Analysis on groups

Problem Set 7 6 April 2017

Exercise 0
(i) Let G be a group and $N \lhd G$ a normal subgroup. Prove that G/N is amenable if and only if the following condition holds:

For any G-action on any set X such that the N-action is amenable,
the G-action is amenable too.

(ii) State and prove a generalisation where $N < G$ is not supposed normal.

Exercise 1
Consider $G = \mathbf{Z}^2 \rtimes \mathbf{SL}_2(\mathbf{Z})$ and view $H = \mathbf{SL}_2(\mathbf{Z})$ as a subgroup of G. Prove that H is not co-amenable in G.

Hint: Prove that δ_0 is the only H-invariant mean on $\mathbf{Z}^2$.

Exercise 2
Two elements u, v in a group G are said to generate a "free semigroup" if any two distinct words in u, v define distinct elements of G.
Note: contrary to free groups, we don't use u^{-1} and v^{-1} in these words. In particular, there is no need to worry about non-reduced words.

i) Show that in that case G is not sub-exponential.

ii) Give an example of an amenable group where this happens.

iii) In a group as in (ii), the elements u, v cannot generate a free subgroup (why?). For your choice of u, v, find a reduced word (in u, v, u^{-1}, v^{-1}) which is trivial.

Exercise 3
We denote by SO(3) the group of 3×3 orthogonal (real) matrices with determinant one. We denote by SU(2) the group of 2×2 unitary (complex) matrices with determinant one.

i) Make sure that you understand these definitions and the fact that SO(3) is the group of rotations around the origin in $\mathbf{R}^3$.

ii) Understand the statement and the proof of the fact that there is a surjective homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$. What is the kernel?

For (ii), the best method is perhaps a trip to the library (asking Maxime does not count as a solution).