Exercise 0
Let $f, g, h$ be functions on a group $G$ and let $x, y \in G$.

(i) Verify that $f \ast (g \ast h) = (f \ast g) \ast h$ holds (as soon as all these sums are absolutely convergent).

(ii) Compute $\delta_x \ast \delta_y$, $\delta_x \ast f$ and $f \ast \delta_x$.

(iii) Find a formula for $(f \ast g) \vee$, where for any function $h$ we write $h \vee(x) := h(x^{-1})$.

(iv) Let $1 \leq p \leq \infty$ and suppose $f \in \ell^p(G)$, $g \in \ell^1(G)$. Verify $\|g \ast f\|_p \leq \|f\|_p \cdot \|g\|_1$. Deduce $\|f \ast g\|_p \leq \|f\|_p \cdot \|g\|_1$. (You might want to separate the case $p = \infty$.)

(v) For $f \in \ell^1(G)$, write $\Sigma := \sum_{x \in G} f(x)$. Assuming $f, g \in \ell^1(G)$, prove $\Sigma(f \ast g) = \Sigma f \Sigma g$. Deduce that $f, g \in \text{Prob}(G)$ implies $f \ast g \in \text{Prob}(G)$.

Exercise 1
(i) On the group $\mathbb{Z}$ with the probability measure $\mu = \frac{1}{3} \delta_{-1} + \frac{2}{3} \delta_1$, determine all $\mu$-harmonic functions.

(ii) Same question for $\mu = \frac{1}{4} (\delta_{-1} + \delta_1 + \delta_2)$.

Exercise 2
In class, we found a function $f: F_2 \to \mathbb{R}$ on $F_2 = \langle a, b \rangle$ that is $\mu$-harmonic for

$$
\mu = \frac{1}{4} (\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}}).
$$

Find your own $\mu$-harmonic function on $F_2$ for the same measure $\mu$; try not to choose just a simple variation of $f$.

Exercise 3
Let $G$ be a group, $\mu \in \text{Prob}(G)$ and suppose that the support of $\mu$ generates $G$ as semigroup (or as monoid). Prove the following maximum principle:

If a $\mu$-harmonic function $f$ has a maximum on $G$, then $f$ is constant.

N.B.: You might want to verify that the bounded harmonic function on $F_2$ seen in class does not achieve its inf nor its sup.

What happens if we only assume that the support of $\mu$ generates $G$ as a group?

Exercise 4
Let $G$ be an uncountable group and $\mu \in \text{Prob}(G)$. Prove that $(G, \mu)$ is not Liouville.