Analysis on groups

Problem Set 5

Exercise 0
Make sure that you understood the definition of $\bigoplus_{i \in I} G_i$ for a family $\{G_i\}_{i \in I}$ of groups $G_i$. Prove that $\bigoplus_{i \in I} G_i$ is amenable if all of the $G_i$ are amenable. Compare to Ex. 4 in Set 4.

Exercise 1
(i) Let $G$ be a finite group. Prove that $G^\mathbb{N} := \prod_{n \in \mathbb{N}} G$ is amenable.
(Compare again to Ex. 4 in Set 4.)
(ii) Give an example of an amenable group $G$ such that $G^\mathbb{N}$ is non-amenable.

Exercise 2
Let $G$ be a group. Prove that $G$ contains a maximal normal amenable subgroup, and that it is unique. We call it the amenable radical of $G$ and denote it by $\text{Ramen}(G)$. Show that the amenable radical of $G/\text{Ramen}(G)$ is trivial.

Exercise 3
(i) If needed, learn what a simple group is and what the alternating group $A_n$ is.
(ii) Find out the truth about the simplicity of $A_n$ — using the library, wikipedia or your brain (or a combination of the above).
(iii) Define $A_\infty$ as the increasing union of $A_n$ as $n \to \infty$; this might require some thinking.
(iv) Prove that $A_\infty$ is an infinite simple amenable group.