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Topology III (IIII) III-III

TOPOLOGY

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Ideal bicomings for hyperbolic groups and applications

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Received 25 June 2003; accepted 13 January 2004

Abstract

For every hyperbolic group and more general hyperbolic graphs, we construct an equivariant ideal bicombing: this is a homological analogue of the geodesic flow on negatively curved manifolds. We then construct a cohomological invariant which implies that several Measure Equivalence and Orbit Equivalence rigidity results established in Monod and Shalom (Orbit equivalence rigidity and bounded cohomology, preprint, to appear) hold for all non-elementary hyperbolic groups and their non-elementary subgroups. We also derive superrigidity results for actions of general irreducible lattices on a large class of hyperbolic metric spaces.

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MSC: 20F65; 20F67; 20J06

Keywords: Hyperbolic groups; Hyperbolic spaces; Ideal bicombing; Bounded cohomology; Superrigidity

1. Introduction

1.1. If M is a simply connected manifold with sectional curvature ≤ -1 , then any two distinct points of the sphere at infinity ∂M can be connected by a unique geodesic line. Moreover, when two such geodesics share a common ideal endpoint, they converge at exponential synchronous rate, and the whole setting is of course equivariant under the group of isometries of M . A similar situation occurs more generally for $\text{CAT}(-1)$ spaces. *The goal of this paper is to establish an analogous phenomenon for Gromov-hyperbolic graphs, and to use it in constructing cohomological invariants that have far-reaching applications in rigidity theory.*

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¹ Partially supported by NSF grant DMS 0132514 (CAREER).

² Partially supported by FNS grant 8220-067641 and NSF grant DMS 0204601.

One of the reasons the theory of hyperbolic groups is so fruitful is that whilst hyperbolicity is a very general and robust property (indeed, it is *generic* in some ways [15,27]), many ideas from large scale differential geometry in negative curvature do find analogues for hyperbolic groups. In the case at hand, geodesics are to be replaced with a homological concept, an *ideal* version of the notion of homological bicombing:

Let $\mathcal{G} = (\mathcal{G}^{(0)}, E)$ be a (connected) hyperbolic graph, for instance a Cayley graph of a hyperbolic group, and let $\partial\mathcal{G}$ be its boundary at infinity (for all terminology see Section 2).

Definition 1. An *ideal bicombing* on \mathcal{G} is a map $q : (\partial\mathcal{G})^2 \rightarrow \ell^\infty(E)$ such that:

- (i) $q[\xi, \eta]$ is a cycle for all $\xi, \eta \in \partial\mathcal{G}$, i.e. $\partial q[\xi, \eta] = 0$.
- (ii) For all distinct $\xi_+, \xi_- \in \partial\mathcal{G}$ and all disjoint graph neighbourhoods V_\pm of ξ_\pm , the function $\partial(q[\xi_-, \xi_+]|_{V_\pm})$ is finitely supported of sum ∓ 1 .

We say that q has *bounded area* if $\|q[\xi, \eta] + q[\eta, \zeta] + q[\zeta, \xi]\|_1$ is finite and uniformly bounded over $\xi, \eta, \zeta \in \partial\mathcal{G}$, and that q is *quasigeodesic* if there exists a constant C such that for all $\xi, \eta \in \partial\mathcal{G}$ the support of $q[\xi, \eta]$ lies in the C -neighbourhood of any geodesic from ξ to η .

Our first result is of geometric nature:

Theorem 2. *Every hyperbolic graph \mathcal{G} of bounded valency admits a weak-*continuous quasigeodesic ideal bicombing of bounded area which is equivariant under the isometry group of \mathcal{G} .*

This theorem, which is our replacement for infinite geodesics with synchronous convergence, occupies half of the paper. The starting point of the construction is a modification of the bicombing introduced in [22]. We point out that the case where \mathcal{G} is a Cayley graph does not seem to be any simpler; even though Gromov introduced a geodesic flow for hyperbolic groups (see [14,7]), his construction is not sufficiently well understood for our further needs. We single out the precise notion of (homological) exponential convergence in Theorem 17 below.

1.2. Our main motivation for Theorem 2 lies in establishing an invariant in bounded cohomology. This type of invariant is used in [24,35] to prove superrigidity results and applied in [25] to the theory of Orbit Equivalence to establish new rigidity phenomena. Of particular importance in that context is the class \mathcal{C}_{reg} of groups Γ for which $H_b^2(\Gamma, \ell^2(\Gamma))$ is non-trivial; it is shown in [24] that this class contains all groups acting properly and non-elementarily on proper $\text{CAT}(-1)$ spaces, various amalgamated products (including those with finite amalgamated subgroup), and hyperbolic groups with vanishing first ℓ^2 -Betti number. We can now complete the picture as conjectured in [24]:

Theorem 3. *Let Γ be a countable group admitting a proper non-elementary action on a hyperbolic graph of bounded valency. Then $H_b^2(\Gamma, \ell^p(\Gamma))$ is non-trivial for all $1 \leq p < \infty$. In particular Γ is in the class \mathcal{C}_{reg} .*

This applies in particular to the case where Γ is a non-elementary hyperbolic group or more generally a non-elementary subgroup of a hyperbolic group. We point out that whilst there are countably many non-isomorphic hyperbolic groups, there is a continuum 2^{\aleph_0} of non-isomorphic sub-

groups of hyperbolic groups (a fact explained to us by I. Kapovich and P. Schupp, see Remark 28); note that only countably many of them are elementary. This adds a substantial collection of groups to the class \mathcal{C}_{reg} .

Using structure theory of locally compact groups, one can combine Theorem 3 with the case of CAT(−1) spaces addressed in [24] and deduce the following:

Corollary 4. *Let X be a hyperbolic proper geodesic metric space on which $\text{Isom}(X)$ acts cocompactly. Then any countable group Γ admitting a proper non-elementary isometric action on X is in \mathcal{C}_{reg} .*

Theorem 3 is obtained by first constructing a specific cocycle at infinity

$$\omega : (\partial\mathcal{G})^3 \rightarrow \ell^1(E \times E)$$

using a “doubling” of the ideal bicombing, see Theorem 18 below. This cocycle is our cohomological invariant mentioned above. Then we obtain a class in bounded cohomology using the functorial approach [6,23] together with Poisson boundary theory.

1.3. Our interest in Theorem 3 lies firstly in the attempt to relate more intimately geometric and measurable group theory (as e.g. in [12,25,33]). Recall that two countable groups Γ and A are called *measure equivalent* (ME) if there is a σ -finite measure space (Σ, m) with commuting measure preserving Γ - and A -actions, such that each one of the actions admits a finite measure fundamental domain. For example, it is evident from the definition that any two lattices in the same locally compact group are ME. The ME relation was suggested by Gromov [15,0.5.E] as a measurable analogue of being quasi-isometric: Indeed, the latter is equivalent to the existence of a locally compact space with commuting continuous Γ - and A -actions, such that each one is proper and cocompact [15]. Thus ME amounts to replacing such a space by the measurable counterpart (Σ, m) .

Whilst the question of which group properties are geometric (i.e. preserved under the quasi-isometry relation) is a well studied one, and of central interest in geometric group theory—much less is known about it in the analogous measurable setting, which seems to require more subtle techniques in general. An intriguing example is provided by the distinguished class of hyperbolic groups. Indeed, whilst being hyperbolic is well known to be a geometric property, it is *not* ME invariant. For instance, in the automorphism group of a regular tree one can find non-Abelian free groups as cocompact lattices, but also non-finitely generated groups as non-uniform lattices. Hence, being ME to a hyperbolic (or even free) group does not even guarantee finite generation (let alone finite presentation or hyperbolicity). Tamer counter-examples arise when considering a non-uniform lattice in a rank one simple Lie group G , which is ME to any uniform (hence hyperbolic) lattice in the same G , but is never hyperbolic itself when $G \neq \text{SL}_2(\mathbf{R})$. Thus, it is natural to inquire what it is in the geometric hyperbolicity property which can be, after all, detected measure-theoretically.

Previously, only two restrictions were known for groups ME to a hyperbolic group: they must have a finite centre [2], and they cannot split as a direct product of two infinite groups [1]. Since the class \mathcal{C}_{reg} is a ME invariant [25], Theorem 3 implies:

Corollary 5. *If Γ is a countable group which is ME to a non-elementary (subgroup of a) hyperbolic group, then Γ is in the class \mathcal{C}_{reg} .*

One immediate consequence of this result is to generalize both restrictions above in two directions: First, every normal amenable subgroup of Γ must be finite. Secondly, this property, as well as the property not to split as a direct product, both hold for *any infinite normal subgroup* $A \triangleleft \Gamma$. (The normality assumption is essential. For proofs see Section 5.3.)

Theorem 3 combined with the results of [25] implies a “prime factorization” phenomenon for torsion-free hyperbolic groups with respect to ME:

Corollary 6. *Let Γ_i and Λ_j be non-elementary torsion-free hyperbolic groups ($1 \leq i \leq n, 1 \leq j \leq m$). If $\Gamma = \prod \Gamma_i$ is ME to $\Lambda = \prod \Lambda_j$ then $n=m$ and after permutation of the indices each Γ_i is ME to Λ_i .*

The *geometric* counterpart of this statement (prime factorization for quasi-isometries instead of ME) follows from arguments of [9,10,20]; an analogue for von Neumann algebras has been proposed recently by Ozawa–Popa [28]. We emphasize that, outside the class \mathcal{C}_{reg} , not only $n \neq m$ can occur, but also the unique factorization of Corollary 6 can fail when $n = m$. A further consequence of Theorem 3 is that all products of hyperbolic groups exhibit the strong rigidity phenomena for Orbit Equivalence studied in [25]; see also [19] for related results in a somewhat different setting.

1.4. Recall that a lattice $\Lambda < G = G_1 \times \cdots \times G_n$ in a product of ($n \geq 2$) locally compact groups G_i is *irreducible* if its projection to each factor G_i is dense. This generalizes the usual notion of irreducibility for lattices in semisimple algebraic groups, where each G_i is simple. However, there are by now important new examples of irreducible lattices outside the linear context (compare the introduction of [32]). *Without any assumption* on the factors of the ambient group, we show that non-elementary homomorphisms extend after possibly factoring out a compact obstruction:

Theorem 7. *Let G_1, \dots, G_n be locally compact σ -compact groups, Λ an irreducible lattice in $G = G_1 \times \cdots \times G_n$ and $\pi : \Lambda \rightarrow \text{Isom}(X)$ a homomorphism, where X is either:*

- (i) *any hyperbolic graph of bounded valency, or*
- (ii) *any hyperbolic proper geodesic metric space on which $\text{Isom}(X)$ acts cocompactly.*

Then either the closure $H = \overline{\pi(\Lambda)} < \text{Isom}(X)$ is amenable (and hence the Λ -action on X is elementary), or there is a compact normal subgroup $K \triangleleft H$ such that the induced homomorphism $\Lambda \rightarrow H/K$ extends to a continuous homomorphism $G \rightarrow H/K$, factoring through some G_i .

To put the result in a better perspective, we note the following comments:

(a) Observe that Theorem 7 applies immediately to homomorphisms to hyperbolic groups. In particular, *for any non-trivial homomorphism from Λ to a torsion-free hyperbolic group, either its image is infinite cyclic or it extends continuously to G .*

(b) Note that if Λ (or equivalently each G_i) has in addition Kazhdan’s property (T), then it follows that without any further assumption π *always* extends continuously (modulo a compact subgroup).

