Analysis on groups

Problem Set 12 18 May 2017

Exercise 0 (topology)
Recall the definitions of "second countable", "metrizable" and "separable".

(i) A metrizable space is second countable iff it is separable.

(ii) A compact space is second countable iff it is metrizable.

*(iii) A compact space K is metrizable iff the Banach space $C(K)$ (with sup-norm) is separable.

Exercise 1
(i) Prove that the compact space $\beta\mathbf{N}$ is separable but not second countable.

Hint: look at some sequence in $\mathbf{N}$.

*(ii) Prove directly that $\ell^\infty(\mathbf{N})$ is not separable; deduce another proof of (i).

Exercise 2
Let $f\colon X \to X$ be a bijection of a set X. Show that $\beta f\colon \beta X \to \beta X$ has a fixed point (if and) only if f does.

Hint: Show first that the extension to $\beta\mathbf{Z} \to \beta\mathbf{Z}$ *of the map* $\mathbf{Z} \to \mathbf{Z}$, $n \mapsto n+1$ *has no fixed point, then try to generalize.*

Moral of the exercise: invariant means in $\mathscr{M}(X)$ *are generally not in* $\beta X \subseteq \mathscr{M}(X)$.

Exercise 3 (Gelfand transform)
Consider the following map "$\widehat{\ }$"

$$\begin{array}{ccl} \ell^\infty(X) & \longrightarrow & C(\beta X) \\ f & \longmapsto & \hat{f} \\ & & \hat{f}(x) := x(f) \end{array}$$

Prove that it is well-defined, bijective and isometric.

***Exercise 4**
If you know what usual ("Radon") probability measures on a compact space are, then Ex. 3 shows that $\mathscr{M}(X)$ can be identified with the (convex compact) space of probability measures on βX. (Why?) Now explain why this does not contradict the fact that means are usually not countably additive!