Analysis on groups

Problem Set 6 30 March 2017

Exercise 0
Let G be an amenable group. Prove that there is $\mu \in \mathscr{M}(G)$ which is invariant under both the left and the right multiplication.

Exercise 1
Let G be a group acting on a set X. Show that the Følner property

(F) $$\forall\, S \subseteq_{\mathrm{f}} G \;\forall\, \epsilon > 0 \;\exists\, A \subseteq_{\mathrm{f}} X \;\forall\, s \in S : |sA \triangle A| < \epsilon |A|$$

is equivalent to:

(F′) $$\forall\, S \subseteq_{\mathrm{f}} G \;\forall\, \epsilon > 0 \;\exists\, A \subseteq_{\mathrm{f}} X : |SA| < (1+\epsilon)|A|.$$

Note: by definition, $SA = \{sa : s \in S, a \in A\}$.

Exercise 2
Let $G = \{x \mapsto ax + b : a \in \mathbf{R}^*, b \in \mathbf{R}\}$ be the group of affine transformations of $\mathbf{R}$. Consider the G-action on $X = \mathbf{R}$.

(i) Find a Følner set A for $\epsilon = 1/10$ and $S = \{+1, \cdot 2\}$.

(ii) Prove that $\mathbf{Z} \subseteq \mathbf{R}$ contains no such Følner set A.

Exercise 3
A group G is called *sub-exponential* if $\lim_{n\to\infty} \sqrt[n]{|C^n|} = 1$ for every finite subset $C \subseteq_{\mathrm{f}} G$, where

$$C^n = \big\{c_1 c_2 \cdots c_n : c_1, c_2, \ldots, c_n \in C\big\}$$

(check that the limit always exists).

(i) Prove that every sub-exponential group is amenable.

(ii) Check that it applies to $G = \mathbf{Z}^d$.

Hint: it might help to replace C by a larger set that you understand better.

***Exercise 4**
Prove that the Reiter property (R_1) implies the Følner property (F).

(Although this follows from later theorems of the lecture, find a direct proof).