Analysis on groups

Problem Set 2 — 1 March 2018

Exercise -1
Read `http://en.wikipedia.org/wiki/Net_(mathematics)` or use the library to understand the concept of *nets*, also known as *generalized sequences*. Understand the connection to open and closed sets, to continuous maps, to compactness. In particular, make sure to understand sub-nets.

Exercise 0
Let X be a set and define $\ell^1(X) = \{f \colon X \to \mathbf{R} : \|f\|_1 < \infty\}$, where $\|f\|_1 = \sum_{x \in X} |f(x)|$.

(i) Make sure that you really understand the definition of $\sum_{x\in X} |f(x)|$ and deduce that $\sum_{x\in X} f(x)$ makes sense for $f \in \ell^1(X)$.

Let moreover $P(X) = \{f \in \ell^1(X) : f \geq 0, \sum_{x\in X} f(x) = 1\}$.

(ii) Explain why we can consider $P(X) \subseteq \mathscr{M}(X)$.

(ii) Suppose that a group G acts on X. Prove that $P(X)^G \neq \varnothing$ (if and) only if G has a finite orbit in X.

Exercise 1
Prove that the group $\mathbf{Z}^2$ is amenable.

This will later be a very special case of a theorem, but for now proceed as follows: construct a sequence of means $\{\mu_n\}$ in $\mathscr{M}(\mathbf{Z}^2)$ such that any accumulation point of this sequence is invariant; try to imitate the case of $\mathbf{Z}$ seen in class.

Exercise 2*
Prove that the sequence $\{\mu_n\}$ defined by

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_j$$

has no convergent subsequence in the compact space $\mathscr{M}(\mathbf{Z})$.

Meditate deeply about the fact that the sequence of Exercise 1 must admit a convergent sub-net.

*An exercise with a star might be more difficult than the others.