Analysis on groups

Problem Set 5 23 March 2017

Exercise 0
After our discussion of *unions of groups* in class, you might wonder:
can a group be the union of two (proper) subgroups? of three?

Exercise 1
The goal of this exercise is to show that if we replace naively "compact" by "bounded and closed" in the Fixed Point Criterion, then "amenable" becomes "finite".

Prove that a group G is finite if and only if:
For every topological vector space V with a continuous linear G-action, every non-empty bounded closed convex G-invariant set contains a fixed point.

Hint: for the harder direction, try to think about some basic normed vector space that you can construct using G.

Exercise 2
Let G be a group. Prove that G contains a *maximal* normal amenable subgroup, and that it is unique. We call it the *amenable radical* of G and denote it by $\mathrm{Ramen}(G)$. Show that the amenable radical of $G/\mathrm{Ramen}(G)$ is trivial.

Exercise 3
i) If needed, learn what a *simple* group is and what the *alternating* group A_n is.

ii) Find out the truth about the simplicity of A_n — using the library, wikipedia or your brain (or a combination of the above).

iii) Define A_∞ as the increasing union of A_n as $n \to \infty$; this might require some thinking.

iv) Prove that A_∞ is an infinite simple amenable group.

Exercise 4
Let G_n be a family of groups indexed by $n \in \mathbf{N}$ and let $G = \prod_{n \in \mathbf{N}} G_n$.

i) Give an example with each G_n amenable but G non-amenable.
Hint: you can even take G_n finite and G containing $\mathbf{SL}_2(\mathbf{Z})$.

ii) Suppose that all G_n are identical to some finite group G_0. Prove that G is amenable.
Hint: write G as a directed union, somehow.

iii) Give an example with all G_n identical to some amenable G_0 but G non-amenable.
Hint: Try to use point (i) but beware of point (ii).