Analysis on groups

Problem Set 10 2 May 2019

Exercise 1
Consider the rotations

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}, \qquad b = \begin{pmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(You might want to check that $a, b \in \mathrm{SO}(3)$ and give a geometric description of a and b in everyday language.) The goal of this exercise is to show that a, b generate a free group F_2. Thus, for a reduced word w of length k in the letters $a^{\pm 1}$, $b^{\pm 1}$, we need to show that w does not represent the trivial rotation. There is no loss of generality in assuming that w ends with $b^{\pm 1}$ (why?). It suffices to prove that $w(1,0,0) \neq (1,0,0)$, where I used the horizontal vector notation to save space.

i) Prove that there are integers $x, y, z \in \mathbf{Z}$ such that $w(1,0,0) = 3^{-k}(x, y\sqrt{2}, z)$.

ii) Prove that y is not divisible by 3; then in particular $y \neq 0$, finishing the proof of the exercise. You can do this by induction by looking carefully at the first two letters of w and how they change the parameter y.

Exercise 2
Give an example of a matching problem $\Sigma\colon X \to \mathscr{P}(Y)$ that does not admit an injective selection even though it satisfies $|\Sigma_A| \geq |A|$ for all $A \subseteq X$.

(This does not contradict the result seen in class since none of the sets is assumed finite here.)

Exercise 3
Find a concrete example of a finite selection problem $\Sigma\colon X \to \mathscr{P}_{\mathrm{f}}(Y)$ such that $|\Sigma_A| \geq 2|A|$ for all $A \subseteq_{\mathrm{f}} X$ and of an injective selection $\sigma\colon X \to Y$ such that the new selection problem

$$X \ni x \longmapsto \Sigma(x) \setminus \sigma(X) \subseteq_{\mathrm{f}} Y$$

does not admit an injective selection anymore.