups. Then $G * H \in \mathcal{A}$ unless both G and H have virtually* (F).

Proof. We can assume that H does not have virtually (F). Let Y be an H-set as in Proposition 2.15. Let $G = D \sqcup R$ be a partition of G into two infinite sets. We can index the H-orbits of Y by R and write

$$Y = \bigsqcup_{g \in R} Y_g.$$

We endow the set $X = H \sqcup Y$ with the natural H-action. Given any injective map $\beta : G \to X$, we obtain a G-action on X as follows. We transport by β the action of G on itself to the corresponding G-action on $\beta(G)$ and let G act trivially on $X \setminus \beta(G)$. We denote by $G_\beta < X!$ the resulting subgroup; since H acts faithfully on X, we can consider it as a subgroup $H < X!$ and thus by universality we have a canonical epimorphism $G * H \to \langle G_\beta, H \rangle$. We shall show that for a suitable choice of β the resulting $G * H$-action on X is faithful, transitive and amenable.

We shall consider only maps β such that $\beta(D) \subseteq H$ and $\beta(R) \subseteq Y$. We start by determining β on R. Let $\{A_n\}_{n \in \mathbf{N}}$ be a sequence of subsets of Y as in Proposition 2.15. For each $g \in R$, we make any choice $\beta(g) \in Y_g$. Since Y_g meets (and then contains) at most one A_n, we can require that $\beta(g) \notin A_n$ for any $n \in \mathbf{N}$ unless $A_n = Y_g$. This choice, together with Proposition 2.15(ii), ensures that

$$\lim_{n \to \infty} \frac{|A_n \cap \beta(G)|}{|A_n|} = 0.$$

Therefore, the sequence $\{A_n\}_{n \in \mathbf{N}}$ is a Følner sequence for the G-action on X, no matter how we define $\beta|_D : D \to H$. It follows (see Remark 2.14) that the $G * H$-action on X is amenable. On the other hand, this action is transitive, regardless of the definition of $\beta|_D$, since the G-orbit $\beta(G)$ meets every H-orbit.

It remains to show that the action is faithful upon suitably determining β on D. We will refer to elements of $G * H$ written in their reduced canonical form as *words*. With a customary abuse of notation, a (non-trivial) word is of the form $w = h_n g_n \dots h_2 g_2 h_1 g_1$, where only g_1 and h_n are allowed to be trivial. Since there are only countably many words, it is enough to prove the following claim. For any w and any finite set $E \subseteq D$ on which β is already determined, we can prescribe β on a finite set $E' \subseteq D$ such that:

(i) $E \cap E' = \varnothing$ and $\beta|_{E \sqcup E'}$ is injective, and

(ii) for any injection $\beta|_D : D \to H$ extending $\beta|_{E'}$, there is $x_0 \in \beta(E')$ with $w x_0 \neq x_0$.

(Hence w is not in the kernel of $G * H \to \langle G_\beta, H \rangle$.)

In order to prove the claim, we assume for definiteness that $g_1, h_n \neq e$ (the other cases are similar). We use indices $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n$. Since H is infinite, we can pick $x_0, y_i \in H$ such that all the $2n+1$ elements $x_0, y_i, h_i y_i$, are distinct and not in $\beta(E)$. Further, we pick $d_j \in D$ such that all the $2n+1$ elements $d_j, g_i d_{i-1}$ are distinct and not in E. We set $E' = \{d_j, g_i d_{i-1}\}$ and define $x_i = h_i y_i$. The prescriptions $\beta(d_j) = x_j$, $\beta(g_i d_{i-1}) = y_i$ extend β injectively; moreover, $w x_0 = x_n$ is indeed different from x_0. This proves the claim, and thus concludes the proof. □

3.C. Another ingredient for Theorem 1.5 is as follows.

