Analysis on groups

Problem Set 2 28 February 2019

Exercise -1 (part (i) is useful for Ex. 3 of Set 1 . . .)
Let $X$ be a set and define $\ell^1(X) = \{ f : X \rightarrow \mathbb{R} : \|f\|_1 < \infty \}$, where $\|f\|_1 = \sum_{x \in X} |f(x)|$.

(i) Make sure that you really understand the definition of $\sum_{x \in X} |f(x)|$ and deduce that $\sum_{x \in X} f(x)$ makes sense for $f \in \ell^1(X)$.

Let moreover $P(X) = \{ f \in \ell^1(X) : f \geq 0, \sum_{x \in X} f(x) = 1 \}$.

(ii) Explain why we can consider $P(X) \subseteq \mathcal{M}(X)$.

(ii) Suppose that a group $G$ acts on $X$. Prove that $P(X)^G \neq \emptyset$ (if and) only if $G$ has a finite orbit in $X$.

Exercise 0
Prove that the group $\mathbb{Z}^2$ is amenable.

This will later be a very special case of a theorem, but for now try to imitate the case of $\mathbb{Z}$ seen in class.

Exercise 1
Prove that the sequence $\{ \mu_n \}$ defined by

$$\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_j$$

has no convergent subsequence in the compact space $\mathcal{M}(\mathbb{Z})$.

Meditate deeply about the fact that it admits a convergent sub-net.

If you are brave, prove the following more general fact:
If a sequence $\{v_n\}_{n \in \mathbb{N}}$ of $\ell^1(\mathbb{Z}) \subseteq \ell^\infty(\mathbb{Z})^*$ converges weak-* to some $v \in \ell^\infty(\mathbb{Z})^*$, then $v \in \ell^1(\mathbb{Z})$.

Exercise 2
Let $X$ be a set and $m \in \ell^\infty(X)^*$. Consider the following properties:

(i) $m(f) \geq 0$ if $f \geq 0$.
(ii) $m(1_X) = 1$, where $1_X$ is the constant function 1.
(iii) $\|m\| = 1$.

Show that each of these properties follows from the two other.