A worth noting feature of Theorem 7 is that it goes beyond the setting of CAT(−1) target spaces, to the more robust framework of hyperbolicity. For example, in [24] it is observed that if Γ is any Gromov-hyperbolic group and X is the Cayley graph associated to the free product of Γ and a free group (with respect to naturally defined generators), then $\text{Isom}(X)$ is a locally compact group which is not discrete (nor an extension of such), acting cocompactly on X , and Theorem 7 applies. Thus this theorem covers completely new situations in terms of the target (and source) groups being involved.

1.5. The results of [24] together with the above Theorem 3 provide us with a large class of groups that are in \mathcal{C}_{reg} because of various instances of negative curvature phenomena. It is therefore natural to study the class \mathcal{C}_{reg} more thoroughly, and particularly to ask: *Is there a characterisation of \mathcal{C}_{reg} in geometric terms? Is being in \mathcal{C}_{reg} a geometric property?*

On the one hand, it would be interesting to find geometric restrictions on this class—in addition to the algebraic restrictions that we know [24]. On the other, one could try to include more groups in \mathcal{C}_{reg} by searching amongst further relatives in the family of “negative curvature”: for instance groups with a non-elementary proper isometric action on some general geodesic Gromov-hyperbolic metric space. Here we have in mind the possibility of non-proper spaces as well, so that the notion of “proper action” should be that of *metrically proper*, or maybe even weaker, as in [4]. This would include mapping class groups, since they act on the curve complex, which is not locally finite but hyperbolic [21]. A positive indication in that direction is the fact [4] that groups with such actions have infinite dimensional $H_b^2(-, \mathbf{R})$; indeed, in concrete situations, it is often possible to recover (infinitely many) classes in $H_b^2(-, \mathbf{R})$ from the constructions that we use for $H_b^2(-, \ell^2(-))$. For more “negative curvature” aspects of the mapping class groups, compare also [11].

Finally, we ask: *What (else) can be said about the class of all groups ME to some non-elementary hyperbolic group?*

In fact, as pointed out in [25], any (countable) non-Abelian free group is already ME to the uncountable class of all free products of any finite collection of (at least two) countable amenable groups.

2. Notation

For any set X we write $\ell^p(X)$ for the space of p -summable functions ($1 \leq p < \infty$) and $\ell^\infty(X)$ for bounded functions; $C(X)$, $C_{00}(X)$ stand for the space of all, respectively all finitely supported functions; $C_0(X)$ for those vanishing at infinity (i.e. along the filtre of cofinite subsets). The Banach spaces $\ell^p(X)$ are endowed with the p -norms $\|\cdot\|_p$, whilst the weak-*topology on $\ell^p(X)$ refers to the canonical isomorphisms $\ell^p(X) \cong \ell^{p/(p-1)}(X)^*$ for $1 < p < \infty$, $\ell^\infty(X) \cong \ell^1(X)^*$ and $\ell^1(X) \cong C_0(X)^*$.

We use the Serre’s notation for graphs, so that a graph $\mathcal{G} = (\mathcal{G}^{(0)}, E)$ consists of a set of vertices $\mathcal{G}^{(0)}$, a set of edges E , an involution $e \mapsto \bar{e}$ of E and initial/terminal maps $E \rightarrow \mathcal{G}^{(0)}$ (which we denote $e \mapsto e_-$, $e \mapsto e_+$) subject to the usual conditions. We say that \mathcal{G} has bounded valency if there is a bound on the number of edges issuing from any vertex. As usual, we endow the geometric realization of \mathcal{G} with the path metric d of unit edge length and often abuse notation in writing \mathcal{G} for the resulting space. Likewise, paths are simplicial paths as well as path in the realization. The “distance” d between subsets of \mathcal{G} is the infimum of the distance between their points.

Elements of $C_{00}(\mathcal{G}^{(0)})$ and $C_{00}(E)$ are also referred to as 0- and 1-chains. The boundary ∂f of a 1-chain f is the 0-chain $\partial f(v) = \sum_{e_+=v} f(e) - \sum_{e_-=v} f(e)$; this definition extends to $C(E)$ if \mathcal{G} is locally finite. Any (vector-valued) function on vertices or edges extends by linearity to a function on chains, and we will use the same notation (considering chains as formal sums). We also use the notation $\langle f, e \rangle$ for the value of $f \in C(E)$ at the edge e , and $\text{supp}(f)$ is the closure of the union of the edges where f does not vanish. Throughout the paper, $\mathcal{G} = (\mathcal{G}^{(0)}, E)$ is a hyperbolic graph of bounded valency; consequently, for every $R > 0$ there is a uniform bound on the number of edges or

vertices in any ball of radius R . For example, the Cayley graph of a hyperbolic group with respect to any finite set of generators would be such.

Let (X, d) be a metric space. Denote by $B(x, r)$, respectively $\bar{B}(x, r)$, the open, respectively closed, r -ball with centre x , and by $N(S, r)$ the *closed r -neighbourhood* of the subset S in X . One calls X *proper* if all closed balls are compact. A background reference for *hyperbolic* metric spaces is [13]. Unless otherwise stated, $H = \text{Isom}(X)$ is the group of isometries of X ; in the graph case $X = \mathcal{G}$, it coincides with the group of automorphisms and all spaces of functions on \mathcal{G} defined above are endowed with the natural H -representation. For any proper metric space X , the compact-open topology turns H into a locally compact topological group and its action on X is proper. One says that X has *at most exponential growth* if there is a constant α such that any ball of any radius r contains at most α^r disjoint balls of radius one.

Hyperbolicity of X guarantees the existence of a constant $\delta \geq 0$ such that all the geodesic triangles in X are δ -fine in the following sense: If a, b, c are points of X and $[a, b]$, $[b, c]$, and $[c, a]$ are geodesics from a to b , from b to c , and from c to a , respectively, if moreover points $\bar{a} \in [b, c]$, $v, \bar{c} \in [a, b]$, $w, \bar{b} \in [a, c]$ satisfy

$$d(b, \bar{c}) = d(b, \bar{a}), \quad d(c, \bar{a}) = d(c, \bar{b}), \quad d(a, v) = d(a, w) \leq d(a, \bar{c}) = d(a, \bar{b}),$$

then $d(v, w) \leq \delta$. We call $\{\bar{a}, \bar{b}, \bar{c}\}$ a triple *inscribed* in the triangle $\{a, b, c\}$. For the rest of the paper, δ denotes a positive integer, depending only on X , such that all the geodesic triangles in X are δ -fine.

We write ∂X for the Gromov boundary and $\partial^n X \subseteq (\partial X)^n$ for the subset of n -tuples of pairwise distinct points. The *Gromov product* of $a, b \in X$ with respect to a basepoint $x_0 \in X$ is

$$(a|b)_{x_0} := \frac{1}{2}(d(a, x_0) + d(b, x_0) - d(a, b)).$$

There is a natural extension of this product to $a, b \in \bar{X} := X \sqcup \partial X$; this extension is not necessarily continuous and in general one can make various choices of such extensions that differ by a bounded amount. The sets $\{x \in \bar{X} : (x|\xi)_{x_0} > r\}$ form a basis of neighbourhoods of $\xi \in \partial X$ in \bar{X} . The various extensions of $(\cdot|\cdot)_{x_0}$ induce the same topology on ∂X . In the graph case, we call a subgraph $U \subseteq \mathcal{G}$ a *graph neighbourhood* of $\xi \in \partial \mathcal{G}$ if U is the restriction to \mathcal{G} of a neighbourhood of ξ in $\bar{\mathcal{G}}$.

A subgroup of a hyperbolic group is said to be *elementary* if it is virtually cyclic; elementary subgroups of hyperbolic groups are exactly the amenable subgroups (see e.g. [13, Theorem 38, p. 21]). This notion has a natural generalisation in view of the following fact:

Proposition 8. *Let X be a hyperbolic proper geodesic metric space and $H < \text{Isom}(X)$ any subgroup. The following are equivalent:*

- (i) H fixes a probability measure on ∂X .
- (ii) X preserves a compact set in X or a point in ∂X or a pair of points in ∂X .

Moreover, if X has at most exponential growth (e.g. if it is a graph of bounded valency or if $\text{Isom}(X)$ acts cocompactly), these are equivalent to.

- (iii) The closure of H in $\text{Isom}(X)$ is an amenable group.

For a proof see Section 5.1; we say that H is *elementary* if it satisfies the equivalent conditions (i) and (ii).

3. The homological bicombing and exponential convergence

Let \mathcal{G} be a hyperbolic graph of bounded valency.

In this section, we elaborate on the construction of a homological bicombing from [22], and establish its further properties which we need for exponential convergence. For every $a, b \in \mathcal{G}^{(0)}$, let $S_{a,b}$ be the (finite) set of all geodesic paths from a to b . We fix once and for all a choice $p[a, b] \in S_{a,b}$ for every vertices $a, b \in \mathcal{G}^{(0)}$. We will view $p[a, b]$ both as a path and as a 1-chain. By $p[a, b](r)$ we mean the point on $p[a, b]$ (viewed as a path) at distance r from a . Contrary to the situation in [22], we cannot assume p to be equivariant; we have however an H -equivariant map $p' : \mathcal{G}^{(0)} \times \mathcal{G}^{(0)} \rightarrow C_{00}(E)$ defined by

$$p'[a, b] := |S_{a,b}|^{-1} \sum_{s \in S_{a,b}} s.$$

(So in particular $p'[a, b](r)$ will be viewed as a 0-chain.) This p' will be used instead of p in the construction in order to get equivariance; however, the non-equivariant choice of geodesics p fixed above will be used as auxilliary tool in some proofs.

Proposition 9 (Proposition 7 in Mineyev [22]). *There exists a function $\bar{f} : \mathcal{G}^{(0)} \times \mathcal{G}^{(0)} \rightarrow C_{00}(\mathcal{G}^{(0)})$ with the following properties:*

- (i) $\bar{f}(b, a)$ is a convex combination, that is its coefficients are non-negative and sum up to 1.
- (ii) If $d(a, b) \geq 10\delta$, then $\text{supp } \bar{f}(b, a) \subseteq \bar{B}(s(10\delta), 8\delta)$ for every $s \in S_{b,a}$.
- (iii) If $d(a, b) \leq 10\delta$, then $\bar{f}(b, a) = a$.
- (iv) \bar{f} is H -equivariant, i.e. $\bar{f}(hb, ha) = h\bar{f}(b, a)$ for any $a, b \in \mathcal{G}^{(0)}$ and $h \in H$.
- (v) There exist constants $L \in [0, \infty)$ and $\lambda \in [0, 1)$ such that, for any $a, a', b \in \mathcal{G}^{(0)}$,

$$\|\bar{f}(b, a) - \bar{f}(b, a')\|_1 \leq L\lambda^{(a|a')_b}.$$

- (vi) There exists a constant $\lambda' \in [0, 1)$ such that if $a, b, b' \in \mathcal{G}^{(0)}$ satisfy $(a|b')_b \leq 10\delta$ and $(a|b)_{b'} \leq 10\delta$, then

$$\|\bar{f}(b, a) - \bar{f}(b', a)\|_1 \leq 2\lambda'.$$

- (vii) Let $a, b, c \in \mathcal{G}^{(0)}$, $s \in S_{a,b}$, and let $c \in N(s, 9\delta)$. Then $\text{supp } \bar{f}(c, a) \subseteq N(s, 9\delta)$.

