Analysis on groups

Problem Set 12 — 24 May 2018

Exercise 1
Prove that every finite group has property (T).

(This finishes the proof that "amenable & (T) $\iff$ finite".)

Exercise 2
Let G be a group with property (T) and let $\delta > 0$. Prove that there is $\epsilon > 0$ and $S \subseteq_{\mathrm{f}} G$ with the following property:

For every orthogonal representation V and every (S, ϵ)-invariant vector $v \in V$, there is $u \in V^G$ with $\|u - v\| < \delta \|v\|$.

Hints: given a family (V_n) of orthogonal representations, we can define a new orthogonal representation using the direct sum of all V_n. Also, given an orthogonal representation V, we can consider the orthogonal complement $V_0 \subseteq V$ of V^G.

Exercise 3
Prove that the product of two groups with property (T) still has property (T).

Exercise 4
Let $H < G$ be a finite index subgroup of G. Prove that H has property (T) if and only if G has property (T).

Hint: we already know the "if" part; for "only if", show first that we can assume H normal in G and then use another exercise of this set.