THEOREM 3.4. *Let G, H be countable groups, and assume that $G * H$ has a transitive amenable action admitting a Følner sequence $\{A_n\}_{n\in\mathbf{N}}$ with $|A_n|$ unbounded. Then $G * H \in \mathcal{A}$.*

COROLLARY 3.5. *Let G and H be finitely generated countable groups, both admitting a finite index proper subgroup. Then $G * H \in \mathcal{A}$.*

Proof of Corollary 3.5. Let $G_0 \lneq G$ and $H_0 \lneq H$ be finite index subgroups; we can assume that they are normal. The free product of the quotients

$$L = (G/G_0) * (H/H_0)$$

admits a (non-trivial, finitely generated) free subgroup $F < L$ of finite index (see, for example, [**31**]). If F has rank one, it is amenable and hence in $\mathcal{A}$. Otherwise, F is of the form $\mathbf{Z} * \ldots * \mathbf{Z}$ and hence is also in $\mathcal{A}$ by Theorem 3.3 (or already by van Douwen's result). Since F is of finite index in L, it is in particular co-amenable, and therefore L is also in $\mathcal{A}$ by Remark 2.9. We thus have a transitive amenable action $G * H \to L \to X!$ and need only show that it satisfies the assumption of Theorem 3.4. If not, there would be a Følner sequence $\{A_n\}_{n\in\mathbf{N}}$ of cardinality bounded by some $N \in \mathbf{N}$. Let S be a finite set generating $G * H$. In particular, since some A_n is a (S, ε)-Følner set for $\varepsilon < 1/N$ we obtain a non-empty finite $G * H$-invariant subset $A_n \subseteq X$, which is impossible since X is countable and the action is transitive. □

Proof of Theorem 3.4. Let Y be a $G * H$-set as in the assumption, and choose any $y_0 \in Y$. Fix two non-trivial elements $g_0 \in G$, $h_0 \in H$, and consider the transitive $G * H$-set $Z = (G * H)/\langle g_0 h_0\rangle$; let $z_0 \in Z$ be the trivial coset $\langle g_0 h_0\rangle$. It follows, for example from elementary Bass–Serre theory [**31**], that:

(i) $z_0 \neq h_0 z_0 = g_0^{-1} z_0$, and

(ii) any non-trivial $w \in G * H$ fixes at most one point in Z.

The latter property can be verified as follows. If w has fixed points, we can assume upon conjugating that one of them is z_0 and hence $w = (g_0 h_0)^n$ for some non-zero $n \in \mathbf{Z}$. Now any w-fixed point $u\langle g_0 h_0\rangle$ satisfies $(g_0 h_0)^n u\langle g_0 h_0\rangle = u\langle g_0 h_0\rangle$. By uniqueness of the normal form for free products (and $n \neq 0$), u is in $\langle g_0 h_0\rangle$ and hence determines the same fixed point.

We now set $X = Y \sqcup Z$ and endow it with the corresponding action $G * H \to X!$. Consider the permutation $\sigma \in X!$ that transposes (y_0, z_0) and is trivial otherwise. We claim that the new $G * H$-action obtained by conjugating H by σ is faithful, transitive and amenable. To be more precise, we consider the original G-action on X and the new H-action given by $(h, x) \mapsto h^\sigma x = \sigma^{-1} h \sigma x$ (note that $\sigma^{-1} = \sigma$). This yields a new $G * H$-action by the universality of free products; by an abuse of notation we denote this action by $(w, x) \mapsto w^\sigma x$ for $w \in G * H$.

Amenability. Upon passing to a subsequence, we can assume that $|A_n| \to \infty$. But for any $s \in G * H$ there are only finitely many points for which the new action is different from the original one, so that we still have $\lim_{n\to\infty} |A_n \vartriangle sA_n|/|A_n| = 0$.

Transitivity. We claim that every point lies in the orbit of y_0, and will use statement (i) above. First pick any $y \in Y$, $y \neq y_0$. Let $w \in G * H$ be the *shortest* word such that $wy_0 = y$ in the original action; w exists since $G * H \to Y!$ is transitive. If the rightmost letter of w is in G, then $w^\sigma y_0 = wy_0 = y$ (because the 'trail' of y_0 under all non-empty right prefixes of w^σ avoids y_0, by minimality of w). Otherwise, the rightmost letter of w is some $h \in H$ with $hy_0 \neq y_0$. But then $hy_0 = h^\sigma g_0 h_0^\sigma y_0$, and thus replacing h with $hg_0 h_0$ yields a new word v with $v^\sigma y_0 = y$.