About the proof. (a) The construction of $\bar{f}(a, b)$ in [22] made use of an equivariant choice of geodesic paths $p[a, b]$. If we replace this p with the above p' , the construction and arguments in [22] can easily be adapted to yield the above \bar{f} ; it is important here that there is a uniform bound on the number of vertices in any ball of a given radius.

(b) In addition, we slightly redefine the function \bar{f} given in [22, Proposition 7] by imposing the above condition (iii). \square

3.1. The homological bicombings q' and q

Taking into account that we use p' instead of a choice of geodesics, the homological bicombing $q' : \mathcal{G}^{(0)} \times \mathcal{G}^{(0)} \rightarrow C_{00}(E)$ constructed in [22] can now be defined as follows. Set $q'[a, a] := 0$; then,

inductively on $d(a, b)$,

$$q'[a, b] := q'[a, \bar{f}(b, a)] + p'[\bar{f}(b, a), b].$$

We recall that the word (homological) *bicombing* refers to the property

$$\partial q'[a, b] = b - a \quad \forall a, b \in \mathcal{G}^{(0)}. \tag{1}$$

Proposition 10 (Proposition 8 in Mineyev [22]). *The homological bicombing q' above satisfies the following conditions:*

- (i) q' is Γ -equivariant, i.e. $\gamma q'[a, b] = q'[\gamma a, \gamma b]$ for $a, b \in \mathcal{G}^{(0)}$ and $\gamma \in \Gamma$.
- (ii) q' is quasigeodesic. More precisely, for all $a, b \in \mathcal{G}^{(0)}$, the support of $q'[a, b]$ lies in the 27δ -neighbourhood of any geodesic from a to b ; moreover $\|q'[a, b]\|_1 \leq 18\delta d(a, b)$.
- (iii) There exist constants $M \in [0, \infty)$ and $N \in [0, \infty)$ such that, for all $a, b, c \in \mathcal{G}^{(0)}$,

$$\|q'[a, b] - q'[a, c]\|_1 \leq Md(b, c) + N. \quad \square$$

The homological bicombing q is now defined as follows:

$$q[a, b] := \frac{1}{2}(q'[a, b] - q'[b, a]).$$

It was shown in [22] that q has bounded area, that is,

$$\sup_{a, b, c \in \mathcal{G}^{(0)}} \|q[a, b] + q[b, c] + q[c, a]\|_1 < \infty. \tag{2}$$

3.2. More properties of q'

Proposition 11. *For all vertices a, b and any edge e in \mathcal{G} , $|\langle q'[a, b], e \rangle| \leq 2003\delta^2$.*

Proof. Fix a vertex a and an edge e in \mathcal{G} .

(a) First assume $d(a, e_-) \geq 50\delta$.

The definition of q' and Proposition 9(vii) imply that the support of $q'[a, b]$ lies in the 27δ -neighbourhood of any geodesic between a and b . So if a vertex b satisfies $d(a, b) \leq d(a, e_-) - 28\delta$, then the support of $q'[a, b]$ is disjoint from e and $|\langle q'[a, b], e \rangle| = 0$.

Now assume

$$d(a, e_-) - 28\delta \leq d(a, b) \leq d(a, e_-) + 20\delta. \tag{3}$$

In this case we prove the inequality

$$|\langle q'[a, b], e \rangle| \leq 18\delta(d(a, b) - d(a, e_-) + 46\delta) \tag{4}$$

inductively on $d(a, b)$.

If b is such that $d(a, e_-) - 46\delta \leq d(a, b) \leq d(a, e_-) - 28\delta$, then by the argument above,

$$\langle q'[a, b], e \rangle = 0 \leq 18\delta(d(a, b) - d(a, e_-) + 46\delta),$$

which provides for the basis of induction. For the inductive step, if b satisfies (3), then each vertex x in the support of $\tilde{f}(b, a)$ satisfies

$$d(a, e_-) - 46\delta \leq d(a, x) \leq d(a, b) - 1,$$

so the induction hypothesis applies to each such x giving

$$\begin{aligned} |\langle q'[a, b], e \rangle| &= |\langle q'[a, \tilde{f}(b, a)] + p'[\tilde{f}(b, a), b], e \rangle| \leq |\langle q'[a, \tilde{f}(b, a)], e \rangle| + |\langle p'[\tilde{f}(b, a), b], e \rangle| \\ &\leq 18\delta(d(a, b) - 1 - d(a, e_-) + 46\delta) + \|p'[\tilde{f}(b, a), b]\|_1 \\ &\leq 18\delta(d(a, b) - 1 - d(a, e_-) + 46\delta) + 18\delta = 18\delta(d(a, b) - d(a, e_-) + 46\delta). \end{aligned}$$

This proves (4) in the case when (3) holds, so this gives the bound

$$|\langle q'[a, b], e \rangle| \leq 18\delta(d(a, b) - d(a, e_-) + 46\delta) \leq 18\delta(20\delta + 46\delta) \leq 2003\delta^2. \quad (5)$$

Finally, assume $d(a, b) \geq d(a, e_-) + 20\delta$. Inductively on $d(a, b)$ we show that the above bound is preserved. The basis of induction is provided by (5), and for the inductive step note that since the support of $p'[\tilde{f}(b, a), b]$ lies in the ball $\tilde{B}(b, 18\delta)$, it is disjoint from e , so

$$|\langle q'[a, b], e \rangle| = |\langle q'[a, \tilde{f}(b, a)] + p'[\tilde{f}(b, a), b], e \rangle| = |\langle q'[a, \tilde{f}(b, a)], e \rangle| \leq 2003\delta^2.$$

(b) Now assume $d(a, e_-) \leq 50\delta$. If $d(a, b) \leq d(a, e_-) + 20\delta$, then $d(a, b) \leq 50\delta + 20\delta = 70\delta$ and Proposition 10(ii) gives the same bound as in (5):

$$|\langle q'[a, b], e \rangle| \leq \|q'[a, b]\|_1 \leq 18\delta \cdot 48\delta \leq 2003\delta^2.$$

In the case $d(a, b) \geq d(a, e_-) + 20\delta$ this bound is preserved, similarly to (a) above. This finishes the proof of Proposition 11. \square

Lemma 12. *There exist constants $Q \in [0, \infty)$ and $\sigma_0 \in [0, 1)$ such that for all $a, b, b' \in \Gamma$ and any edge e in \mathcal{G} , if $d(b, b') \leq 56\delta$ then*

$$|\langle q'[a, b] - q'[a, b'], e \rangle| \leq (d(b, b') + Q)\sigma_0^{d(e_-, b) + d(e_-, b')}.$$

Proof. First we define the constants. Let $\lambda' \in [0, 1)$ be the constant from Proposition 9(vi). Choose $\sigma_0 \in [0, 1)$ close enough to 1 so that

$$1 - \lambda'\sigma_0^{-148\delta} > 0. \quad (6)$$

Next, choose Q large enough so that

$$2 \cdot 2003\delta^2 \leq Q\sigma_0^{312\delta} \quad \text{and} \quad \lambda' \cdot 56\delta\sigma_0^{-148\delta} \leq Q(1 - \lambda'\sigma_0^{-148\delta}). \quad (7)$$

Then make σ_0 closer to 1, if needed, so that

$$(56\delta + Q)(\sigma_0^{-74\delta} - 1) \leq \sigma_0^{-74\delta}. \quad (8)$$

Note that (6) and (7) still hold for this new σ_0 . The second inequality in (7) rewrites as

$$\lambda'(56\delta + Q)\sigma_0^{-148\delta} \leq Q. \quad (9)$$

Each of these constants depends only on \mathcal{G} . Now we proceed with the proof.

Fix a vertex a and an edge e in \mathcal{G} . We will use induction on $d(a, b) + d(a, b')$ for vertices b and b' .

If e is not in the 28δ -neighbourhood of either $p[a, b]$ or $p[a, b']$, then by Proposition 10(ii), $|\langle q'[a, b] - q'[a, b'], e \rangle| = 0$ and the result follows. If e is in the 28δ -neighbourhood of $p[a, b]$ but not of $p[a, b']$, then e_- is δ -close to $p[b, b']$ and by (7),

$$\begin{aligned} |\langle q'[a, b] - q'[a, b'], e \rangle| &\leq |\langle q'[a, b], e \rangle| \\ &\leq 2003\delta^2 \leq Q\sigma_0^{312\delta} \leq Q\sigma_0^{d(b, b') + 2\delta} \leq Q\sigma_0^{d(e_-, b) + d(e_-, b')}. \end{aligned}$$

Similarly for b and b' interchanged, so from now on we can assume that e is in the 28δ -neighbourhoods of $p[a, b]$ and $p[a, b']$.

First assume that

$$d(a, b) + d(a, b') \leq 2d(a, e_-) + 100\delta. \tag{10}$$

If $d(a, b) > d(a, e_-) + 100\delta$ were true, then $d(a, b') > d(a, b) - 100\delta \geq d(a, e_-)$ and $d(a, b) + d(a, b') > 2d(a, e_-) + 100\delta$, which is a contradiction. Therefore, $d(a, b) \leq d(a, e_-) + 100\delta$. Similarly, we obtain $d(a, b') \leq d(a, e_-) + 100\delta$.

Let u be a vertex on the geodesic $p[a, b']$ closest to e_- . Then

$$\begin{aligned} d(e_-, b') &\leq d(e_-, u) + d(u, b') \leq 28\delta + d(a, b') - d(a, u) \\ &\leq 28\delta + d(a, b') - (d(a, e_-) - 28\delta) \leq d(a, b') - d(a, e_-) + 56\delta \\ &\leq 100\delta + 56\delta = 156\delta. \end{aligned}$$

Similarly, $d(e_-, b) \leq 156\delta$. Then using Proposition 11 and (7),

$$\begin{aligned} |\langle q'[a, b] - q'[a, b'], e \rangle| &\leq |\langle q'[a, b], e \rangle| + |\langle q'[a, b'], e \rangle| \leq 2 \cdot 2003\delta^2 \\ &\leq Q\sigma_0^{312\delta} \leq Q\sigma_0^{d(e_-, b) + d(e_-, b')} \leq (d(b, b') + Q)\sigma_0^{d(e_-, b) + d(e_-, b')}, \end{aligned}$$

which proves the desired inequality assuming (10).

From now on we can assume $d(a, b) + d(a, b') \geq 2d(a, e_-) + 100\delta$. Since $d(b, b') \leq 56\delta$, this implies

$$d(a, b) \geq d(a, e_-) + 20\delta \quad \text{and} \quad d(a, b') \geq d(a, e_-) + 20\delta \tag{11}$$

(by contradiction). Consider the following two cases.

Case 1: $(a|b')_b > 10\delta$ or $(a|b)_{b'} > 10\delta$ (see Fig. 1).

