Exercise 0
Consider a map \( f : X \to Y \) between sets \( X, Y \). What can you say if \( X \) is empty, or \( Y \), or both? Can there be an action of a group \( G \) on \( X = \emptyset \)? is it amenable? does it admit a paradoxical decomposition?

(For the rest of the class/exercises, we always assume \( X \neq \emptyset \))

Exercise 1
We will prove in class that the closed unit ball \( B \) of \( \mathbb{R}^3 \) is equidecomposable with any subset \( A \subseteq \mathbb{R}^3 \) that is (1) bounded and (2) with non-empty interior.

(i) Check that condition (1) is necessary.

(ii) Find an example showing that condition (2) is not necessary.

Exercise 2
Let \( G \) be a group acting on a set \( X \).

(i) Check that the action on \( X \) is amenable as soon as there exists some orbit \( Gx_0 \subseteq X \) on which the \( G \)-action is amenable.

*(ii) Can you imagine an example of an amenable \( G \)-action such that the action on every orbit is non-amenable? (even without giving a complete proof.)

Exercise 3
Consider the action of \( G = \text{SO}(3) \) on \( X = \mathbb{R}^3 \setminus \{0\} \). Prove that all stabilisers are amenable. Same question for \( G = \text{O}(3) \).

Exercise 4
The group \( G \) of isometries of \( \mathbb{R}^n \) consists by definition of all bijections preserving the Euclidean distance. Prove \( G = \mathbb{R}^n \rtimes \text{O}(n) \).

*Hint: show first that any isometry fixing 0 must be linear.*

P.S.: For which \( n \) is this group amenable?