

# Analysis on groups

Problem Set 4

15 March 2018

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## Exercise 1

(i) Make sure that you understand the proof of the theorem about barycenters. What happens if  $\mu \in \mathcal{M}(K)$  is a finite convex combination of Dirac masses?

(ii\*) Can you find an example of a convex set  $K$  in a Banach space  $V$  which is closed and bounded (instead of compact) and of a mean  $\mu \in \mathcal{M}(K)$  such that no  $x \in K$  satisfies  $\lambda(x) = \mu(\lambda|_K)$  for all  $\lambda \in V^*$ ?

## Exercise 2

*The goal of this exercise is to show that if we replace naively “compact” by “bounded and closed” in the Fixed Point Criterion, then “amenable” becomes “finite”.*

Prove that a group  $G$  is finite if and only if:

For every topological vector space  $V$  with a continuous linear  $G$ -action, every non-empty bounded closed convex  $G$ -invariant set contains a fixed point.

*Hint: for the harder direction, try to think about some basic normed vector space that you can construct using  $G$ .*

## Exercise 3

Let  $G$  be an amenable group. Prove that there is  $\mu \in \mathcal{M}(G)$  which is invariant under both the left and the right multiplication.

## Exercise 4

Find a sequence  $G_n$  of amenable groups such that the product  $\prod_n G_n$  is non-amenable.