Analysis on groups

Problem Set 10 — 4 May 2017

Exercise 1
Let f, g, h be functions on a group G and let $x, y \in G$.

(i) Verify that $f * (g * h) = (f * g) * h$ holds (as soon as all these sums are absolutely convergent).

(ii) Compute $\delta_x * \delta_y$, $\delta_x * f$ and $f * \delta_x$.

(iii) Find a formula for $(f * g)^\vee$, where for any function h we write $h^\vee(x) := h(x^{-1})$.

(iv) Let $1 \leq p \leq \infty$ and suppose $f \in \ell^p(G)$, $g \in \ell^1(G)$. Verify $\|g * f\|_p \leq \|f\|_p \cdot \|g\|_1$. Deduce $\|f * g\|_p \leq \|f\|_p \cdot \|g\|_1$. (You might want to separate the case $p = \infty$.)

(v) For $f \in \ell^1(G)$, write $\Sigma f := \sum_{x \in G} f(x)$. Assuming $f, g \in \ell^1(G)$, prove $\Sigma(f * g) = \Sigma f \Sigma g$. Deduce that $f, g \in \mathrm{Prob}(G)$ implies $f * g \in \mathrm{Prob}(G)$.

Exercise 2
(i) On the group $\mathbf{Z}$ with the probability measure $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$, check that a function is μ-harmonic if and only if it is affine.

(ii) For $\mu = \frac{1}{3}(\delta_{-1} + \delta_1 + \delta_2)$, determine all μ-harmonic functions.

Exercise 3
In class, we found a function $f\colon F_2 \to \mathbf{R}$ on $F_2 = \langle a, b \rangle$ that is μ-harmonic for

$$\mu = \frac{1}{4}(\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}}).$$

Find your own μ-harmonic function on F_2 for the same measure μ; try not to choose just a simple variation of f.

Exercise 4
Let G be a group, $\mu \in \mathrm{Prob}(G)$ and suppose that the support of μ generates G as semigroup (or as monoid). Prove the following *maximum principle*:

If a μ-harmonic function f has a maximum on G, then f is constant.

You might want to verify that the bounded harmonic function on F_2 seen in class does not achieve its inf nor its sup.