Without loss of generality, $(a|b')_b > 10\delta$, the other case being similar. Let x be an arbitrary vertex in the support of $f(b', a)$ and $v := p[b, a](10\delta)$. Then by Proposition 9(ii) we have

$$\begin{aligned} d(e_-, x) &\geq d(u, v) - 8\delta - 28\delta \geq d(u, b) - 10\delta - 8\delta - 28\delta \\ &= d(u, b) - 46\delta \geq d(e_-, b) - 28\delta - 46\delta = d(e_-, b) - 74\delta. \end{aligned} \tag{12}$$

Inequality (8) implies

$$(d(b, b') + Q)(\sigma_0^{-74\delta} - 1) \leq (56\delta + Q)(\sigma_0^{-74\delta} - 1) \leq \sigma_0^{-74\delta},$$

which is equivalent to

$$(d(b, b') - 1 + Q)\sigma_0^{-74\delta} \leq d(b, b') + Q. \tag{13}$$

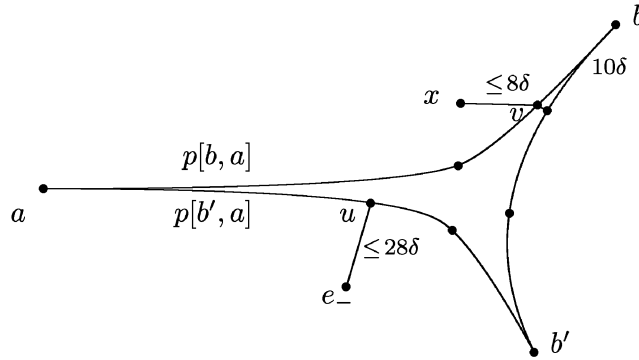


Fig. 1. Case 1.

It is easy to check that

$$d(b', x) \leq d(b, b') - 1 \leq 56\delta \quad \text{and} \quad d(a, x) + d(a, b') < d(a, b) + d(a, b'), \quad (14)$$

therefore the induction hypotheses are satisfied for the vertices x and b' . Using the induction hypotheses, (12), (14) and (13), we have

$$\begin{aligned} |\langle q'[a, b'] - q'[a, x], e \rangle| &\leq (d(b', x) + Q)\sigma_0^{d(e_-, x) + d(e_-, b')} \\ &\leq (d(b, b') - 1 + Q)\sigma_0^{d(e_-, b) - 74\delta + d(e_-, b')} \\ &= (d(b, b') - 1 + Q)\sigma_0^{-74\delta} \sigma_0^{d(e_-, b) + d(e_-, b')} \\ &\leq (d(b, b') + Q)\sigma_0^{d(e_-, b) + d(e_-, b')}. \end{aligned} \quad (15)$$

By (11), e is disjoint from $\text{supp}(p'[x, b])$ for each $x \in \text{supp}(\bar{f}(b, a))$, therefore e is disjoint from $\text{supp}(p'[\bar{f}(b, a), b])$. Since $q'[a, x]$ is linear in x , (15) implies the desired inequality:

$$\begin{aligned} |\langle q'[a, b] - q'[a, b'], e \rangle| &= |\langle q'[a, \bar{f}(b, a)] + p'[\bar{f}(b, a), b] - q'[a, b'], e \rangle| \\ &= |\langle q'[a, \bar{f}(b, a)] - q'[a, b'], e \rangle| \leq (d(b, b') + Q)\sigma_0^{d(e_-, b) + d(e_-, b')}. \end{aligned}$$

Case 2: $(a|b')_b \leq 10\delta$ and $(a|b)_{b'} \leq 10\delta$.

In particular, $d(b, b') = (a|b')_b + (a|b)_{b'} \leq 20\delta$. Since

$$d(a, b) + d(a, b') \geq 2d(a, e_-) + 100\delta > 40\delta,$$

one can easily see that $d(a, b) > 10\delta$ and $d(a, b') > 10\delta$.

The 0-chain $\bar{f}(b, a) - \bar{f}(b', a)$ has the form $f_+ - f_-$, where f_+ and f_- are 0-chains with non-negative coefficients and disjoint supports. By Proposition 9(6),

$$|f_+|_1 + |f_-|_1 = |f_+ - f_-|_1 = |\bar{f}(b, a) - \bar{f}(b', a)|_1 \leq 2\lambda',$$

where $\lambda' \in [0, 1)$ is independent of a, b, b' . The coefficients of the 0-chain $f_+ - f_- = \bar{f}(b, a) - \bar{f}(b', a)$ sum up to 0, because $\bar{f}(b, a)$ and $\bar{f}(b', a)$ are convex combinations. It follows that

$$|f_+|_1 = |f_-|_1 \leq \lambda'. \quad (16)$$

Also,

$$\text{supp } f_+ \subseteq \text{supp } \bar{f}(b, a) \subseteq \bar{B}(p[b, a](10\delta), 8\delta)$$

and

$$\text{supp } f_- \subseteq \text{supp } \bar{f}(b', a) \subseteq \bar{B}(p[b', a](10\delta), 8\delta),$$

hence, by the hypotheses of Case 2, for all $x \in \text{supp } f_+$ and $x' \in \text{supp } f_-$,

$$d(x, x') \leq d(x, b) + d(b, b') + d(b', x') \leq 18\delta + 20\delta + 18\delta = 56\delta.$$

Also $d(a, x) + d(a, x') < d(a, b) + d(a, b')$, so all such pairs x and x' satisfy the induction hypotheses. Similar to (12), we obtain

$$d(e_-, x) \geq d(e_-, b) - 74\delta \quad \text{and} \quad d(e_-, x') \geq d(e_-, b') - 74\delta,$$

then using the induction hypotheses,

$$\begin{aligned} |\langle q'[a, x] - q'[a, x'], e \rangle| &\leq (d(x, x') + Q)\sigma_0^{d(e_-, x) + d(e_-, x')} \\ &\leq (56\delta + Q)\sigma_0^{d(e_-, b) - 74\delta + d(e_-, b') - 74\delta} \\ &= (56\delta + Q)\sigma_0^{-148\delta} \sigma_0^{d(e_-, b) + d(e_-, b')}. \end{aligned} \tag{17}$$

Using (11) we see that e is disjoint from $p'[\bar{f}(b, a), b]$ and $p'[\bar{f}(b', a), b']$, then by linearity of $q'[a, x]$ in x , (16), (17), and (9),

$$\begin{aligned} &|\langle q'[a, b] - q'[a, b'], e \rangle| \\ &\leq |\langle q'[a, \bar{f}(b, a)] + p'[\bar{f}(b, a), b] - q'[a, \bar{f}(b', a)] - p'[\bar{f}(b', a), b'], e \rangle| \\ &\leq |\langle q'[a, \bar{f}(b, a)] - q'[a, \bar{f}(b', a)], e \rangle| \leq |\langle q'[a, f_+] - q'[a, f_-], e \rangle| \\ &\leq \lambda' \cdot (56\delta + Q)\sigma_0^{-148\delta} \sigma_0^{d(e_-, b) + d(e_-, b')} \leq Q\sigma_0^{d(e_-, b) + d(e_-, b')} \\ &\leq (d(b, b') + Q)\sigma_0^{d(e_-, b) + d(e_-, b')}, \end{aligned}$$

which is the desired inequality. This finishes the proof of Lemma 12. \square

Proposition 13. *There exist constants $S_1 \in [0, \infty)$ and $\sigma_1 \in [0, 1)$ such that, for all vertices a, b, b' with $d(b, b') \leq 1$ and any edge e in \mathcal{G} ,*

$$|\langle q'[a, b] - q'[a, b'], e \rangle| \leq S_1 \sigma_1^{d(e_-, b)}.$$

Proof. Immediate from Lemma 12: since $d(b, b') \leq 56\delta$, we have

$$\begin{aligned} |\langle q'[a, b] - q'[a, b'], e \rangle| &\leq (d(b, b') + Q)\sigma_0^{d(e_-, b) + d(e_-, b')} \\ &\leq (1 + Q)\sigma_0^{d(e_-, b) + d(e_-, b) - 1} = (1 + Q)\sigma_0^{-1} (\sigma_0^2)^{d(e_-, b)}, \end{aligned}$$

so we set $S_1 := (1 + Q)\sigma_0^{-1}$ and $\sigma_1 := \sigma_0^2$. \square

Now we would like to have a similar result with the roles of a 's and b 's interchanged (Proposition 15). Since the construction of $q'[a, b]$ is not symmetric in a and b , a different argument is required. We start with the following lemma.

Lemma 14. *There exist constants $K \in [0, \infty)$ and $\lambda \in [0, 1)$ such that for all vertices a, a', b with $d(a, a') \leq 1$ and any edge e in \mathcal{G} , the following conditions hold:*

- (i) *If $d(a, b) \leq d(a, e_-) - 30\delta$, then $\langle q'[a, b] - q'[a', b], e \rangle = 0$.*
- (ii) *If $d(a, b) \geq d(a, e_-) - 60\delta$, then*

$$|\langle q'[a, b] - q'[a', b], e \rangle| \leq K(\lambda^{d(a, e_-) - 60\delta} + \lambda^{d(a, e_-) - 60\delta + 1} + \dots + \lambda^{d(a, b)}).$$

Proof. Let $\lambda \in [0, 1)$ and $L \in [0, \infty)$ be the constants from Proposition 9(v), and choose K large enough so that

$$2 \cdot 2003\delta^2 \leq K \quad \text{and} \quad 2L\lambda^{-1} \cdot 2003\delta^2 \leq K. \tag{18}$$

Fix vertices a and a' with $d(a, a') \leq 1$ and an edge e in \mathcal{G} .

(i) Assume that b satisfies $d(a, b) \leq d(a, e_-) - 30\delta$. By Proposition 10(ii), $q'[a, b]$ and $q'[a', b]$ lie in the 27δ -neighbourhoods of the geodesics $p[a, b]$ and $p[a', b]$, respectively. It is easy to see that e is disjoint from these neighbourhoods, so (i) follows.

(ii) Assume that b satisfies $d(a, b) \geq d(a, e_-) - 60\delta$.