Now pick any $z \in Z$. If $z = z_0$, we attain our objective, since $z_0 = g_0 h_0^\sigma y_0$. Otherwise, let w be the shortest word such that $wz_0 = z$ in the original action. If the rightmost letter of w is in G, then again $w^\sigma z_0 = wz_0 = z$ and hence $z = (wg_0 h_0)^\sigma y_0$. Otherwise this letter is some $h \in H$, in which case $w^\sigma y_0 = z$ since $h^\sigma y_0 = hz_0 \neq z_0$.

Faithfulness. Let $w \in G * H$ be any non-trivial word. In view of (ii) above, there is some $z \in Z$ such that in the original action not only $wz \neq z$, but also the 'trail' of z under all right prefixes of w avoids z_0. Therefore $w^\sigma z = wz \neq z$ and the action is faithful. □

3.D. We are now ready to finish the proof of our main theorem.

Proof of Theorem 1.5. In view of Theorem 3.3, we can assume without loss of generality that both G and H have finite index subgroups $G_0 \lneq G$ and $H_0 \lneq H$ with the fixed point property (F). By Lemma 2.16, a non-finitely-generated countable group cannot have virtually (F). Therefore, both G and H are finitely generated, and we conclude from Corollary 3.5 that $G * H \in \mathcal{A}$ in this case too. The converse was established as Lemma 3.1. □

Proof of Corollary 1.6. Keep the notation of the corollary. If $n = \infty$, then $H = *_{i=2}^{\infty} G_i$ is not finitely generated, and in particular by Lemma 2.16 does not virtually have (F). Thus we can apply Theorem 1.5 and conclude that $G = G_1 * H$ is in $\mathcal{A}$.

Assume now that $n \neq \infty$. If all G_i with $1 < i \leqslant n$ have property (F) and G_1 virtually has (F), then Lemmas 3.1 and 3.2 show that $G = G_1 * (G_2 * \ldots * G_n)$ has virtually (F). If not, we can assume upon reordering that either:

(i) both G_1 and G_2 fail to have property (F), or

(ii) G_1 does not virtually have (F).

In case (i), $G_2 * \ldots * G_n$ also fails to have property (F), and hence $G = G_1 * (G_2 * \ldots * G_n)$ is in $\mathcal{A}$ by Theorem 1.5. In case (ii), Theorem 1.5 implies that $G \in \mathcal{A}$ as well. □

4. *Remaining proofs*

4.A. In this section, all Roman numerals refer to the properties listed in Proposition 1.7. Property (i) follows by definition. Property (iii) follows from Theorem 1.5. For Property (ii), let $G \to X!$ and $H \to Y!$ be faithful transitive amenable actions. Then the $G \times H$-action on $X \times Y$ is faithful and transitive. If $A_n \subseteq X$ and $B_n \subseteq Y$ form Følner sequences, then $A_n \times B_n$ yields a Følner sequence for the product. Conversely, let $G \times H \to X!$ be amenable, transitive and faithful. Since H permutes the G-orbits transitively, it follows that X, as a G-set, is isomorphic to $X_0 \times X_1$, where X_0 is any G-orbit in X and $X_1 = G \setminus X$ with trivial G-action. The G-action on X_0 is faithful, transitive and amenable, by Lemma 2.1, since there is a (non-canonical) G-map $X \to X_0$.

The first part of Proposition 1.7(vi) has already been recorded in Remark 2.9. For the second part, let Q be any group, and consider $G = H \rtimes \mathbf{Z}$ where $H = \bigoplus_{\mathbf{Z}} Q$ denotes the direct sum (equal to the restricted product) and $\mathbf{Z}$ acts on it by a shift. It was observed in [**22**] that the subgroup $K = \bigoplus_{\mathbf{N}} Q$ is co-amenable in G. On the other hand, the G-action on $X = G/K$ is faithful and transitive; thus $G \in \mathcal{A}$. Since $H \cong Q \times H$, it is enough now by (ii) to chose for Q any group not in $\mathcal{A}$ (for example, using Lemma 4.2 below).

4.B. For the sake of our discussion, it is convenient to introduce the following definition.

Definition 4.1. Let $\mathcal{B}$ be the class of all countable groups admitting some amenable action on a countable set without finite orbits.

