Analysis on groups

Problem Set 7 4 April 2019

Exercise 1
Let G be a group acting on a set X. Prove that the following condition is equivalent to (R_2):
For every $S \subseteq_{\mathrm{f}} G$ and every $\varepsilon > 0$ there is $v \in \ell^2(X)$ with

$$\left\| \frac{1}{|S|} \sum_{s \in S} sv \right\|_2 > (1 - \varepsilon)\|v\|_2.$$

*(*Meditate about the corresponding statement for (R_p), especially when $p = 1$.)*

Exercise 2
Let $G = \{x \mapsto ax + b : a \in \mathbf{R}^*, b \in \mathbf{R}\}$ be the group of affine transformations of $\mathbf{R}$. Consider the G-action on $X = \mathbf{R}$.

(i) Find a Følner set A for $\epsilon = 1/10$ and $S = \{+1, \cdot 2\}$.

(ii) Prove that $\mathbf{Z} \subseteq \mathbf{R}$ contains no such Følner set A.

Exercise 3
Prove that the Reiter property (R_1) implies the Følner property (F).

This follows from the Tarski theorem proved later, but find a direct proof.

Exercise 4*
Let G be a group with an amenable action on a set X. Prove that the Reiter property (R_1) holds.

Again, this follows from the Tarski theorem, but try to find a direct proof.
The first step is to show that there is a net $(\varphi_j)_{j \in J}$ in $\ell^1(X)$ with $\|\varphi_j\|_1 = 1$ and such that for all $g \in G$ the net $g\varphi_j - \varphi_j$ tends to zero weakly.