(a) First suppose that $d(a, e_-) \leq 60\delta$. Then by Proposition 11 and (18),

$$\begin{aligned} |\langle q'[a, b] - q'[a', b], e \rangle| &\leq |\langle q'[a, b], e \rangle| + |\langle q'[a', b], e \rangle| \leq 2 \cdot 2003\delta^2 \\ &\leq K \leq K\lambda^{d(a, e_-) - 60\delta} \leq K(\lambda^{d(a, e_-) - 60\delta} + \lambda^{d(a, e_-) - 60\delta + 1} + \dots + \lambda^{d(a, b)}). \end{aligned}$$

(b) Now suppose that $d(a, e_-) \geq 60\delta$. We use induction on $d(a, b)$ to show (ii). Part (i) provides the base of induction: if b satisfies

$$d(a, e_-) - 60\delta \leq d(a, b) \leq d(a, e_-) - 30\delta,$$

then obviously

$$|\langle q'[a, b] - q'[a', b], e \rangle| = 0 \leq K(\lambda^{d(a, e_-) - 60\delta} + \lambda^{d(a, e_-) - 60\delta + 1} + \dots + \lambda^{d(a, b)}).$$

For the inductive step, assume that b satisfies $d(a, b) \geq d(a, e_-) - 30\delta$. We have

$$\begin{aligned} (a|a')_b &= \frac{1}{2}(d(a, b) + d(a', b) - d(a, a')) \\ &\geq \frac{1}{2}(d(a, b) + d(a, b) - 1 - 1) = d(a, b) - 1. \end{aligned} \tag{19}$$

Since, for any 0-chain f , $p'[f, b]$ is a convex combination of geodesics, and any edge occurs in any geodesic at most once, it follows that

$$|\langle p'[f, b], e \rangle| \leq \|f\|_1. \tag{20}$$

Represent the 0-chain $\bar{f}(b, a) - \bar{f}(b, a')$ as $f_+ - f_-$ where f_+ and f_- have non-negative coefficients and disjoint supports. Then there exists a 0-chain f_0 with non-negative coefficients such that

$$\bar{f}(b, a) = f_0 + f_+, \quad \bar{f}(b, a') = f_0 + f_- \quad \text{and} \quad \|f_0\|_1 \leq 1.$$

Moreover, by Proposition 9(v),

$$\|f_-\|_1 = \|f_+\|_1 = \frac{1}{2} \|\bar{f}(b, a) - \bar{f}(b, a')\|_1 \leq \frac{1}{2} L \lambda^{(a|a')b}. \quad (21)$$

By Proposition 9(ii),

$$\text{supp } f_0 = \text{supp}(\bar{f}(b, a)) \cap \text{supp}(\bar{f}(b, a')) \subseteq \bar{B}(p[b, a](10\delta), 8\delta)$$

therefore each vertex $x \in \text{supp } f_0$ satisfies $d(a, x) < d(a, b)$, so by the induction hypotheses,

$$\begin{aligned} |\langle q'[a, x] - q'[a', x], e \rangle| &\leq K(\lambda^{d(a, e_-) - 60\delta} + \dots + \lambda^{d(a, x)}) \\ &\leq K(\lambda^{d(a, e_-) - 60\delta} + \dots + \lambda^{d(a, b) - 1}). \end{aligned} \quad (22)$$

Using (18)–(21) we obtain

$$\begin{aligned} &|\langle q'[a, b] - q'[a', b], e \rangle| \\ &= |\langle q'[a, \bar{f}(b, a)] + p'[\bar{f}(b, a), b] - q'[a', \bar{f}(b, a')] - p'[\bar{f}(b, a'), b], e \rangle| \\ &= |\langle q'[a, \bar{f}(b, a)] - q'[a', \bar{f}(b, a')] + p'[\bar{f}(b, a) - \bar{f}(b, a'), b], e \rangle| \\ &= |\langle q'[a, f_0 + f_+], e \rangle - \langle q'[a, f_0 + f_-], e \rangle + \langle p'[f_+ - f_-, b], e \rangle| \\ &\leq |\langle q'[a, f_0] - q'[a', f_0], e \rangle| + |\langle q'[a, f_+], e \rangle| + |\langle q'[a, f_-], e \rangle| \\ &\quad + |\langle p'[f_+, b], e \rangle| + |\langle p'[f_-, b], e \rangle| \\ &\leq \|f_0\|_1 \cdot K(\lambda^{d(a, e_-) - 60\delta} + \dots + \lambda^{d(a, b) - 1}) \\ &\quad + \|f_+\|_1 \cdot 2003\delta^2 + \|f_-\|_1 \cdot 2003\delta^2 + \|f_+\|_1 + \|f_-\|_1 \\ &\leq K(\lambda^{d(a, e_-) - 60\delta} + \dots + \lambda^{d(a, b) - 1}) + \|f_+\|_1 \cdot 2003\delta^2 \cdot 4 \\ &\leq K(\lambda^{d(a, e_-) - 60\delta} + \dots + \lambda^{d(a, b) - 1}) + \frac{1}{2} L \lambda^{(a|a')b} \cdot 2003\delta^2 \cdot 4 \\ &\leq K(\lambda^{d(a, e_-) - 60\delta} + \dots + \lambda^{d(a, b) - 1}) + 2L \lambda^{d(a, b) - 1} \cdot 2003\delta^2 \\ &\leq K(\lambda^{d(a, e_-) - 60\delta} + \dots + \lambda^{d(a, b) - 1}) + K \lambda^{d(a, b)} \\ &\leq K(\lambda^{d(a, e_-) - 60\delta} + \dots + \lambda^{d(a, b)}). \end{aligned}$$

This proves Lemma 14. \square

Proposition 15. *There exist constants $S_2 \in [0, \infty)$ and $\sigma_2 \in [0, 1)$ such that, for all vertices a, a', b with $d(a, a') \leq 1$ and any edge e in \mathcal{G} ,*

$$|\langle q'[a, b] - q'[a', b], e \rangle| \leq S_2 \sigma_2^{d(a, e_-)}.$$

Proof. Immediate from Lemma 14: if $d(a, b) \leq d(a, e_-) - 30\delta$, then the statement obviously follows, and if $d(a, b) \geq d(a, e_-) - 60\delta$, then

$$\begin{aligned} |\langle q'[a, b] - q'[a', b], e \rangle| &\leq K(\lambda^{d(a, e_-) - 60\delta} + \dots + \lambda^{d(a, b)}) \\ &\leq K \lambda^{d(a, e_-) - 60\delta} \sum_{i=0}^{\infty} \lambda^i = \frac{K \lambda^{-60\delta}}{1 - \lambda} \lambda^{d(a, e_-)}. \end{aligned}$$

So we set $S_2 := K \lambda^{-60\delta} / (1 - \lambda)$ and $\sigma_2 := \lambda$. \square

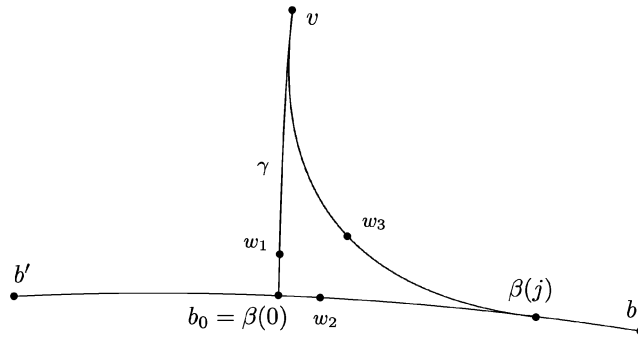


Fig. 2. Illustration for Lemma 16.

Our goal is to combine Propositions 13 and 15 into one stronger statement (Theorem 17).

Let $v, b, b' \in \mathcal{G}^{(0)}$, and let $\beta: J \rightarrow \mathcal{G}$ be a geodesic (i.e. an isometric embedding of an interval J) connecting b to b' . Abusing the notation we will identify β with the image of β . Pick a distance-minimizing vertex $b_0 \in \beta$, i.e. such that $d(v, b_0) = d(v, \beta)$. Let $\gamma = [v, b_0]$ be any geodesic between a_0 and b_0 , then γ is a shortest geodesic connecting v to β . Choose the interval J in \mathbf{R} containing 0 so that $\beta(0) = b_0, \beta(d(b_0, b)) = b, \beta(-d(b_0, b')) = b'$.

Lemma 16. *With the above notations, for all $j \in J$,*

$$d(v, \beta(j)) \geq d(v, \beta) + |j| - 2\delta.$$

Proof. From symmetry, it suffices to show the lemma when $j \geq 0$. Inscribe a triple of points in the triangle $\{v, b_0, \beta(j)\}$ as shown in Fig. 2. It is an easy exercise to see from the figure that $d(b_0, w_2) = d(b_0, w_1) \leq \delta$, therefore

$$\begin{aligned} d(v, \beta(j)) &= d(v, w_3) + d(w_3, \beta(j)) = d(v, w_1) + d(w_2, \beta(j)) \\ &\geq (d(v, b_0) - \delta) + (d(b_0, \beta(j)) - \delta) = d(v, \beta) + |j| - 2\delta. \quad \square \end{aligned}$$

We recall that q was defined by

$$q[a, b] = \frac{1}{2}(q'[a, b] - q'[b, a]) \tag{23}$$

and proceed to state the exponential convergence:

Theorem 17. *Let q be the above bicombing. Then there exist constants $S \in [0, \infty)$ and $\sigma \in [0, 1)$ such that, for all vertices a, a', b, b' and any edge e in \mathcal{G} ,*

$$|\langle q[a, b] - q[a', b'], e \rangle| \leq S(\sigma^{(a|a')e_-} + \sigma^{(b|b')e_-}).$$

Proof. In view of (23) and by symmetry, it is enough to prove the statement of the theorem for q' instead of q . Draw geodesics β and γ as in Fig. 2, where we take $v := e_-$. Similarly to (19) we

obtain

$$(\beta(j)|\beta(j+1))_{e_-} \geq d(e_-, \beta(j)) - 1.$$

Using Proposition 13 and Lemma 16,

$$\begin{aligned} |\langle q'[a, b] - q'[a, b_0], e \rangle| &= \left| \left\langle \sum_{j=0}^{d(b_0, b)-1} q'[a, \beta(j)] - q'[a, \beta(j+1)], e \right\rangle \right| \\ &\leq \sum_{j=0}^{\infty} |\langle q'[a, \beta(j)] - q'[a, \beta(j+1)], e \rangle| \leq \sum_{j=0}^{\infty} S_1 \sigma_1^{(\beta(j)|\beta(j+1))_{e_-}} \\ &\leq \sum_{j=0}^{\infty} S_1 \sigma_1^{d(e_-, \beta(j))-1} \leq \sum_{j=0}^{\infty} S_1 \sigma_1^{d(e_-, \beta) + |j| - 2\delta - 1} = \frac{S_1 \sigma_1^{-2\delta - 1}}{1 - \sigma_1} \sigma_1^{d(e_-, \beta)}. \end{aligned}$$

Similarly,

$$|\langle q'[a', b_0] - q'[a', b'], e \rangle| \leq \frac{S_1 \sigma_1^{-2\delta - 1}}{1 - \sigma_1} \sigma_1^{d(e_-, \beta)}.$$

Draw a geodesic α from a to a' , and let $a_0 \in \alpha$ be a vertex nearest to e_- . Interchanging the roles of a 's and b 's, a symmetric argument using Proposition 15 gives

$$|\langle q'[a, b_0] - q'[a_0, b_0], e \rangle| \leq \frac{S_2 \sigma_2^{-2\delta - 1}}{1 - \sigma_2} \sigma_2^{d(e_-, \alpha)} \quad \text{and}$$

$$|\langle q'[a_0, b_0] - q'[a', b_0], e \rangle| \leq \frac{S_2 \sigma_2^{-2\delta - 1}}{1 - \sigma_2} \sigma_2^{d(e_-, \alpha)}.$$

It is an easy exercise using the triangle inequality to see that

$$(a|a')_{e_-} \leq d(e_-, \alpha) \quad \text{and} \quad (b|b')_{e_-} \leq d(e_-, \beta).$$

Combining the bounds above, we obtain

$$\begin{aligned} |\langle q'[a, b] - q'[a', b'], e \rangle| &\leq |\langle q'[a, b] - q'[a, b_0], e \rangle| + |\langle q'[a, b_0] - q'[a_0, b_0], e \rangle| \\ &\quad + |\langle q'[a_0, b_0] - q'[a', b_0], e \rangle| + |\langle q'[a', b_0] - q'[a', b'], e \rangle| \\ &\leq 2 \frac{S_2 \sigma_2^{-2\delta - 1}}{1 - \sigma_2} \sigma_2^{d(e_-, \alpha)} + 2 \frac{S_1 \sigma_1^{-2\delta - 1}}{1 - \sigma_1} \sigma_1^{d(e_-, \beta)} \\ &\leq 2 \frac{S_2 \sigma_2^{-2\delta - 1}}{1 - \sigma_2} \sigma_2^{(a|a')_{e_-}} + 2 \frac{S_1 \sigma_1^{-2\delta - 1}}{1 - \sigma_1} \sigma_1^{(b|b')_{e_-}}. \end{aligned}$$

Now the desired inequality follows if we set

$$\sigma := \max\{\sigma_1, \sigma_2\} \quad \text{and} \quad S := \max \left\{ 2 \frac{S_1 \sigma_1^{-2\delta - 1}}{1 - \sigma_1}, 2 \frac{S_2 \sigma_2^{-2\delta - 1}}{1 - \sigma_2} \right\}.$$

This proves Theorem 17. \square

4. The ideal bicombing and a cocycle

4.1. Extending the bicombing

We fix homological bicombings q', q as above. We proceed now to extend q to infinity.

