Analysis on groups

Problem Set 4 15 March 2018

Exercise 1
(i) Make sure that you understand the proof of the theorem about barycenters. What happens if $\mu \in \mathscr{M}(K)$ is a finite convex combination of Dirac masses?

(ii*) Can you find an example of a convex set K in a Banach space V which is closed and bounded (instead of compact) and of a mean $\mu \in \mathscr{M}(K)$ such that no $x \in K$ satisfies $\lambda(x) = \mu(\lambda|_K)$ for all $\lambda \in V^*$?

Exercise 2
The goal of this exercise is to show that if we replace naively "compact" by "bounded and closed" in the Fixed Point Criterion, then "amenable" becomes "finite".

Prove that a group G is finite if and only if:
For every topological vector space V with a continuous linear G-action, every non-empty bounded closed convex G-invariant set contains a fixed point.

Hint: for the harder direction, try to think about some basic normed vector space that you can construct using G.

Exercise 3
Let G be an amenable group. Prove that there is $\mu \in \mathscr{M}(G)$ which is invariant under both the left and the right multiplication.

Exercise 4
Find a sequence G_n of amenable groups such that the product $\prod_n G_n$ is non-amenable.