Whilst $\mathcal{B}$ contains $\mathcal{A}$, it is much wider; notice for instance that any group with a quotient in $\mathcal{B}$ is itself in $\mathcal{B}$. Moreover (Lemma 2.16), Kazhdan's observation [**20**] shows that any countable group that is not finitely generated is in $\mathcal{B}$. For example, let $G = \bigoplus_{n=1}^{\infty} G_n$, where G_n are any countable groups with $G_1 \notin \mathcal{A}$; then $G \notin \mathcal{A}$ by Property (ii), but $G \in \mathcal{B}$.

A group in $\mathcal{B}$ cannot have virtually (F); however, the class of groups to which Theorem 1.5 applies is yet much wider than $\mathcal{B}$, since it contains notably every countable group without a minimal finite index subgroup (for example, $\mathrm{SL}_3(\mathbf{Z})$). In summary:

$$\mathcal{A} \subsetneq \mathcal{B} \subsetneq \big\{\text{not virtually (F)}\big\}.$$

4.C. Another well-known criterion for amenability is that the G-action on X is amenable if and only if the associated unitary representation on $\ell^2(X)$ *almost has invariant vectors* in the sense of Kazhdan [**20**]; see also [**18**] for this notion, Kazhdan's property (T) and the Kazhdan–Margulis relative property (T). Since ℓ^2-functions have finite level-sets, the next lemma follows.

LEMMA 4.2. *Every amenable action of a group with Kazhdan's property* (T) *has a finite orbit. In particular, any Kazhdan group without non-trivial finite quotients has property* (F). □

Thus a Kazhdan group can never be in $\mathcal{B}$, and Proposition 1.7(vii) follows.

Notice that the second statement of Lemma 4.2 applies in particular to infinite simple Kazhdan groups; we point out that such groups do indeed exist, as follows from the work of Gromov [**16**] and Ol′shanskiĭ [**25**] on quotients of hyperbolic groups (see [**26**] for precise references).

There exist, however, groups with property (F) that are not Kazhdan groups; indeed, property (F) is stable under free product (Lemma 3.2) whilst a (non-trivial) free product is never a Kazhdan group [**18**]. Since property (F) passes to quotients, we remark further that any group generated by finitely many subgroups with property (F) still enjoys this property. (See also Lemma 4.5 below.)

4.D. Proposition 1.7(viii) follows from an argument similar to that for (vii), as follows.

LEMMA 4.3. *Let $H \lhd G$ be a normal subgroup of $G \in \mathcal{A}$. If the pair (H, G) has the relative property* (T), *then H has finite exponent.*

Proof. Let $G \to X!$ be a faithful transitive amenable action. Since G almost has invariant vectors in $\ell^2(X)$, the relative property (T) implies that H fixes a non-zero vector in $\ell^2(X)$. Therefore H preserves a non-empty finite set in $F \subseteq X$. Since H is normal in G and G acts transitively, it follows that X has a partition into G-translates gF of F, and that this partition is preserved by H. Therefore we have a natural morphism

$$H \to \prod_{gH \in G/H} (gF)!.$$

Since the G-action is faithful, this morphism is injective and thus realizes H as a subgroup of a group of exponent $|F|!$. □

Now Proposition 1.7(iv) also follows. Since $\mathrm{SL}_2(\mathbf{Z})$ is virtually free, it is in $\mathcal{A}$ by Property (vi); the group $\mathbf{Z}^2$ is in $\mathcal{A}$ since it is amenable. On the other hand, the natural semi-direct product $\mathbf{Z}^2 \rtimes \mathrm{SL}_2(\mathbf{Z})$ is not in $\mathcal{A}$ since it has the relative property (T) (see [**18**]). This is yet another example of a group in $\mathcal{B} \setminus \mathcal{A}$, since its quotient $\mathrm{SL}_2(\mathbf{Z})$ is in $\mathcal{A}$.

4.E. At this point we have a number of alternative proofs of Property (v), namely that any countable group Q embeds into some $G \in \mathcal{A}$. For instance, take $G = Q * \mathbf{Z}$, or $G = *_{n=1}^{\infty} Q$ (by Corollary 1.6), or the group $G = H \rtimes \mathbf{Z}$ with $H = \bigoplus_{\mathbf{Z}} Q$ considered earlier. (See also [**15**].)

We also notice that a group in $\mathcal{A}$ can be the increasing union of groups with property (F). Indeed, let $\{Q_n\}$ be a family of groups with property (F) and $G = *_{n=1}^{\infty} Q_n$. Then G is in $\mathcal{A}$ by Corollary 1.6, but it is the increasing union over k of (the canonical images of) $*_{n=1}^{k} Q_n$, each of which has property (F) by Lemma 3.2.