Since $\overline{\mathcal{G}} = \mathcal{G} \sqcup \partial\mathcal{G}$ is compact, it has an *unique* uniform structure \mathcal{U} compatible with its topology. We shall use the general principle that uniformly continuous maps defined on a dense subset extend continuously if the range is a complete Hausdorff uniform space [5, Part II, Section 3, No. 6]. For a basepoint $x_0 \in \mathcal{G}^{(0)}$, the restriction of \mathcal{U} to $\mathcal{G}^{(0)}$ is given by the base of entourages

$$\{(x, y) \in \mathcal{G}^{(0)} \times \mathcal{G}^{(0)} : x = y \text{ or } (x|y)_{x_0} > r\}, \quad r > 0.$$

Therefore, as this holds for any x_0 , Theorem 17 states that for all $e \in E$ the function $\langle q[\cdot, \cdot], e \rangle$ is uniformly continuous in both variables, and thus uniformly continuous on $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ for the product uniform structure. Recall that any topological vector space has a canonical uniform structure. The preceding comments amount to saying that the map $q : \mathcal{G}^{(0)} \times \mathcal{G}^{(0)} \rightarrow \ell^\infty(E)$ is uniformly continuous for the uniform structure corresponding to the topology of pointwise convergence. This topology is generally weaker than the weak-*topology, but they coincide on norm-bounded sets (as follows e.g. from the Banach-Alaoglu Theorem [31, I.3.15]). By Proposition 11 the map q' , hence also q , does indeed range in the ball B of radius $2003\delta^2$. But B is weak-*complete and $\mathcal{G}^{(0)}$ is dense in $\mathcal{G}^{(0)} \sqcup \partial\mathcal{G}$, so we conclude that q extends (uniquely) to a weak-*continuous map:

$$q : (\mathcal{G}^{(0)} \sqcup \partial\mathcal{G}) \times (\mathcal{G}^{(0)} \sqcup \partial\mathcal{G}) \rightarrow \ell^\infty(E).$$

We claim that the restriction of q to $\partial\mathcal{G}$ is an ideal bicombing as sought for Theorem 2:

Equivariance follows from the uniqueness of the extension. It is easy to check that the image of a closed $\|\cdot\|_1$ -ball in ℓ^1 under the inclusion $\ell^1 \rightarrow \ell^\infty$ is weak-*closed in ℓ^∞ . Therefore, bounded area follows from fact (2) that q has bounded area on $\mathcal{G}^{(0)}$. To be quasigeodesic is a pointwise condition, so it follows from weak-*continuity and the corresponding property on $\mathcal{G}^{(0)}$, since we may in particular write $q[\xi, \eta]$ at the weak-*limit of $q[a_n, b_n]$ where a_n, b_n tend to ξ, η along a fixed geodesic connecting ξ to η (unless $\xi = \eta$, in which case q vanishes anyway); indeed with this choice $q[a_n, b_n]$ is supported in a 27δ -neighbourhood of any such geodesic, by Proposition 10(ii).

Thus it remains only to show that q satisfies the two conditions of Definition 1. For the first condition, fix a vertex x of \mathcal{G} ; then (1) shows that $\partial q[a, b](x)$ vanishes as a, b tend to the boundary. The condition $\partial q[\cdot, \cdot](x) = 0$ involves only a fixed finite set of edges, namely the edges incident to x . Therefore the weak-*continuity of q implies $\partial q[\xi, \eta](x) = 0$ for all $\xi, \eta \in \partial\mathcal{G}$.

Turning to the second condition, let ξ_\pm and V_\pm be as in Definition 1, see Fig. 3. Let D_0 be the set of vertices $x \in \mathcal{G}^{(0)}$ in the 27δ -neighbourhood of some geodesic from ξ_- to ξ_+ such that x is incident to an edge in V_+ as well as to an edge not in V_+ . Since V_+ lies in the complement of a neighbourhood of ξ_- (and since all geodesics connecting ξ_\pm are uniformly close), the set D_0 is finite. Let $D \subseteq \mathcal{G}$ be the finite subgraph spanned by $N(D_0, 1)$. Let a_n^\pm be sequences tending to ξ_\pm along a given geodesic connecting ξ_\pm . Since q is 27δ -quasigeodesic, the condition $\partial q[a_n^-, a_n^+] = a_n^+ - a_n^-$ shows that

$$\text{supp}(\partial q[a_n^-, a_n^+]|_{V_+}) \subseteq D \cup \{a_n^+\} \quad \text{for } n \text{ big enough.} \tag{24}$$

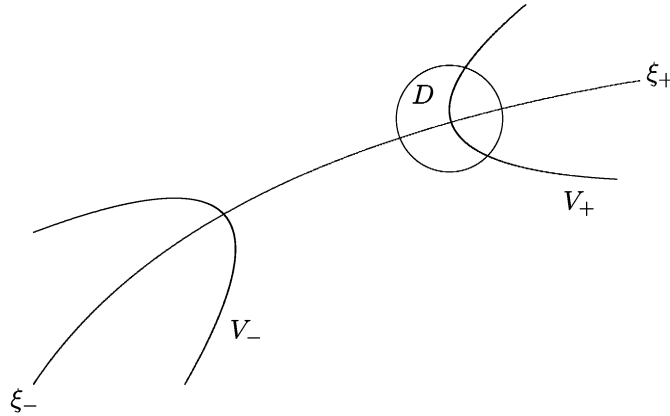


Fig. 3. Illustration for Definition 1(ii).

By the definition of the boundary operator ∂ , the boundary of any 1-chain (on any subgraph of \mathcal{G}) sums to zero. Thus (24) together with $\partial q[a_n^-, a_n^+](a_n^+) = 1$ shows that $\partial(q[\xi_-, \xi_+]|_{V_+})$ is supported on D and sums to -1 . The argument for V_- is the same. This completes the proof of Theorem 2. \square

4.2. A cocycle at infinity

We now construct a non-vanishing cocycle at infinity. The idea is to “double” an ideal bicombing into a map α ranging in functions on pairs of edges in such a way that its coboundary ω never vanishes; this is analogous to a construction given in [24] for trees and $\text{CAT}(-1)$ spaces (compare [35]).

Theorem 18. *Let $\mathcal{G} = (\mathcal{G}^{(0)}, E)$ be a hyperbolic graph of bounded valency. Then there is a weak- $*$ continuous $\text{Isom}(X)$ -equivariant alternating cocycle*

$$\omega : (\partial\mathcal{G})^3 \rightarrow \ell^1(E \times E)$$

vanishing nowhere on $\partial^3\mathcal{G}$.

Remark 19. Such a cocycle will have uniformly bounded ℓ^1 -norm on $(\partial\mathcal{G})^3$ since the latter is compact and since weak- $*$ compact sets are always bounded in norm [8, Part II, Section 3]. In the construction below, however, the boundedness is granted anyway because we use an ideal bicombing of bounded area.

First, an independent basic lemma on ideal bicomblings which shows that they indeed “connect” points at infinity:

Lemma 20. *Let q be an ideal quasigeodesic bicombing. There is a constant D such that for every distinct $\xi_-, \xi_+ \in \partial\mathcal{G}$, every geodesic γ from ξ_- to ξ_+ and every vertex $x \in \gamma$ there is an edge e in $\bar{B}(x, D)$ with $\langle q[\xi_-, \xi_+], e \rangle \neq 0$.*

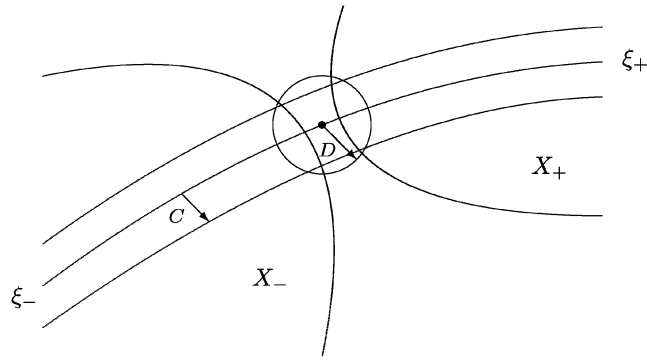


Fig. 4. Illustration for Lemma 20.

Proof. Let C be a constant as in the definition of quasigeodesic ideal bicombings and fix $D \geq C + 2 + 3\delta$ (Fig. 4). Given a geodesic $\gamma: \mathbf{R} \rightarrow \mathcal{G}$ and $t \in \mathbf{R}$, let $X^+ \subseteq E$ (respectively X^-) be the set of edges $e \in E$ such that some $s \geq t + D - C - 1$ (resp. some $s \leq t - (D - C - 1)$) realizes the minimum over \mathbf{R} of the function $s \mapsto d(\gamma(s), e)$. Then $X^+ \cup X^- \cup \bar{B}(\gamma(t), D)$ covers $N(\gamma, C)$. On the other hand, by δ -hyperbolicity, $X^+ \cap X^- \neq \emptyset$. Apply this to any geodesic as in the statement with $x = \gamma(t)$; assuming for a contradiction $q[\xi_-, \xi_+]|_{\bar{B}(x, D)} = 0$, we deduce $\partial(q[\xi_-, \xi_+]|_{X^\pm}) = 0$ since $0 = \partial(q[\xi_-, \xi_+]) = \partial(q[\xi_-, \xi_+]|_{X^-}) + \partial(q[\xi_-, \xi_+]|_{X^+})$ and $X^+ \cap X^- = \emptyset$. This contradicts the definition of an ideal bicombing because it can be readily checked that X^\pm is a graph neighbourhood of ξ_\pm using δ -hyperbolicity. \square

Proof of Theorem 18. Let q be an equivariant weak-*continuous quasigeodesic ideal bicombing of bounded area, as granted by Theorem 2, and let C be as above. We may assume q alternating upon replacing $q[\xi, \eta]$ with $(q[\xi, \eta] - q[\eta, \xi])/2$ (the particular bicombing q constructed above is alternating anyway). Given a constant R we define $\alpha: (\partial\mathcal{G})^2 \rightarrow \ell^\infty(E \times E)$ by

$$\alpha(\xi, \eta)(e, e') = \begin{cases} \langle q[\xi, \eta], e \rangle \langle q[\xi, \eta], e' \rangle & \text{if } d(e, e') \leq R, \\ 0 & \text{otherwise.} \end{cases}$$

Now we set $\omega = d\alpha$, that is, by alternation, $\omega(\xi, \eta, \zeta) := \alpha(\xi, \eta) + \alpha(\eta, \zeta) + \alpha(\zeta, \xi)$. We shall check (i) that ω ranges in $\ell^1(E \times E)$ and even in a closed ball thereof, and (ii) that (for a suitable choice of R) it vanishes nowhere on $\partial^3\mathcal{G}$, as the other properties follow from the definition. In particular, weak-*continuity of ω for ℓ^1 (in duality to C_0) follows from pointwise continuity because it ranges within a closed $\|\cdot\|_1$ -ball (Remark 19 and (i) below); we used this trick for ℓ^∞ in Section 4.1.

