Analysis on groups

Problem Set 11 11 May 2017

Exercise 0 (This fact was used implicitly in Tuesday's class)
Let G be a group, $S \subseteq_{\mathrm{f}} G$ and $\epsilon > 0$. Let $\mu \in \mathrm{Prob}(G)$ be (S, ϵ)-invariant.
Prove that $\|\vartheta * \mu - \mu\|_1 < \epsilon$ for any $\vartheta \in \mathrm{Prob}(G)$ supported on S.

Exercise 1
What happens to Exercise 4 of last week's Problem Set 10 if we only assume that the support of μ generates G *as a group*?

Exercise 2
Suppose that $\mu \in \mathrm{Prob}(G)$ for an uncountable group G. Prove that (G, μ) is not Liouville.

Exercise 3
Let G be a countable amenable group. Prove that there is a probability measure μ on G such that $\mu = \check{\mu}$ and

$$\lim_{n\to\infty} \|g\mu^{*n} - \mu^{*n}\| = 0$$

for all $g \in G$.

(Notice that this implies also $\lim_{n\to\infty} \|\mu^{*n}g - \mu^{*n}\| = 0$.)

***Exercise 4** (Optional because it involves integration)
Let G be a group acting on a (usual) measure space (Z, ν) and let $\mu \in \mathrm{Prob}(G)$. We assume that ν is *$\check{\mu}$-stationary*, which means by definition $\check{\mu} * \nu = \nu$ (make sure that you understand what this assumption means).

Given any integrable function $f \in L^1(Z, \nu)$, define $\mathfrak{P}f \colon G \to \mathbf{R}$ by

$$\mathfrak{P}f(g) = \nu(g^{-1}f) = \int_Z f(gz)\, d\nu(z).$$

Prove that $\mathfrak{P}f$ is μ-harmonic and that $\mathfrak{P}$ is G-equivariant.

Remarks. (i) This $\mathfrak{P}$ is a version of the Poisson transform. *(ii) If for instance Z is a compact topological space, one can prove that a μ-stationary measure ν always exists; do you see why?*