Analysis on groups

Problem Set 13 May 2017

Exercise 1
Let G be a group with property (T) and let $\delta > 0$. Prove that there is $\epsilon > 0$ and $S \subseteq_{\mathrm{f}} G$ with the following property:

For every orthogonal representation V and every (S, ϵ)-invariant vector $v \in V$, there is $u \in V^G$ with $\|u - v\| < \delta \|v\|$.

Hints: given a family (V_n) of orthogonal representations, we can define a new orthogonal representation using the direct sum of all V_n. Also, given an orthogonal representation V, we can consider the orthogonal complement of V^G in V.

Exercise 2
Prove that every finite group has property (T).

(This finishes the proof that "amenable & (T) $\iff$ finite".)

Exercise 3
Prove that the product of two groups with property (T) still has property (T).

Exercise 4
Let G be a group and $H \lhd G$ be a normal subgroup of finite index. Suppose that H has property (T) and prove that G also has property (T).

Hint: use a previous exercise from this problem set.

Remark: if you do Ex. 5, then you can remove the normality assumption in the current Ex. 4.

***Exercise 5**
Let G be a group with property (T) and $H < G$ be a subgroup of finite index. Prove that H also has property (T).

Hint: given an orthogonal representation V of H, we can define an orthogonal representation W of G as follows. The Hilbert space W is the direct sum of copies of V indexed by $H \backslash G$. To define a representation of G on W, identify W with the set of H-equivariant maps

$$\Big\{ w \colon G \to V \;\Big|\; w(hx) = hw(x) \;\forall h \in H, x \in G \Big\}$$

and let G act on w by $(gw)(x) = w(xg)$.

Remark: You can deduce from this exercise that $\mathbf{SL}_2(\mathbf{Z})$ does not have property (T) (using a previous problem set).