(i) By hyperbolicity, there is a constant r such that for any distinct $\xi, \eta, \zeta \in \partial\mathcal{G}$ and any three geodesics joining them, the union of the C -neighbourhoods of the three geodesics can be covered by some ball of radius r together with three subgraphs A_ξ, A_η, A_ζ such that A_ξ intersects only the C -neighbourhoods of the two geodesics leading to ξ —and likewise for A_η, A_ζ . Thus, by the symmetry of the situation, it is enough to show that the restriction of $\omega(\xi, \eta, \zeta)$ to pairs of edges in A_ξ is summable (observe indeed that there are only uniformly finitely many pairs (e, e') with $\omega(\xi, \eta, \zeta)(e, e') \neq 0$ for which e is in A_ξ but e' is not). Let M be a bound on the number of edges

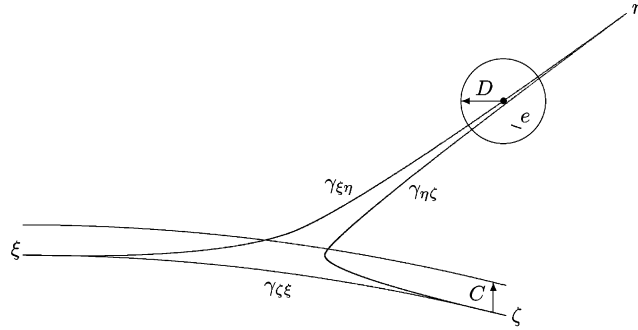


Fig. 5. Point (ii).

in any ball of radius $R + 1$; we have:

$$\begin{aligned} \|\omega(\xi, \eta, \zeta)|_{A_\xi}\|_1 &= \sum_{e, e' \in A_\xi} |\alpha(\xi, \eta)(e, e') - \alpha(\xi, \zeta)(e, e')| \\ &\leq \sum_{e \in A_\xi} \sum_{e' \in N(e, R)} |\langle q[\xi, \eta], e \rangle| \cdot |\langle q[\xi, \eta], e' \rangle - \langle q[\xi, \zeta], e' \rangle| \\ &\quad + \sum_{e' \in A_\xi} \sum_{e \in N(e', R)} |\langle q[\xi, \eta], e \rangle - \langle q[\xi, \zeta], e \rangle| \cdot |\langle q[\xi, \zeta], e' \rangle| \\ &\leq 2M \|q\|_\infty \sum_{e \in A_\xi} |\langle q[\xi, \eta], e \rangle - \langle q[\xi, \zeta], e \rangle| \\ &= 2M \|q\|_\infty \sum_{e \in A_\xi} |\langle q[\xi, \eta], e \rangle + \langle q[\eta, \zeta], e \rangle + \langle q[\zeta, \xi], e \rangle|. \end{aligned}$$

The last term is finite and uniformly bounded since q has bounded area.

(ii) Let D be as in Lemma 20 and choose $L > 2(C + D + \delta)$. We claim that whenever $R > 2L + 2D$ the cocycle ω vanishes nowhere on $\partial^3 \mathcal{G}$. Indeed, choose any three distinct points ξ, η, ζ in $\partial \mathcal{G}$. Let $\gamma_{\xi\eta}$ be a geodesic from ξ to η , and similarly $\gamma_{\eta\zeta}, \gamma_{\xi\zeta}$ (see Fig. 5). There is $t \in \mathbf{R}$ such that $\gamma_{\xi\eta}(t)$ is at distance at most δ from $\gamma_{\eta\zeta}$ as well as from $\gamma_{\xi\zeta}$. By Lemma 20, we can find an edge e in the ball $\bar{B}(\gamma_{\xi\eta}(t + L), D)$ with $\langle q[\xi, \eta], e \rangle \neq 0$. Likewise, there is an edge e' in $\bar{B}(\gamma_{\xi\eta}(t - L), D)$ with $\langle q[\xi, \eta], e' \rangle \neq 0$. However, by the choice of t and L we have $\bar{B}(\gamma_{\xi\eta}(t + L), D) \cap N(\gamma_{\xi\zeta}, C) = \emptyset$ and thus $\langle q[\zeta, \xi], e \rangle = 0$. Likewise, $\langle q[\eta, \zeta], e' \rangle = 0$. Therefore, $\omega(\xi, \eta, \zeta)(e, e') = \langle q[\xi, \eta], e \rangle \langle q[\xi, \eta], e' \rangle$ is non-zero because $d(e, e') < R$. \square

5. Superrigidity and remaining proofs

One of the reasons why one has to work on the boundary at infinity is that there is no known technique to show that general cocycles defined on the group itself yield non-trivial classes in

bounded cohomology when one is dealing with arbitrary coefficients. Extending cocycles to infinity settles this problem, see 5.2 below.

5.1. More general spaces

In order to tackle more general hyperbolic spaces, we establish the following alternative:

Theorem 21. *Let X be a proper geodesic hyperbolic space and $H < \text{Isom}(X)$ a non-elementary closed subgroup acting cocompactly on X . Then either:*

- (i) H has a proper non-elementary vertex-transitive action on a hyperbolic graph of bounded valency (thus quasi-isometric to X), or
- (ii) there is a finite index open subgroup $H^* \triangleleft H$ and a compact normal subgroup $K \triangleleft H$ contained in H^* such that H^*/K is (isomorphic to) a connected simple Lie group of rank one.

We first give the

Proof of Proposition 8. Assume that H fixes a probability measure μ on ∂X . If the support of μ contains at least three points, then H is relatively compact [3, 5.3] and hence preserves a compact set (any orbit of the closure of H) in X . Otherwise, the support of μ is an H -invariant set of one or two points in ∂X . The converse is clear as the stabiliser of a compact set is compact and hence fixes a measure on ∂X .

Condition (iii) always implies (i), and the converse follows from universal amenability of the boundary action, which is granted by a result of Adams [3, 6.8] when X has at most exponential growth. Note that the growth condition is automatically satisfied for graphs of bounded valency, and in the case of a proper geodesic space with cocompact $\text{Isom}(X)$ -action it can be deduced e.g. from Lemma 22 below. \square

For lack of a reference, we prove the following:

Lemma 22. *Let X be a proper geodesic metric space and $H < \text{Isom}(X)$ a closed subgroup. If H acts cocompactly on X , then H is compactly generated.*

Proof. Fix a point $x \in X$. There is $r > 0$ such that $HB(x, r) = X$. It follows, since $\bar{B}(x, 2r)$ is compact, that there is a finite set $L \subseteq H$ such that $\bar{B}(x, 2r) \subseteq LB(x, r)$. Let $C \subseteq H$ be the compact set of elements $h \in H$ such that $hx \in \bar{B}(x, r)$. We claim that H is generated by $C \cup L$. Indeed, the fact that X is geodesic implies by induction on n that $B(x, nr) \subseteq \langle L \rangle B(x, r)$. Thus $\langle L \rangle B(x, r) = X$ and the claim follows. \square

Proof of Theorem 21. Let $K \triangleleft H$ be the maximal amenable normal subgroup; we claim that K is compact. Indeed, K fixes a probability measure on ∂X . If the support F of this measure contains at least three points, then K is compact [3, 5.3]. If on the other hand F contains one or two points, then H cannot preserve the set F since H is non-elementary. Thus there is $h \in H$ such that $hF \cup F$ contains at least three points. But since h normalises K , the set $hF \cup F$ is preserved by K . This implies again that K is compact for the same reason since K fixes the uniform distribution on that set.

Let now $L=H/K$ and let L^0 be its connected component. As explained e.g. in [23, 11.3.4], the solution to Hilbert’s fifth problem and the classification of outer automorphisms of adjoint semi-simple Lie groups implies that there is a finite index (normal) open subgroup $H^* \triangleleft H$ containing K such that H^*/K splits as a direct product $L^0 \times Q$ where Q is totally disconnected and L^0 is a semi-simple connected Lie group without compact factors. Assume first that L^0 is non-compact; we claim that Q is trivial and L^0 simple.

Indeed, otherwise H^*/K would be a direct product of two non-compact groups since H^*/K has no normal compact subgroups (see [23, 11.3.3]). Let $H_1, H_2 < H^*$ be the preimages of these two factors. Since H_1 is non-compact, its limit set $S \subseteq \partial X$ is non-empty. But H_2 acts trivially on S because the H_i commute up to the compact subgroup K ; therefore, S contains only one or two points since otherwise H_2 would be compact. Both H_i preserve S , so that now H^* would be elementary; in view of the first characterisation in Proposition 8, this implies that H is elementary, a contradiction.

Thus, in the case L^0 non-compact, it remains only to see that H^*/K has rank one; this is the case because the associated symmetric space is quasi-isometric to X .

The other case is when L^0 is compact, and hence trivial. Thus $L = H/K$ is totally disconnected; since it is compactly generated by Lemma 22, it admits a locally finite Schreier graph (see e.g. [23, p. 150]). The H -action on this graph is proper and vertex-transitive, hence the graph has bounded valency and is quasi-isometric to X , which implies both hyperbolicity and non-elementarity. \square

We are going to need the existence of a distance-like function for pairs of pairs of points at infinity. Since the invariant cross-ratio construction (as e.g. in [29]) is not continuous even for hyperbolic groups, we construct a continuous invariant analogue by applying the following lemma to $C = \partial X$ and $G < \text{Isom}(X)$:

Lemma 23. *Let G be a locally compact group with a continuous action on a compact second countable space C such that the diagonal action on the space of distinct triples of C is proper. Denote by D_2 the space of subsets of cardinality two in C and by D the space of distinct (not necessarily disjoint) pairs $(a, b) \in D_2 \times D_2$. Then there is a continuous G -invariant function $\delta : D \rightarrow \mathbf{R}_+$ such that $\delta(a, b) = 0$ if and only if $a \cap b \neq \emptyset$.*

Proof. The space C is metrisable, so let ϱ be a metric inducing the topology. For two disjoint elements $a = \{\alpha_1, \alpha_2\}$ and $b = \{\beta_1, \beta_2\}$ of D_2 , let

$$\delta'(a, b) = \left| \ln \frac{\varrho(\alpha_1, \beta_1) \cdot \varrho(\alpha_2, \beta_2)}{\varrho(\alpha_1, \beta_2) \cdot \varrho(\alpha_2, \beta_1)} \right|^{-1}.$$

This extends to a continuous function $\delta' : D \rightarrow \mathbf{R}_+$ such that $\delta'(a, b) = 0$ if and only if $a \cap b \neq \emptyset$; indeed the definition of D_2 excludes the case where both numerator and denominator are zero. Our assumption implies that G acts properly on D . Thus, fixing a left Haar measure on G , there exists a generalized Bruhat function, that is, a continuous function $h : D \rightarrow \mathbf{R}_+$ such that (i) for all $(a, b) \in D$ we have $\int_G h(g^{-1}(a, b)) dg = 1$ and (ii) for every compact subset $L \subset D$ the intersection of the support of h with the saturation GL is compact (see e.g. [23, 4.5.4]). The function $\delta : D \rightarrow \mathbf{R}_+$ defined by

$$\delta(a, b) = \int_G h(g^{-1}(a, b)) \delta'(g^{-1}(a, b)) dg$$

is G -invariant and continuous (for continuity, see e.g. the proof of [23, 4.5.5]; the integrand vanishes outside a compact set). Moreover, $\delta(a, b) > 0$ if a, b are disjoint because of (ii). \square

5.2. Boundary theory and bounded cohomology

We shall prove the following more general form of Theorem 3. For background on (continuous) bounded cohomology, see [6,23].