4.F. It is unknown whether R. Thompson's group F is amenable or not. This group is defined in detail in [**7**]; all that we need to know here is that it satisfies the assumptions of the following lemma.

Lemma 4.4. *Let F be a group of orientation-preserving piecewise linear homeomorphisms of the interval $(0, 1)$. If the derived subgroup F' has a dense orbit, then F' and F are in $\mathcal{A}$.*

Proof. Let $X = F'x$ be such an orbit, and notice that F' acts faithfully on X. Choose a sequence $\{x_n\}$ in X converging to 0. Let μ be a limit point of the sequence of point-measures δ_{x_n} in the space of means on X. Then μ is F'-invariant because for every $g \in F'$ there is $\varepsilon > 0$ such that g is trivial on $(0, \varepsilon)$ (this is where we use the assumptions on F). Therefore $F' \in \mathcal{A}$ and thus $F \in \mathcal{A}$ by Proposition 1.7(vi). □

4.G. A *Tarski monster* is a non-cyclic group such that all its proper subgroups are cyclic. Ol′shanskiĭ constructed various Tarski monsters and then proved that his groups are non-amenable [**24**]. Therefore the following statement applies to them.

Lemma 4.5. *Let G be a non-amenable group such that all its proper subgroups are amenable. Then G has property* (F).

Proof. Suppose, for a contradiction, that G has an amenable action on the countable set X without fixed points. We will obtain a contradiction by constructing an (S, ε)-Følner set in G for any finite set $S \subseteq G$ and any $\varepsilon > 0$. To this end, choose an orbit $G/L \subseteq X$ containing an $(S, \varepsilon/2)$-Følner set $A \subseteq G/L$, as granted by Theorem 2.11 and Remark 2.12. Fix a section $\sigma : G/L \to G$ of the natural map $G \to G/L$. By our assumptions, L is amenable. Thus there is a $(T, \varepsilon/2)$-Følner set $B \subseteq L$ for $T = \{\sigma(sgL)^{-1} s\sigma(gL) : s \in S, gL \in A\} \subseteq L$. One verifies that $\{\sigma(gL)\ell : gL \in A, \ell \in B\}$ is an (S, ε)-Følner set in G. □

4.H. To appreciate the strength of property (F), it is interesting to reformulate it without reference to invariant means, as follows.

> *A group G has property* (F) *if and only if any G-action on any set admits a paradoxical decomposition of its non-fixed points.*

More precisely, let X be any G-set. There is of course nothing to say about the subset $X^G \subseteq X$ of fixed points, on which G acts trivially. The above condition states that its complement admits a decomposition

$$X \setminus X^G = A_1 \sqcup \ldots \sqcup A_n \sqcup B_1 \sqcup \ldots \sqcup B_m$$

such that two other partitions are given by

$$X \setminus X^G = g_1 A_1 \sqcup \ldots \sqcup g_n A_n,$$
$$X \setminus X^G = h_1 B_1 \sqcup \ldots \sqcup h_m B_m,$$

for some $g_i, h_j \in G$. The equivalence with property (F) follows readily from Tarski's theorem; see [**27**, 3.15, p. 120].

4.I. Finally, we pose a problem: What can be said about amenable actions of the Hilbert modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$? In particular, does Γ belong to any of the classes considered in this paper?

More generally, let Γ be a lattice in a product $G = G_1 \times G_2$ of locally compact groups, and assume that both projections $\Gamma \to G_i$ have dense image. Are there natural conditions on the groups G_i that imply that $\Gamma \in \mathcal{A}$ or $\Gamma \notin \mathcal{A}$? (Not desired here are conditions so coarse as to apply to *any* lattice in G, such as a product of lattices $\Gamma_i < G_i$.)

In another direction, are there interesting amenable actions of lattices in $\mathrm{SO}(n, 1)$ or in $\mathrm{SU}(n, 1)$?

To remain closer to the present paper, one could ask when free products with amalgamations and HNN-extensions belong to $\mathcal{A}$. The very interesting examples of amalgamated products of free groups constructed in [**5**] and [**6**] arise as lattices $\Gamma < G_1 \times G_2$, as above.

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