Analysis on groups

Problem Set 3 9 March 2017

Exercise 1
Let X be a set and $m \in \ell^\infty(X)^*$. Consider the following properties:

(i) $m(f) \geq 0$ if $f \geq 0$.
(ii) $m(\mathbf{1}_X) = 1$, where $\mathbf{1}_X$ is the constant function 1.
(iii) $\|m\| = 1$.

Show that each of these properties follows from the two other.

Exercise 2
The goal of this exercise is to prove that the group $\mathrm{SL}_2(\mathbf{Z})$ contains a free group F_2.

Consider the elements $u = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $v = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$. Analyse again what they do to a vector $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ in $\mathbf{Z}^2$ by discussing whether $|x| < |y|$ etc. You might want to draw a picture.

Now prove that the subgroup $\langle u, v \rangle$ generated by $\{u, v\}$ is the free group on $\{u, v\}$.

Hint: first, understand that this amounts simply to proving that every (non-empty) reduced word using u, v represents a non-trivial element of $\mathrm{SL}_2(\mathbf{Z})$.

Exercise 3
(i) Prove that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ generate $\mathrm{SL}_2(\mathbf{Z})$ and satisfy a non-trivial relation.

(ii) Deduce that $\mathrm{SL}_2(\mathbf{Z})$ itself is not free.

(iii) Prove that $\mathrm{SL}_2(\mathbf{Z})$ is "virtually free" in the sense that the quotient $\mathrm{SL}_2(\mathbf{Z})/\langle u, v \rangle$ is finite.

***Exercise 4**
Prove the following:
If a sequence $\{v_n\}_{n \in \mathbf{N}}$ of $\ell^1(\mathbf{Z}) \subseteq \ell^\infty(\mathbf{Z})^*$ converges weak-* to some $v \in \ell^\infty(\mathbf{Z})^*$, then $v \in \ell^1(\mathbf{Z})$.

(Notice that this statement implies in particular the result of Exercise 2 from Problem Set 2.)

Why does this not contradict Exercise 2(iii) from Problem Set 1?!?

*Exercises marked with a star are harder than what I would expect at the exam; if you don't manage to solve them, focus your energy on the other exercises…