Theorem 24. *Let \mathcal{G} be a hyperbolic graph of bounded valency and $H < \text{Isom}(\mathcal{G})$ a non-elementary closed subgroup. Then $H_{\text{cb}}^2(H, L^p(H))$ is non-zero for all $1 \leq p < \infty$.*

Using Theorem 21, we can combine Theorem 24 with [24] and deduce:

Corollary 25. *Let X be a hyperbolic proper geodesic metric space such that $\text{Isom}(X)$ acts cocompactly. For any non-elementary closed subgroup $G < \text{Isom}(X)$, the space $H_{\text{cb}}^2(G, L^2(G))$ is non-zero.*

Remark 26. In fact the proof works for coefficients in $L^p(G)$ for all $1 < p < \infty$, but contrary to Theorem 24 the method does not apply to $p = 1$.

Proof of the corollary. Let $H = \text{Isom}(X)$. Since H contains the non-elementary subgroup G , it is itself non-elementary and we may apply Theorem 21. In case (i), we are done by Theorem 24 applied to G . In case (ii), there is a finite index subgroup of G with a proper non-elementary action on a rank one symmetric space, a situation covered by Monod and Shalom [24]. \square

We need the notion of (double) \mathcal{X}^{sep} -ergodicity introduced by Burger–Monod in [6]. Recall that if H is any locally compact σ -compact group acting on a measure space (B, ν) by measurable transformations leaving ν quasi-invariant, then B is called \mathcal{X}^{sep} -ergodic if for every separable coefficient H -module F , any measurable H -equivariant function $f: B \rightarrow F$ is essentially constant. (Here, a *coefficient module* is the contragradient of a continuous isometric representation on a separable Banach space.) Further, B is said *doubly* \mathcal{X}^{sep} -ergodic if the diagonal action on $B \times B$ is \mathcal{X}^{sep} -ergodic. It is proved in [6,23] that if A is compactly generated then a certain Poisson boundary is a doubly ergodic H -space. This has been generalized by Kaimanovich [18] to the σ -compact case. On the other hand, Poisson boundaries are amenable in the sense of Zimmer, see [34]. (For comments about amenable spaces for possibly not second countable groups, see [23, 5.3]; as to measurability issues, all our H -actions considered here factor through a second countable quotient.) Using the cohomological characterization of amenability established in [6,23], it follows that if B is a Poisson boundary as above, then for any separable coefficient H -module F there is a canonical isometric isomorphism

$$H_{\text{cb}}^2(H, F) \cong ZL_{\text{alt}}^\infty(B^3, F)^H, \quad (25)$$

where the right-hand side denotes the space of equivariant essentially bounded measurable alternating cocycles; see [6,23].

Proposition 27. *Let X be a proper hyperbolic geodesic metric space and H a locally compact σ -compact group with a continuous non-elementary action on X . Let B be a doubly ergodic*

amenable H -space. Then (after possibly discarding a null-set in B) there is a Borel H -equivariant map $f : B \rightarrow \partial X$.

Proof. An examination of the proof given in [24] for the $\text{CAT}(-1)$ case (in the more general cocycle setting) shows that the only two ingredients about ∂X that are really needed are (i) the existence of a function δ as in Lemma 23 (for $C = \partial X$ and G the closure in $\text{Isom}(X)$ of the image of H), and (ii) the fact [3, 5.3] that $\text{Isom}(X)$ acts properly on the space of probabilities on ∂X whose support contains at least three points. \square

End of the proof of Theorem 24. Let (B, ν) be a doubly \mathcal{X}^{sep} -ergodic amenable H -space and $f : B \rightarrow \partial X$ as in Proposition 27. Let now $F = \ell^1(E \times E)$ and $\omega : (\partial \mathcal{G})^3 \rightarrow F$ be as in Theorem 18; consider the resulting map $f_* \omega : B^3 \rightarrow F$. We observe that F is a separable coefficient module. The support of $f_* \nu$ is an H -invariant set in $\partial \mathcal{G}$, so that it contains at least three points since H is non-elementary. The Theorem of Fubini–Lebesgue implies that $\partial^3 \mathcal{G}$ is not $(f_* \nu)^3$ -null, hence $f_* \omega$ defines a non-zero element in the right-hand side of (25) because ω is nowhere vanishing on $\partial^3 \mathcal{G}$. We claim that $H_{\text{cb}}^2(H, \ell^1(E))$ is also non-vanishing. Indeed this follows readily from (25) because $\ell^1(E \times E)$ is isomorphic (as coefficient module) to the completed sum of copies of $\ell^1(E)$. Similarly, if $K < H$ is the stabiliser of a given edge in E , the space $H_{\text{cb}}^2(H, \ell^1(H/K))$ is non-zero. Using [23, 11.4.1], and the fact that $\ell^p(H/K)$ is dual even for $p = 1$ since K is open, this implies

$$H_{\text{cb}}^2(H, \ell^p(H/K)) \neq 0 \quad \forall 1 \leq p < \infty. \tag{26}$$

(This is false for $p = \infty$, and indeed [23, 11.4.1] does not apply by lack of separability.) Since K is compact, the submodule $\ell^p(H/K)$ is equivariantly complemented in $L^p(H)$. Now the conclusion follows from [23, 8.2.9]. \square

Observe that the properties of the map of Proposition 27 are the only point of the whole proof where we use non-elementarity. If on the other hand H is elementary, then it is amenable by Proposition 8 and thus $H_{\text{cb}}^2(H, \ell^p(E))$ vanishes altogether since $\ell^p(E)$ is a dual module (compare e.g. [17, 2.5]).

5.3. Other proofs

End of the proof of Theorem 3. The situation of Theorem 3 corresponds to a homomorphism $\pi : \Gamma \rightarrow \text{Isom}(\mathcal{G})$ with $\Gamma_0 := \pi(\Gamma)$ a discrete non-elementary subgroup. Thus Theorem 24 implies that $H_{\text{b}}^2(\Gamma_0, \ell^p(\Gamma_0))$ is non-zero. The conclusion follows because the kernel of π is finite. \square

Proof of Corollary 4. Keep the notation of Corollary 4 and let $H = \text{Isom}(X)$; we may apply Theorem 21 since H contains a non-elementary subgroup, namely the image of Γ . In case (i), we are done by Theorem 3. In case (ii), there is a finite index subgroup of Γ with a proper non-elementary action on a rank one symmetric space. Then the results of [24] imply that Γ is in \mathcal{C}_{reg} . \square

Previously, only two restrictions were known for groups ME to a hyperbolic group: they must have a finite centre [2], and they cannot split as a direct product of two infinite groups [1]. Since the class \mathcal{C}_{reg} is a ME invariant [25], Theorem 3 implies:

Proof of the claims following Corollary 5. Let Γ be any countable group which is ME to a non-elementary (subgroup of a) hyperbolic group and let $A \triangleleft \Gamma$ be any infinite normal subgroup. Since the class \mathcal{C}_{reg} is a ME invariant [25], Theorem 3 implies that Γ is in \mathcal{C}_{reg} . But according to [25] this implies in turn that A is in \mathcal{C}_{reg} . If, on the one hand, A has a normal amenable subgroup N (e.g. its centre), then *any* dual A -module with non-vanishing H_b^2 must have a non-zero vector invariant under N ; see e.g. [26] or [23, 8.5.4]. Applying this to the module $\ell^2(A)$ shows that N must be finite. On the other hand, if $A = A_1 \times A_2$, then any separable dual A -module with non-vanishing H_b^2 must have a non-zero vector invariant under some A_i ; see e.g. [6, Theorem 14; 18]. Again, this shows that the corresponding factor A_i is finite by considering the module $\ell^2(A)$. \square

The normality assumption in the above argument is essential as can be readily seen on the example of $\mathbf{Z} * \mathbf{Z}^2$, which is ME to the hyperbolic group $\mathbf{Z} * \mathbf{Z}$.

Proof of Theorem 7. Keep the notation of Theorem 7 and suppose H non-amenable, so that it is non-elementary (see the Proof of Proposition 8 above). Assume first that we are in case (i) with $X = \mathcal{G} = (\mathcal{G}^{(0)}, E)$. Arguing as in the proof of Theorem 24 but with a doubly \mathcal{X}^{sep} -ergodic amenable A -space (and A replacing H), we deduce that $H_b^2(A, \ell^2(E))$ is non-zero, and thus $H_b^2(A, L^2(H))$ does not vanish either. In case (ii), Theorem 21 allows us to apply results from [24] and deduce again that $H_b^2(A, L^2(H))$ is non-zero. Now the superrigidity formula for bounded cohomology of irreducible lattices [6, Theorem 6] implies that there is a non-zero A -invariant subspace of $L^2(H)$ on which the A -representation extends to a continuous G -representation that factors through $G \rightarrow G_i$ for some i . (In order to relax the compact generation assumption in [6] to σ -compactness, apply [18].) Using the arguments of the proof of Theorem 0.3 in [32], one deduces that there is a compact normal subgroup $K \triangleleft H$ such that the homomorphism $\Gamma \rightarrow H/K$ extends to a continuous homomorphism $G \rightarrow H/K$ factoring through G_i . \square

Remark 28. We mentioned in the introduction that there are uncountably many non-isomorphic subgroups of hyperbolic groups. Here is the sketch of the argument given to us by Kapovich and Schupp; we thank them for communicating it. Let $A = \bigoplus_p \mathbf{Z}/p$ be the sum of all cyclic groups of prime order. Since A is countable and recursively presented, it can be embedded in a finitely presented group Q by a result of Higman's [16]. The construction of Rips in [30] can be modified so as to give a hyperbolic group Γ with a homomorphism π onto Q such that the kernel K of π is perfect:

$$A \xhookrightarrow{i} Q \xleftarrow{\pi} \Gamma \triangleright K.$$

Now observe that for any subgroup $B < A$ the subgroup $\Gamma_B = \pi^{-1}(i(B))$ of Γ has Abelianisation isomorphic to B since K is perfect. Since there is a continuum of non-isomorphic $B < A$ (by taking sums restricted to any set of primes), the conclusion follows.

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