Analysis on groups

Problem Set 4 — 16 March 2017

Exercise 0
Consider a map $f\colon X \to Y$ between sets X, Y. What can you say if X or Y or both are empty? Can there be an action of a group G on $X = \varnothing$? is it amenable? does it admit a paradoxical decomposition?

(For the rest of the class/exercises, we always assume $X \neq \varnothing$.)

Exercise 1
Let X be a set and define $\ell^1(X) = \{f\colon X \to \mathbf{R} : \|f\|_1 < \infty\}$, where $\|f\|_1 = \sum_{x\in X} |f(x)|$.

(i) Make sure that you really understand the definition of $\sum_{x\in X} |f(x)|$ and deduce that $\sum_{x\in X} f(x)$ makes sense for $f \in \ell^1(X)$.

Let moreover $P(X) = \{f \in \ell^1(X) : f \geq 0, \sum_{x\in X} f(x) = 1\}$.

(ii) Explain why we can consider $P(X) \subseteq \mathscr{M}(X)$.

(ii) Suppose that a group G acts on X. Prove that $P(X)^G \neq \varnothing$ (if and) only if G has a finite orbit in X.

Exercise 2
(i) Make sure that you understand the proof of the theorem about barycenters. What happens if $\mu \in \mathscr{M}(K)$ is a finite convex combination of Dirac masses?

*(ii) Can you find an example of a convex set K in a Banach space V which is closed and bounded (instead of compact) and of a mean $\mu \in \mathscr{M}(K)$ such that no $x \in K$ satisfies $\lambda(x) = \mu(\lambda|_K)$ for all $\lambda \in V^*$?

Exercise 3
Consider the notation

$$X = A_1 \sqcup \ldots \sqcup A_n \sqcup B_1 \sqcup \ldots \sqcup B_m = g_1A_1 \sqcup \ldots \sqcup g_nA_n = h_1B_1 \sqcup \ldots \sqcup h_mB_m$$

used in class for a paradoxical decomposition. The number $n + m$ is called the **Tarski number** of this decomposition.

i) Show that the Tarski number is ≥ 4.

ii) If the Tarski number is 4, prove that the group contains a free subgroup F_2.
Hint: show first that you can suppose $g_1 = h_1 = e$.